CROSS-INTERSECTING PAIRS OF HYPERGRAPHS

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Abstract. Two hypergraphs $H_1,\ H_2$ are called *cross-intersecting* if $e_1 \cap e_2 \neq \emptyset$ for every pair of edges $e_1 \in H_1,\ e_2 \in H_2$. Each of the hypergraphs is then said to *block* the other. Given integers n, r, m we determine the maximal size of a sub-hypergraph of $[n]^r$ (meaning that it is r-partite, with all sides of size n) for which there exists a blocking sub-hypergraph of $[n]^r$ of size m. The answer involves a self-similar sequence, first studied by Knuth. We also study the same question with $\binom{n}{r}$ replacing $[n]^r$. These results yield new proofs of some known Erdős-Ko-Rado type theorems.

1. Blockers in r-partite hypergraphs

1.1. **Blockers.** For a set A and a number r let $\binom{A}{r}$ be the set of all subsets of size r of A, in other words the complete r-uniform hypergraph on A. Given numbers r and n let $[n] = \{1, 2, \ldots, n\}$, and let $[n]^r$ be the complete r-partite hypergraph with all sides being equal to $[n]$. Let U be either $\binom{[n]}{r}$ or $[n]^r$, and let F be a sub-hypergraph of U. The *blocker* $B(F) = B(U, F)$ of F is the set of those edges of U that meet all edges of F. For a number t we denote by $b_p(t)$ (resp. $b_c(t)$ - reference to the uniformity r is suppressed in both notations) the maximal size of $|B([n]^r, F)|$ (resp. $|B(\binom{[n]}{r}, F)|$), where F ranges over all sets of t edges in $[n]^r$ (resp. $\binom{[n]}{r}$). The subscript p alludes at "partite", and the subscript c alludes at "complete". The aim of this paper is to calculate $b_p(t)$ and $b_c(t)$ for all values of n, r and t. As a side benefit, this will enable us to give new proofs of some well-known Erdős-Ko-Rado type results.

1.2. **Cross intersecting versions of the Erdős-Ko-Rado theorem.** The famous Erdős-Ko-Rado (EKR) theorem [9] states that if $r \leq \frac{n}{2}$ and a hypergraph $H \subseteq \binom{[n]}{r}$ has more than $\binom{n-1}{r-1}$ edges, then H contains two disjoint sets. Many extensions of this theorem have been proved for pairs of hypergraphs. In [17, 20] the following was proved:

Theorem 1.1. *If $r \leq \frac{n}{2}$, and $H_1, H_2 \subseteq \binom{[n]}{r}$ satisfy $|H_1||H_2| > \binom{n-1}{r-1}^2$ (in particular if $|H_i| > \binom{n-1}{r-1},\ i = 1, 2$), then there exist disjoint edges, $e_1 \in H_1,\ e_2 \in H_2$.*

In [17] this was also extended to hypergraphs of different uniformities. Versions of this result were proved for cross t-intersecting pairs of hypergraphs, in [13, 21, 23].

The EKR theory has been also extended to sets living in $[n]^r$, rather than $\binom{[n]}{r}$. An easy observation is that any subset of $[n]^r$ of size larger than n^{r-1} contains two disjoint edges. This can be proved from the fact that $[n]^r$ is the union of n^{r-1} perfect matchings. More interesting are cross-intersecting type results:

Theorem 1.2. *A pair F_1, F_2 of subsets of $[n]^r$ satisfying $|F_1| > n^{r-1}$ and $|F_2| \geq n^{r-1}$ has a rainbow matching.*

And the even stronger:

Theorem 1.3. *If $F_1, F_2 \subseteq [n]^r$ and $|F_1||F_2| > n^{2(r-1)}$ then the pair (F_1, F_2) has a rainbow matching.*

Theorem 1.3 was proved in [18]. It was generalized to cross t-intersecting pairs of hypergraphs and to hypregaphs of different uniformities in [1, 3, 4, 13, 19, 22] ([1, 22] use spectral methods).

At the end of the next section we shall use the techniques of the present paper to give new proofs for these results.

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2. A self-similar sequence

Denote the sides of $[n]^r$ by $V_1, \dots, V_r$ (so, all V_i's are of size n). Choose one vertex v_i from each V_i. Let Ψ_r be the set of (possibly empty) sequences σ of length at most $r-1$ consisting of $\wedge$'s and $\vee$'s. Let $\Sigma_r = \Psi_r \cup \{\alpha, \omega\}$, where $\alpha = \alpha_r$ and $\omega = \omega_r$ are new elements. Note that $|\Sigma_r| = 2^r + 1$. We define hypergraphs $F_r(\sigma)$ for all $\sigma \in \Sigma_r$, as follows. Let $F_r(\alpha) = \emptyset$ and $F_r(\omega) = [n]^r$. For a sequence $\sigma \in \Psi_r$ having length $m \geq 0$, and whose j-th term is denoted by σ_j ($j \leq m$), let:

$$F_r(\sigma) = \{e \in [n]^r \mid v_1 \in e \ \sigma_1(v_2 \in e \ \sigma_2(v_3 \in e \dots \sigma_m(v_{m+1} \in e) \dots)\}$$

For example, $F_r(\emptyset) = \{e \in [n]^r \mid v_1 \in e\}$ and $F_r(\wedge, \wedge, \vee)$ is the set of edges $e \in [n]^r$ satisfying:

$$v_1 \in e \wedge (v_2 \in e \wedge (v_3 \in e \vee (v_4 \in e)))$$

Let $f_r(\sigma) = |F_r(\sigma)|$. Note that $\Psi_{r-1} \subseteq \Psi_r$.

Lemma 2.1.

If $\sigma \in \Psi_{r-1}$ then

(1) $f_r(\sigma) = n f_{r-1}(\sigma)$
(2) $f_r(\wedge, \sigma) = f_{r-1}(\sigma)$
(3) $f_r(\vee, \sigma) = n^{r-1} + (n-1) f_{r-1}(\sigma)$

Part 1 is true since $F_r(\sigma) = F_{r-1}(\sigma) \times V_r$. Part 2 is true since an edge in $F_r(\wedge, \sigma)$ is obtained from an edge $f \in F_{r-1}(\sigma)$, with indices shifted by 1, by adding v_1. Part 3 is true since $F_r(\vee, \sigma) = \{v_1\} \times V_2 \times \dots \times V_r \cup (V_1 \setminus \{v_1\}) \times F_{r-1}(\sigma)$ (where, again, edges in $F_{r-1}(\sigma)$ have their indices shifted by 1).

Order $f_r(\sigma)$ by size:

$$0 = f_r(\alpha) < f_r(\sigma_1) < f_r(\sigma_2) < \dots < f_r(\sigma_{2^r}) = n^r$$

Let $N(i) = f_r(\sigma_i)$ $(0 \leq i \leq 2^r)$ (the mention of r is suppressed in this notation).

Example 2.2.

(1) $N(0) = f_r(\alpha) = 0$.
(2) $N(1) = f_r(\wedge, \wedge, \dots, \wedge)$ $(r-1$ times), which is 1.
(3) $N(2) = f_r(\wedge, \wedge, \dots, \wedge)$ $(r-2$ times), which is n.
(4) $N(2^{r-1}) = f_r(\emptyset) = n^{r-1}$.
(5) $N(2^r) = f_r(\omega) = n^r$.

In accord we order Σ_r as $\sigma(i)$ $(0 \leq i \leq 2^r)$. For example $\sigma(0) = \alpha$, $\sigma(2^r) = \omega$. We also define the inverse function, which we name "i": if $\sigma(q) = \tau$, then $i(\tau) = q$.

Clearly, for every $\beta, \gamma, \delta \in \Psi_r$ such that $(\beta, \wedge, \gamma)$ and $(\beta, \vee, \delta)$ belong to Ψ_r we have:

$$i((\beta, \wedge, \gamma)) < i(\beta) < i((\beta, \vee, \delta)) \tag{1}$$

The elements of Ψ_r can be viewed as the nodes of a binary tree, the depth of a node being the length of the sequence (so the root, with depth 0, is the empty sequence). The order on Ψ_r, uniquely determined by (1), is known as the "in-order depth first search" on the tree, where $\wedge$ ("left") precedes $\vee$ ("right").

This description of the order on Ψ_r entails an explicit formula for $\sigma(i)$. Represent $i \neq 0, 2^r$ in binary form: $i = 2^{k_0} + 2^{k_1} + \dots + 2^{k_s}$, where $k_0 > k_1 > \dots > k_s$. Then $\sigma(i)$ is of length $r - k_s - 1$, and it consists of s symbols of $\vee$ and $r - k_s - 1 - s$ symbols of $\wedge$. It starts with $r - k_0 - 1$ (possibly zero) $\wedge$'s; if $s > 0$ these are followed by a $\vee$; this is followed by $k_0 - k_1 - 1$ (possibly zero) $\wedge$'s, and if $s > 1$ this is followed by a $\vee$, followed by $k_1 - k_2 - 1$ $\wedge$'s, and so forth.

For example, $\sigma_6(13) = \sigma_6(2^3 + 2^2 + 2^0) = (\wedge, \wedge, \vee, \vee, \wedge)$.

The numbers $N(i)$ can also be written explicitly:

$$N(i) = \sum_{s \leq i} n^{k_s}(n-1)^s$$

The explicit description of $\sigma(i)$ and the formula for $N(i)$ will not be used below, and hence their proofs are omitted.

Example 2.3. The values of N_3 are:

$0,\ 1,\ n,\ n+(n-1),\ n+n(n-1) = n^2,\ n^2+(n-1),\ n^2+(n-1)n,\ n^2+(n-1)(2n-1),\ n^2+(n-1)n^2 = n^3.$

Lemma 2.4.

(1) For $i \leq 2^{r-1}$ we have $N_r(i) = N_{r-1}(i)$, namely the sequence $N_{r-1}(i)$ is an initial segment of $N_r(i)$.
(2) $\sigma(2^p) = (\wedge, \wedge, \ldots, \wedge)$, a sequence of $r-p-1$ $\wedge$'s, and $N(2^p) = n^p$.
(3) For $i < 2^p$ the sequences $\sigma(i)$ are of the form $(\sigma(2^p), \wedge, \beta)$ (β being some sequence), and for $2^p < i < 2^{p+1}$ the sequences $\sigma(i)$ are of the form $(\sigma(2^p), \vee, \beta)$.
(4) For $p \leq r-1$ and $i \leq 2^p$, we have

$$N(2^p + i) = N(2^p) + (n-1)N(i) = n^p + (n-1)N(i)$$

Part 1 is true by part 2 of Lemma 2.1, since $\sigma(1), \ldots, \sigma(2^{r-1}-1)$ all start with a $\wedge$. Parts 2 and 3 follow from Equation (1) and the remark following it. Part 4 follows from part 3 of Lemma 2.1.

Part 4 says that for fixed n, the numbers $N_r(i)$ have a self-similar pattern. Each sequence N_r is obtained from the sequence N_{r-1} by concatenating it with the sequence M defined by $M(i) = n^r + (n-1)N_{r-1}(i)$. The concatenation includes also an identification of elements: the first element of M is identified with the last element of N_{r-1}, both being equal to n^{r-1}. This entails:

Lemma 2.5. *If $b, c \leq 2^p$ then $N(2^p + b) - N(2^p + c) = (n-1)(N(b) - N(c))$.*

2.1. **Shifting.** *Shifting* is an operation on a hypergraph H, defined with respect to a specific linear ordering "$<$" on its vertices. For $x < y$ in $V(H)$ define $s_{xy}(e) = e \cup x \setminus \{y\}$ if $x \notin e$ and $y \in e$, provided $e \cup x \setminus \{y\} \notin H$; otherwise let $s_{xy}(e) = e$. We also write $s_{xy}(H) = \{s_{xy}(e) \mid e \in H\}$. If $s_{xy}(H) = H$ for every pair $x < y$ then H is said to be *shifted*.

Given an r-partite hypergraph G with sides M and W together with linear orders on each of its sides, an *r-partite shifting* is a shifting s_{xy} where x and y belong to the same side. G is said to be *r-partitely shifted* if $s_{xy}(H) = H$ for all pairs $x < y$ that belong to the same side.

Given a collection $\mathcal{H} = (H_i,\ i \in I)$ of hypergraphs, we write $s_{xy}(\mathcal{H})$ for $(s_{xy}(H_i),\ i \in I)$.

As observed in [8] (see also [2]), shifting does not increase the matching number of a hypergraph. This can be generalized to rainbow matchings.

Lemma 2.6. *Let $\mathcal{F} = (F_i \mid\ i \in I)$ be a collection of hypergraphs, sharing the same linearly ordered ground set V, and let $x < y$ be elements of V. If $s_{xy}(\mathcal{F})$ has a rainbow matching, then so does $\mathcal{F}$.*

Proof. Let $R = s_{xy}(e_i),\ i \in I$, be a rainbow matching for $s_{xy}(\mathcal{F})$. We may assume that $s_{xy}(e_i) \neq e_i$ for some $i \in I$, meaning that $y \in e_i$ and $x \notin e_i$. Since R is a matching, only one edge in R can contain x, so $s_{xy}(e_j) = e_j$ for all $j \neq i$.

Assume first that $y \notin s_{xy}(e_j)$ for any $j \in I$. Then replacing $s_{xy}(e_i)$ by e_i as a representative for F_i results in a rainbow matching for $\mathcal{F}$. So, we may assume that $y \in s_{xy}(e_j)$ for some $j \neq i$. Then $e'_j = e_j \setminus \{y\} \cup \{x\} \in F_j$, or else $s_{xy}(e_j) \neq e_j$, contrary to our previous conclusion. Then replacing $e_j = s_{xy}(e_j)$ in R by e'_j as a representative for F_j and replacing $s_{xy}(e_i)$ by e_i as a representative for F_i results in a rainbow matching for $\mathcal{F}$.

□

2.2. **The size of blocking hypergraphs.** For $\sigma \in \Psi_r$ we denote by $\overline{\sigma}$ the sequence obtained by replacing each $\wedge$ by a $\vee$ and vice versa. We also define $\overline{\alpha} = \omega$ and $\overline{\omega} = \alpha$. Clearly, $i(\sigma) > i(\tau)$ if and only if $i(\overline{\sigma}) < i(\overline{\tau})$, and hence we have:

$$i(\overline{\sigma}) = 2^r - i(\sigma) \tag{2}$$

By De Morgan's law, we have:

Lemma 2.7. *$B(F_r(\sigma)) = F_r(\overline{\sigma})$.*

Lemma 2.8.

(1) $N(j+i) \geq N(j) + N(i)$.
(2) If $i \leq j$ then $N(j+i) \geq N(j) + (n-1)N(i)$.

Proof. Clearly, (2) implies (1). So, it suffices to prove (2) using an induction assumption on both (1) and (2), where the induction is on $i+j$. Assume that the lemma is true for all i', j' satisfying $i'+j' < i+j$. Let $j = 2^p + s$, where $s < 2^p$. Assume first that $j+i \leq 2^{p+1}$, and write $j+i = 2^p + t$, where $t \leq 2^p$. By part 4 of Lemma 2.4 N-distances beyond 2^p are the N-distances below 2^p magnified $(n-1)$-fold. Hence we have

$$N(j+i) - N(j) = (n-1)(N(t) - N(s)).$$

By the induction hypothesis on (1) $N(t) - N(s) \geq N(t-s) = N(i)$, and thus $N(j+i) - N(j) \geq (n-1)N(i)$.

Assume next that $j+i > 2^{p+1}$. Let $j+i = 2^{p+1}+w$. Then $i = 2^p + w - s$. Since $2^p + w < 2^{p+1} + w = i+j$, we can apply the induction hypothesis on (1) to the pair (i, s), to obtain

$$N(2^p + w) - N(s) \geq N(i).$$

By Lemma 2.5 $N(2^{p+1}) - N(2^p + s) = (n-1)(N(2^p) - N(s))$ and $N(2^{p+1}+w) - N(2^{p+1}) = (n-1)(N(2^p + w) - N(2^p))$. Adding the last two equalities gives $N(j+i) - N(j) = (n-1)(N(2^p+w) - N(s))$, which by the above is at least $(n-1)N(i)$, completing the proof.

□

A converse inequality is also true, namely for every $k > 1$ it is true that:

$$N(k) = \max\{N(j) + (n-1)N(i) \mid j+i = k,\ i \leq j\} \tag{3}$$

Proof. Let p be maximal such that $2^p < k$, and let $k = 2^p + j$. By Lemma 2.4 (4) $N(k) = N(i) + (n-1)N(j)$. Combining this with Lemma 2.8 proves the desired equality. □

In [16] (3) was used as a defining recursion rule for the sequence $N(i)$ (which appeared there in a different context.)

For a number $t \leq n^r$ denote by $N^*(t)$ the number q such that $N(q-1) < t \leq N(q)$. This is an approximate inverse of N.

Theorem 2.9. *$b_p(t) = N(2^r - N^*(t))$ for every $t \leq n^r$.*

Proof. Let $F = F_r(\sigma(N^*(t))$. Then $|F| \geq t$, and since $B(F) = F_r(\bar{\sigma})$, we have $|B(F)| = N(2^r - N^*(t))$. This proves that $b_p(t) \geq N(2^r - N^*(t))$. To complete the proof we have to show that for every $F \subseteq [n]^r$ of size t we have $|B(F)| \leq N(2^r - N^*(t))$. Write $q = N^*(t)$. We wish to show that $|B(F)| \leq N(2^r - q)$. We do this by induction on r. The case $r = 1$ is easy, so assume that we know the result for $r-1$ and we wish to prove it for r.

Let $F^+ = \{e \setminus V_r \mid v_r \in e \in F\}$ and $F^- = \{e \setminus V_r \mid e \in F,\ v_r \notin e\}$.

By Lemma 2.6 we may assume that F is r-partitely shifted, with v_i being the first element in V_i in the order used for the shifting. In particular, this entails $F^- \subseteq F^+$. Let $B^+ = B_{[n]^{r-1}}(F^+)$ and $B^- = B_{[n]^{r-1}}(F^-)$, and let $f^+ = |F^+|,\ f^- = |F^-|,\ b^+ = |B^+|,\ b^- = |B^-|$. Then $b^- \leq b^+$.

Notation 2.10. If H is a hypergraph and S a set disjoint from $V(H)$, we define $H \diamond S = \{h \cup S \mid h \in H\}$.

Clearly:

$$B(F) = (B^- \diamond \{v_r\}) \cup (B^+ \diamond (V_r \setminus \{v_r\}))$$

and hence

$$|B(F)| = b^- + (n-1)b^+ \tag{4}$$

Let $i = N^*(f^-)$ and $j = N^*(f^+)$. Also let $i' = N^*(b^+)$, $j' = N^*(b^-)$. By Lemma 2.8 we have:

$$|F| \leq f^+ + (n-1)f^- \leq N(i+j)$$

and hence $i + j \geq q$. By the inductive hypothesis $j' \leq 2^{r-1} - i$, and $i' \leq 2^{r-1} - j$, and hence $i' + j' \leq 2^r - (i+j) \leq 2^r - q$. By (4) and Lemma 2.8 , $|B(F)| \leq N(i' + j') \leq N(2^r - q)$, as desired.

□

Our result readily implies Theorem 1.2. Let $t = n^{r-1}$. Then $n^{r-1} = N(2^{r-1})$ and $N^*(t) = 2^{r-1}$. So it follows that $b_p(t) = N(2^r - N^*(t)) = N(2^r - 2^{r-1}) = N(2^{r-1}) = n^{r-1}$, yielding the theorem.

Theorem 1.3 also follows from Theoerm 2.9. The proof requires yet another lemma:

Lemma 2.11. *$N(a)N(b) \leq N(ab)$.*

Proof. By induction on $a + b$. The case $a + b = 0$ is trivial. By (3) $N(a) = N(c) + (n-1)N(d)$ for some $c \leq d < a$ such that $c + d = a$, and $N(b) = N(e) + (n-1)N(f)$ for some $e \leq f < b$ such that $e + f = b$. Then

$$N(a)N(b) = N(c)N(e) + (n-1)[N(d)N(e) + N(c)N(f)] + (n-1)^2 N(d)N(f)$$

Using the induction hypothesis, we get:

$$N(a)N(b) \leq N(ce) + (n-1)[N(d)N(e) + N(c)N(f)] + (n-1)^2 N(df)$$

Using Lemma 2.8 twice we get:

$$N(a)N(b) \leq N(ce + cf) + (n-1)N(de + df) \leq N(ce + cf + de + df) = N(ab).$$

□

The lemma implies that $N(2^{r-1} - q)N(2^{r-1} + q) \leq N(2^{2(r-1)})$ for every $q \leq 2^{r-1}$, meaning that $tb_p(t) \leq n^{2(r-1)}$ for every $t \leq n^{r-1}$, which is another way of formulating Theorem 1.3.

3. Blockers in $\binom{[n]}{r}$

3.1. **Sequences of $\vee$'s and $\wedge$'s and the sets they define.** Let n be a positive integer. For a sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ of $\wedge$'s and $\vee$'s ($m < n$) let $T(\sigma)$ be the set of subsets e of $[n]$, satisfying

$$1 \in e \ \sigma_1 \ (2 \in e \ \sigma_2 \ (3 \in e \ldots \sigma_m \ (m+1 \in e)) \ldots)$$

For a number $r \leq n$ let $T_r(\sigma) = T(\sigma) \cap \binom{[n]}{r}$. Let also $t_r(\sigma) = |T_r(\sigma)|$ (this is the analogue of $f_r(\sigma)$ of the first section).

Example 3.1.

(1) If $\sigma = (\vee, \wedge, \vee, \wedge)$ then $T(\sigma) = \{e \in [n] \mid 1 \in e \ \vee \ (2 \in e \ \wedge \ (3 \in e \ \vee \ (4 \in e \wedge 5 \in e)))\}$.
(2) $T_r(\emptyset) = \{e \in \binom{[n]}{r} \mid 1 \in e\}$, and thus $t_r(\emptyset) = \binom{n-1}{r-1}$.
(3) If $\sigma = \wedge^{r-1}$ (meaning that $\sigma_i = \wedge$ for all $i < r$) then $T_r(\sigma) = \{e \in \binom{[n]}{r} \mid \{1, 2, \ldots, r\} \subseteq e\} = \{[r]\}$, meaning that $t_r(\sigma) = 1$.

For a positive integer r, let Υ_r be the set of sequences $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ consisting of fewer than r symbols of $\wedge$ and fewer than r symbols of $\vee$. Let $\Theta_r = \Upsilon_r \cup \{\alpha\} \cup \{\omega\}$, where α and ω are two new elements. Define $T_r(\alpha) = \emptyset$ and $T_r(\omega) = \binom{[n]}{r}$.

Lemma 3.2. $|\Theta_r| = \binom{2r}{r} + 1$.

Proof. define a map from $\Upsilon_r \setminus \{\emptyset\}$ to the set of sequences of r symbols $\wedge$ and r symbols $\vee$, in which σ goes to a sequence $\psi(\sigma)$ obtained by appending to it at its end a sequence of the form $\wedge\wedge\ldots\wedge\vee\vee\ldots\vee$ or $\vee\vee\ldots\vee\wedge\wedge\ldots\wedge$, in which the first symbol is the opposite of the last symbol of σ. Clearly, σ is reconstructible from $\psi(\sigma)$, since the last symbol of σ is recognizable - it is the first symbol, going from right to left, in the third stretch of identical symbols in $\psi(\sigma)$. The two sequences $\vee\vee\ldots\vee\wedge\wedge\ldots\wedge$ and $\wedge\wedge\ldots\wedge\vee\vee\ldots\vee$ are missing from the image, and remembering that $\emptyset \in \Upsilon_r$ this proves that $|\Upsilon_r| = \binom{2r}{r} - 1$. □

We now wish to order Θ_r. For this purpose we extend every sequence in Υ_r by appending a symbol $*$ at its end, and then ordering Υ_r lexicographically, with the convention $\wedge < * < \vee$ (the "*" is then discarded). We also define α to be the minimal element and ω to be the largest element of Θ_r.

3.2. **The sequence $M_r(i)$.** Write $m = \binom{2r}{r}$. Let $\sigma_0 = \alpha < \sigma_1 < \sigma_2 < \ldots < \omega = \sigma_m$ be the order defined above on Θ_r, and let $M_r(i) = t_r(\sigma_i)$. The identity of r being assumed to be known, we omit its mention and write $M(i)$. This is the analogue of the sequence $N(i)$ in the r-partite case.

Observation 3.3. *The sequence $M(i)$ is strictly ascending.*

Here is for example the sequence for $r = 3$ and general n:

$0, 1, 2, 3, n-2, n-1, n, 2n-5, 2n-4, 3n-9, \binom{n-1}{2}, \binom{n-1}{2}+1, \binom{n-1}{2}+2, \binom{n-1}{2}+n-3, \binom{n-1}{2}+n-2, \binom{n-1}{2}+2n-7, \binom{n-1}{2}+\binom{n-2}{2}, \binom{n-1}{2}+\binom{n-2}{2}+1, \binom{n-1}{2}+\binom{n-2}{2}+n-4, \binom{n-1}{2}+\binom{n-2}{2}+\binom{n-3}{2}, \binom{n}{3}$.

This sequence does not seem to behave as nicely as the sequence $N(i)$, but like the sequence $N(i)$ it has landmarks.

Theorem 3.4.

(1) $\sigma(\binom{2r-i}{r}) = \wedge^{i-1}$.
(2) $\sigma(\binom{2r}{r} - \binom{2r-i}{r-i}) = \vee^{i-1}$.
(3) $M(\binom{2r-i}{r}) = \binom{n-i}{r-i}$.
(4) $M(\binom{2r}{r} - \binom{2r-i}{r-i}) = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{n-i}{r-1}$.

Proof. Part (1): the sequences preceding $\wedge^{i-1}$ are those that start with $\wedge^i$. Using the same idea as in the proof of Lemma 3.2, we define a map between the set of the sequences σ preceding $\wedge^{i-1}$ and the set of sequences of r symbols of $\vee$ and $r-i$ symbols of $\wedge$: we complete σ to a sequence of r symbols $\vee$ and r symbols $\wedge$ by appending to σ at its end a sequence $\vee\vee\ldots\vee\wedge\wedge\ldots\wedge$ or $\wedge\wedge\ldots\wedge\vee\vee\ldots\vee$, where the first symbol of the appended sequence is the opposite of the last symbol of σ. The only sequence that is not in the image of this map is $\wedge^r\vee^r$, and hence the number of sequences preceding $\wedge^{i-1}$ is $\binom{2r-i}{r}$-1.

Part (2) follows by symmetry. Parts (3) and (4) follow by simple counting. □

3.3. **Calculating $b_c(t)$ for $t \le \binom{n}{r}$.** For $\sigma \in \Upsilon_r$ denote by $\overline{\sigma}$ the sequence obtained from σ by replacing each $\wedge$ by a $\vee$ and vice versa. Also define $\overline{\alpha} = \omega$ and $\overline{\omega} = \alpha$. By De Morgan's law, we have:

Lemma 3.5. $B(T_r(\sigma)) = T_r(\overline{\sigma})$.

The main result of this section is:

Theorem 3.6. *For every number $0 \le t \le \binom{n}{r}$ there exists $0 \le i \le \binom{2r}{r}$ such that $b_c(t) = M(i)$.*

The proof uses an already mentioned idea of Daykin [5], who gave a proof of the EKR theorem using the Kruskal-Katona theorem.

For a hypergraph F and a number r, the *r-shadow* of F, denoted by $S_r(F)$, is $\bigcup_{f\in F}\binom{f}{r}$. A hypergraph F of uniformity k is said to be in "cascade form" if there exist sets $B_0 = [n] \supseteq B_1 \supsetneq \ldots \supsetneq B_{s+1}$ and elements $x_i \in B_{i-1} \setminus B_i$ $(1 \le i \le s)$, such that

$$F = \binom{B_1}{k} \cup x_1 \diamond \binom{B_2}{k-1} \cup x_1 \diamond x_2 \diamond \binom{B_3}{k-2} \cup \ldots \cup x_1 \diamond x_2 \diamond \ldots \diamond x_s \diamond \binom{B_{s+1}}{k-s}$$

Here $x \diamond S$ stands for $\{x\} \diamond S$ (for the meaning of the latter, see Notation 2.10).

Theorem 3.7. [14, 15] *Given numbers m, n and $r \leq k$, the minimum of $|S_r(H)|$ over all $H \subseteq \binom{H}{k}$ is attained at a hypergraph H having cascade form.*

Proof of Theorem 3.6 We have to show that there exists $\beta \in \Upsilon_r$ satisfying the following condition: the maximum of $|B(H)|$ over all hypergraphs $H \subseteq \binom{n}{r}$ of cardinality t is attained at a hypergraph H for which $B(H) = T_r(\beta)$ for some sequence $\beta \in \Upsilon_r$.

Clearly, $B(H) = S_r(\bar{H})^c$, where $\bar{H}$ is the set of complements of edges in H, and $S_r(\bar{H})^c$ denotes the set of all edges of size r that do not belong to $S_r(\bar{H})$. By Theorem 3.7 the maximal value of $|B(H)|$ over all $H \subseteq \binom{n}{r}$ is attained at a hypergraph H for which $\bar{H}$ has cascade form. Let this form be

$$(5) \qquad \bar{H} = \binom{B_1}{n-r} \cup x_1 \diamond \binom{B_2}{n-r-1} \cup x_1 \diamond x_2 \diamond \binom{B_3}{n-r-2} \cup \ldots \cup x_1 \diamond x_2 \diamond \ldots \diamond x_s \diamond \binom{B_{s+1}}{n-r-s}$$

Here possibly $s = 0$. As above, we define $B_0 = [n]$. For each $0 \leq i \leq s$ let $B_i \setminus (B_{i+1} \cup \{x_i\}) = \{z_1^i, \ldots, z_{t_i}^i\}$, where $t_i = |B_i \setminus B_{i+1}| - 1$ (Here possibly $t_i = 0$).

Assertion 3.8. $B(H) = T(\theta)$, *where*

$$\theta = z_1^0 \vee (z_2^0 \ldots \vee (z_{t_0}^0 \vee (x_1 \wedge (z_1^1 \vee (z_2^1 \vee \ldots \vee (z_{t_1}^1 \vee (x_2 \wedge (z_1^2 \vee (z_2^2 \vee \ldots \vee (z_{t_2}^2 \ldots$$

if $B_1 \neq [n]$ and $\theta = \alpha$ if $B_1 = [n]$.

To prove the assertion, we have to show that a set e of size r belongs to $S_r(\bar{H})^c$ if and only if it satisfies the conditions imposed by θ. If e contains one of $z_1^0, z_2^0 \ldots, z_{t_0}^0$ then it does not belong to $S_r(\bar{H})$ because edges in $\bar{H}$ are contained in $\{x_1\} \cup B_0$. If e does not contain any of these vertices, it may still belong to $S_r(\bar{H})^c$, if it contains x_1. In such a case if e also contains none of $z_1^1, z_2^1 \ldots, z_{t_1}^1, x_2$ then it belongs to $S_r(\bar{H})$. So, we may assume that e contains one of these vertices or it contains x_2 together with x_1, and so on. This completes the proof of the assertion.

Next note that since e is of size r, it suffices to stop just after x_r, and obtain a condition that is satisfied by e if and only if $e \in T(\theta)$. For example, for $r = 2$ a set of size 2 satisfies the condition

$$x_1 \wedge (z_1^1 \vee (x_2 \wedge (z_1^2 \vee x_3)))$$

if and only if it satisfies the condition

$$x_1 \wedge (z_1^1 \vee x_2)$$

Let β be the formula obtained by truncating θ after x_r, if indeed x_r appears, and let $\beta = \theta$ otherwise.

Note also that the number of $\vee$'s in θ is equal to the number of z_i^j's in θ. The assumption is that the set $\binom{B_{s+1}}{n-r-s}$ appearing in (5) is non-empty, which implies that $|B_{s+1}| \geq n - r - s$. This is easily seen to imply that the number of z_i^j's is at most r. Thus $\beta \in \Upsilon_r$, which completes the proof of Theorem 3.6.

We can now achieve our aim - the calculation of $b_c(t)$ for every $t \leq \binom{n}{r}$.

Theorem 3.9. *If $M(i-1) < t \leq M(i)$ then $b_c(t) = M(\binom{2r}{r} - i)$.*

Proof. By Lemma 3.5 $b_c(M(j)) = M(\binom{2r}{r} - j)$ for all $0 \leq j \leq \binom{2r}{r}$. Since $b_c(c) \leq b_c(d)$ whenever $c \geq d$, this implies that $M(\binom{2r}{r} - i) \leq b_c(t) \leq M(\binom{2r}{r} - i + 1)$, and by Theorem 3.6 it follows that either $b_c(t) = M(\binom{2r}{r} - i + 1)$ or $b_c(t) = M(\binom{2r}{r} - i)$. By the definition of the function b we have $b_c(b_c(t)) \geq t$, and hence if $b_c(t) = M(\binom{2r}{r} - i + 1)$ then $t \leq b_c(M(\binom{2r}{r} - i + 1) = M(i-1)$, contradicting the assumption of the theorem. Thus $b_c(t) = M(\binom{2r}{r} - i)$. $\square$

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