WHEN LINEAR AND WEAK DISCREPANCY ARE EQUAL

DAVID M. HOWARD AND STEPHEN J. YOUNG

Abstract. The linear discrepancy of a poset P is the least k such that there is a linear extension L of P such that if x and y are incomparable, then $|h_L(x) - h_L(y)| \leq k$. Whereas the weak discrepancy is the least k such that there is a weak extension W of P such that if x and y are incomparable, then $|h_W(x) - h_W(y)| \leq k$. This paper resolves a question of Tanenbaum, Trenk, and Fishburn on characterizing when the weak and linear discrepancy of a poset are equal. Although it is shown that determining whether a poset has equal weak and linear discrepancy is NP-complete, this paper provides a complete characterization of the minimal posets with equal weak and linear discrepancy. Further, these minimal posets can be completely described as a family of interval orders.

1. Introduction

In [10] Fishburn, Tanenbaum, and Trenk introduce the notion of the linear discrepancy of a poset as a measure of the "distance" of a poset from a linear order. In essence, the linear discrepancy of a poset measure how far apart incomparable elements are forced in a linear extension of the poset. One can analogously define weak discrepancy as how far apart incomparable elements of a poset are forced in a weak extension [4]. Intuitively, it is clear that the weak discrepancy should be at most the linear discrepancy, and in fact this bound is tight. In this paper we answer a question of Fishburn, Tanenbaum, and Trenk [10] and characterize the tight examples. More precisely, we expand upon the idea of irreducibility with respect to linear discrepancy, introduced in [1] and expanded upon in [6, 7], to define and characterize the class of irreducible posets with equal linear and weak discrepancy.

1.1. Preliminaries.

More formally, if P is a poset let $\mathcal{O}(P)$ be the collection of order preserving maps from P to $\mathbb{N}$ and let $\mathcal{I}(P)$ be the collection of injective order preserving maps from P to $\mathbb{N}$. Then the *linear discrepancy* of P, denoted $\mathrm{ld}(P)$, is

$$\min_{f \in \mathcal{I}(P)} \max_{x \| y} |f(x) - f(y)|,$$

where $x \| y$ means that x is incomparable to y in P. Similarly, the *weak discrepancy* of P, denoted $\mathrm{wd}(P)$, is

$$\min_{f \in \mathcal{O}(P)} \max_{x \| y} |f(x) - f(y)|.$$

Now since $\mathcal{I}(P) \subseteq \mathcal{O}(P)$ it is clear that $\mathrm{wd}(P) \leq \mathrm{ld}(P)$. Tanenbaum, et al. provide an explicit formula for the linear and weak discrepancy of the disjoint union of chains in [10]. From these formula it is easy to see that the disjoint union of a chain of length $2d-1$ and a chain of length 1 has linear and weak discrepancy equal to d, and thus the inequality is tight.

At this point it is worth noting that calculating the linear discrepancy of a poset is NP-complete via a reduction to the bandwidth of its co-comparability graph [3, 10] while the weak discrepancy can be calculated in polynomial time [4, 9]. Thus it is natural to hope that the answer to the question of Tanenbaum, et al. is in the form of a polynomial time algorithm, however, the following reduction indicates that this is unlikely to be the case. That is, there is not a polynomial time algorithm unless P = NP.

A key component of the reduction is the following lemma from [10].

Lemma 1. *If P can be partitioned into two sets U and V such that for all $u \in U$ and $v \in V$, $u < v$, then* $\mathrm{ld}(P) = \max\{\mathrm{ld}(U), \mathrm{ld}(V)\}$ *and* $\mathrm{wd}(P) = \max\{\mathrm{wd}(U), \mathrm{wd}(V)\}$.

Theorem 2. *Determining whether* $\mathrm{ld}(P) = \mathrm{wd}(P)$ *is* NP*-complete.*

Proof. Since determining the linear discrepancy is in NP and determining the weak discrepancy is polynomial, determining whether they are equal is clearly in NP. Thus it suffices to show that there is an NP-complete problem that can be reduced in polynomial time to determining whether the linear and weak discrepancy are equal. The natural candidate for this is determining the linear discrepancy of a poset P. If $\mathrm{ld}(P) = \mathrm{wd}(P)$ the linear discrepancy may be determined by finding the weak discrepancy of P, therefore we may assume that $\mathrm{wd}(P) < \mathrm{ld}(P)$.

Now for all j, let P_j be the poset consisting of a chain of length $2j$ and a single isolated point and observe that $\mathrm{ld}(P_j) = \mathrm{wd}(P_j) = j$. Let X be the ground set of P and let Y_j be the ground set of P_j. For each j from 1 to $|X|$ define the poset P'_j on the ground set $X \cup Y_j$ by letting P'_j be equal to P on X, equal to P_j on Y_j and letting $y < x$ for every $y \in Y_j$ and $x \in X$. Now by Lemma 1, $\mathrm{ld}(P'_j) = \max\{\mathrm{ld}(P), \mathrm{ld}(P_j)\}$ and $\mathrm{wd}(P'_j) = \max\{\mathrm{wd}(P), \mathrm{wd}(P_j)\}$. Thus for $1 \leq j < \mathrm{ld}(P)$ we have $\mathrm{wd}(P'_j) \neq \mathrm{ld}(P'_j)$ and for $j \geq \mathrm{ld}(P)$ we have $\mathrm{wd}(P'_j) = \mathrm{ld}(P'_j)$. Thus $\mathrm{ld}(P)$ is the first j such that $\mathrm{ld}(P'_j) = \mathrm{wd}(P'_j)$. Hence if calculating whether linear and weak discrepancy are equal were polynomial, then determining the linear discrepancy of P would be as well, and thus determining whether linear and weak discrepancy are equal is NP-complete. □

Thus, rather than attempting to explicitly characterize all posets for which linear and weak discrepancy are the same, we follow the work in [1, 6, 7] and determine essential characteristics of posets with equal linear and weak discrepancy. To that end, we recall that a poset P is *d-linear-discrepancy-irreducible* if $\mathrm{ld}(P) = d$ and for any $x \in P$ we have $\mathrm{ld}(P - \{x\}) < d$. We define *$d$-weak-discrepancy-irreducible* analogously. Additionally, we say a poset P is *(s, t)-discrepancy irreducible* (or simply *(s, t)-irreducible*) if $\mathrm{ld}(P) = s$ and $\mathrm{wd}(P) = t$ and for any point $x \in P$ either $\mathrm{ld}(P - \{x\}) < s$ or $\mathrm{wd}(P - \{x\}) < t$. If $s = t$ then we may replace, without loss of generality, the second condition with for any $x \in P$, $\mathrm{wd}(P - \{x\}) < t$. That is, if a poset is (d, d)-irreducible then it is also d-weak-discrepancy-irreducible. Further, we note that if a poset P is such that $\mathrm{ld}(P) = s$ and $\mathrm{wd}(P) = t$ then there are induced subposets of P, denoted P_s, P_t and $P_{(s,t)}$, such that P_s is s-linear-discrepancy-irreducible, P_t is t-weak-discrepancy-irreducible, and $P_{(s,t)}$ is (s, t)-irreducible. With these definitions in hand we review some preliminary work on weak discrepancy.

1.2. **Weak Discrepancy Preliminaries.** In a poset P a *forcing cycle* is a sequence of elements $C = c_1, c_2, \ldots, c_k$ such that for all i either $c_i < c_{i+1}$ or $c_i \parallel c_{i+1}$

and (without loss of generality) $c_1 \parallel c_k$. Given a forcing cycle C, define $\mathrm{up}(C)$ as $|\{i \mid c_i < c_{i+1}, 1 \leq i \leq k-1\}|$ and $\mathrm{side}(C)$ as $1 + |\{i \mid c_i \parallel c_{i+1}, 1 \leq i \leq k-1\}|$. That is, $\mathrm{up}(C)$ is the number of up steps along the cycle and $\mathrm{side}(C)$ is the number of incomparable steps when viewing C cyclically since $c_1 \parallel c_k$. Using this notation, Gimbel and Trenk, prove the following theorem [4].

Theorem 3. *Let P be a poset and $\mathcal{C}$ be the set of forcing cycles on P, then* $\mathrm{wd}(P) = \max_{C\in\mathcal{C}} \left\lceil \frac{\mathrm{up}(C)}{\mathrm{side}(C)} \right\rceil$. *Furthermore, if $C = c_1, c_2, \ldots, c_k$ is a maximal forcing cycle and f is a fractional labelling of P where $f(c_1) = 0$ and $f(c_{i+1}) = f(c_i) + 1$ if $c_i < c_{i+1}$ and $f(c_{i+1}) = f(c_i) - \frac{\mathrm{up}(C)}{\mathrm{side}(C)}$ if $c_i \parallel c_{i+1}$. Then $\lceil f \rceil$ is an optimal weak discrepancy labelling.*

In fact, Gimbel and Trenk prove the stronger result that the f provided is in fact optimal over all fractional weak order preserving maps, yielding a fractional weak discrepancy of $\max_{C\in\mathcal{C}} \frac{\mathrm{up}(C)}{\mathrm{side}(C)}$.

In addition to Theorem 3 which provides combinatorial certification for $\mathrm{wd}(P) \leq k$, the following theorem, which is implicit in the work of Choi and West [2], will be key in characterizing the (d,d)-irreducible posets.

Theorem 4. *A poset P on n points is d-weak-discrepancy irreducible if and only if every forcing cycle C that is maximal with respect to $\frac{\mathrm{up}(C)}{\mathrm{side}(C)}$ has size t side steps and $(d-1)t+1$ up steps and $n = t + (d-1)t + 1$.*

2. (d,d)-irreducible Posets

Let $\mathcal{W}_d$ be the collection of d-weak-discrepancy-irreducible posets where there exists a maximal forcing cycle with all the up steps consecutive, in particular, there exists a forcing cycle $C = a_1, a_2, \ldots, a_{(d-1)t+2}, b_1, b_2, \ldots, b_{t-1}$ using all the elements where $a_i < a_j$ if $i < j$, $b_j \parallel b_{j+1}$ for $1 \leq j \leq t-2$, $a_{(d-1)t+2} \parallel b_1$, $a_1 \parallel b_{t-1}$. We claim that $\mathcal{W}_d$ is the set of all (d,d)-irreducible posets. First we show that all elements of $\mathcal{W}_d$ are (d,d)-irreducible. Since the elements of $\mathcal{W}_d$ are d-weak-discrepancy-irreducible by construction, it suffices to show that they all have linear discrepancy d.

Lemma 5. *If $W \in \mathcal{W}_d$, then* $\mathrm{ld}(W) = d$.

Proof. Let $W \in \mathcal{W}_d$ have $td+1$ points and let $C = a_0 < c_1^1 < c_1^2 < \cdots < c_1^{d-1} < c_2^1 < \cdots < c_2^{d-1} < \cdots < c_t^1 < \cdots < c_t^{d-1} < a_t \parallel a_{t-1} \parallel a_{t-2} \parallel \cdots \parallel a_1$ be the optimal forcing cycle. Now since W is d-weak-discrepancy irreducible let f be the function witnessing the optimal fractional weak discrepancy of $(d-1) + \frac{1}{t}$ as provided in Theorem 3. In particular, $f(a_i) = \left(d - 1 + \frac{1}{t}\right) i$ and $f(c_i^j) = (i-1)(d-1)+j$. Define the function $g : W \longrightarrow \{0, \ldots, dt\}$ by $g(a_i) = di$ and $g(c_i^j) = (i-1)d + j$. We claim g is an order preserving map of W witnessing linear discrepancy at most d. First we observe that by construction if $g(x) = g(y)$, then $x = y$. Now if $f(a_i) < f(c_i^j)$, then

$$\begin{aligned}
\left\lceil \frac{f(a_i)}{d-1} \right\rceil &\leq \left\lceil \frac{f(c_{\hat{i}}^j)}{d-1} \right\rceil \\
\left\lceil \frac{(d-1+\frac{1}{t})i}{d-1} \right\rceil &\leq \left\lceil \frac{(\hat{i}-1)(d-1)+j}{d-1} \right\rceil \\
\left\lceil i + \frac{i}{t(d-1)} \right\rceil &\leq \left\lceil \hat{i} - 1 \frac{j}{d-1} \right\rceil \\
i+1 &\leq \hat{i}.
\end{aligned}$$

Thus $i < \hat{i}$ so $g(a_i) < g(c_{\hat{i}}^j)$. Similarly, if $f(c_{\hat{i}}^j) < f(a_i)$, then

$$\begin{aligned}
\frac{f(c_{\hat{i}}^j)}{d-1} &< \frac{f(a_i)}{d-1} \\
\hat{i} - 1 + \frac{j}{d-1} &< i + \frac{i}{t(d-1)} \\
\hat{i} - 1 + \frac{tj-i}{t(d-1)} &< i.
\end{aligned}$$

But then, since $tj \geq i$, we have $\hat{i} - 1 < i$ and hence $g(c_{\hat{i}}^j) < g(a_i)$. Thus since f is a weak extension and for any $x, y \in W$ if $f(x) < f(y)$, then $g(x) < g(y)$, then g is a weak order preserving map of W. But, since g is one-to-one, this implies that g is an order preserving map of W.

Now suppose $x \parallel y$ and $|g(x) - g(y)| > d$. If $x, y \in \{a_0, a_1, \ldots, a_t\}$, then $|g(x) - g(y)| > d$ implies that the indices of x and y differ by at least two and hence $|f(x) - f(y)| \geq 2\left(d - 1 + \frac{1}{t}\right)$ and so x and y are comparable since f is witnesses fractional weak discrepancy at most $d - 1 + \frac{1}{t}$. Thus precisely one of $\{x, y\}$ is a point of the form c_i^j and the other is a point of the form a_k with $1 \leq k \leq t-1$. We will show that if $\left|g(c_i^j) - g(a_k)\right| > d$ then c_i^j and a_k are comparable. In particular, we wish to show that if $g(c_i^j) - g(a_k) > d$ then $c_i^j > a_k$ and if $g(a_k) - g(c_i^j) > d$ then $a_k > c_i^j$. Since the c_i^j form a chain, it suffices to consider the minimal c_i^j such that $g(c_i^j) - g(a_k) > d$ and the maximal c_i^j such that $g(a_k) - g(c_i^j) > d$. We note that

$$\begin{aligned}
\left|g(a_k) - g(c_i^j)\right| &= |dk - (i-1)d - j| \\
&= |d(k - i + 1) - j| \\
&\leq d\,|k - i + 1| + j \\
&\leq d\,|k - i + 1| + (d-1),
\end{aligned}$$

and thus, if $g(c_i^j) - g(a_k) > d$ then $i \geq k+2$ and if $g(a_k) - g(c_i^j) > d$ then $i \leq k$. However, for $i = k$ we have

$$\left|g(a_k) - g(c_i^j)\right| = |d - j| < d,$$

and thus we need only consider $i < k$. Since $g(c_{k+2}^1) - g(a_k) = d+1 = g(a_k) - g(c_{k-1}^{d-1}$ it suffice to only consider c_{k+2}^1 and c_{k-1}^{d-1}. Now observe that c_{k+2}^1 exists only if

$k \le t-2$, we have

$$\begin{aligned} f(c^1_{k+2}) - f(a_k) &= (k+1)(d-1) + 1 - (d-1)k + \frac{k}{t} \\ &= (d-1) + \frac{t-k}{t} \\ &> (d-1) + \frac{1}{t}. \end{aligned}$$

Thus $a_k < c^1_{k+2}$ since f witnesses fractional weak discrepancy at most $d-1+\frac{1}{t}$. Similarly, c^{d-1}_{k-1} exists only if $k \ge 2$ and then

$$\begin{aligned} f(a_k) - f(c^{d-1}_{k-1}) &= (d-1)k + \frac{k}{t} - (k-2)(d-1) - (d-1) \\ &= (d-1) + \frac{k}{t} \\ &> (d-1) + \frac{1}{t}. \end{aligned}$$

Thus $c^{d-1}_{k-1} < a_k$ and hence g is an order preserving map of W that witnesses linear discrepancy at most d. But then since $d = \mathrm{wd}(W) \le \mathrm{ld}(W) \le d$, the linear discrepancy of W is exactly d. □

The following theorem shows that not only are all elements of $\mathcal{W}_d$ (d,d)-irreducible, every (d,d)-irreducible poset is a member of $\mathcal{W}_d$.

Theorem 6. *Let P be a poset with* $\mathrm{ld}(P) = d$. *Then* $\mathrm{wd}(P) = d$ *if and only if there exists a subposet W of P such that $W \in \mathcal{W}_d$.*

Proof. First suppose there is some subposet W of P such that $W \in \mathcal{W}_d$. Then since $d = \mathrm{ld}(P) \ge \mathrm{wd}(P) \ge \mathrm{wd}(W) = d$, we have $\mathrm{wd}(P) = d$.

Suppose then that $\mathrm{ld}(P) = \mathrm{wd}(P) = d$. Then it is clear that there is some subposet W' of P such that W' is (d,d)-irreducible. Now since the removal of any point from W' decreases either the weak discrepancy or the linear discrepancy and $\mathrm{wd}(P) \le \mathrm{ld}(P)$ for all P, we know that W' is d-weak-discrepancy irreducible. Thus it suffices to show that the maximal forcing cycle has all the up steps consecutive.

Since W' is d-weak-discrepancy irreducible, $|W'| = dt+1$ for some t and there is a maximal forcing cycle C using $dt+1$ points. This forcing cycle naturally partitions the elements of W' into chains $C_1, C_2, \ldots, C_t$ by using the side steps as break points in the chain. For all chains C_i, let a_i be the minimal element and let b_i be the maximal element (note that it is not necessarily the case that $a_i \ne b_i$). We say that a side move $(b,a) \in \{(b_i, a_{i+1}) \mid 1 \le i \le t-1\} \cup \{(b_t, a_1)\}$, *encompasses* a point x with respect to a linear extension L if $b <_L x <_L a$ or $a <_L x <_L b$.

Fix an arbitrary linear extension L of W'. Suppose $x \in C_i$ and $a_i \le x < b_i$ (and hence x is not in a trivial chain) and x is not encompassed by any side move. Then, since $x < b_i$, by traversing the cycle we can conclude that $x \le_L a_j$ for any $1 \le j \le t$. But then $x \le_L y$ for any $y \in W'$ and hence is the minimum element of L. Similarly if $a_i < x \le b_i$, then x is the maximum element of L. Thus the only elements of P that are not encompassed by a side step with respect to L are the minimum and maximum elements of L and the elements belonging to a trivial chain. Now let $\mathcal{T}$ be the set of trivial chains. Then, as there are t side steps, there exists some side move (b_L, a_L) encompassing at least $\left\lceil \frac{dt+1-(2+|\mathcal{T}|)}{t} \right\rceil = d - \left\lfloor \frac{1+|\mathcal{T}|}{t} \right\rfloor$ elements in the linear extension L. Thus if $|\mathcal{T}| < t-1$, then (b_L, a_L) encompasses at least d elements with respect to L and hence $|h_L(b_L) - h_L(a_L)| \geq d+1$. But since L was an arbitrary linear extension this implies that $\mathrm{ld}(W') \geq d+1$, a contradiction. Thus $|\mathcal{T}| = t-1$ and so all but one of the chains is trivial and hence all the up steps are consecutive in the forcing cycle. □

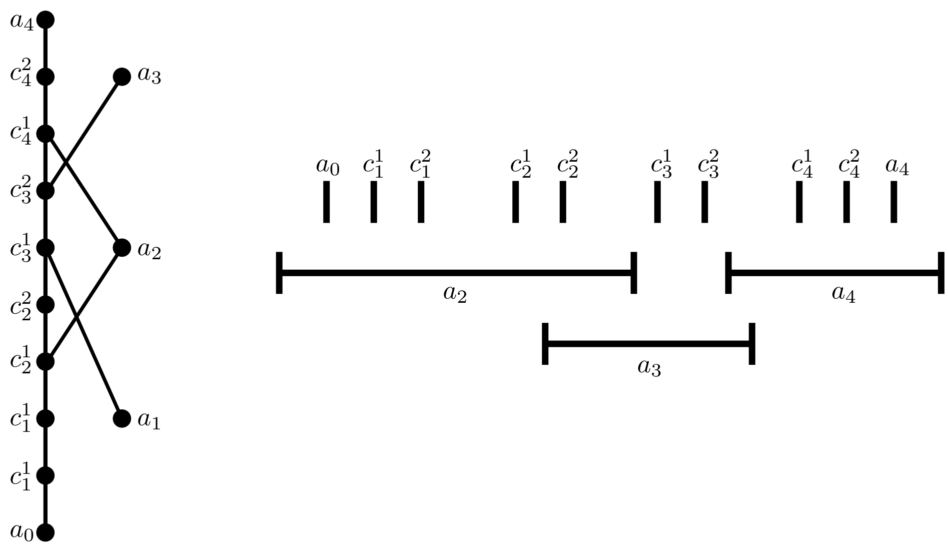
3. Characterization of $\mathcal{W}_d$

In examining the nature of $\mathcal{W}_d$ it is clear that, contrary to most results on posets, $\mathcal{W}_d$ is specified through explicit local restrictions on the set of comparabilities and incomparabilities rather than global restriction on the structure of the poset. That is, $\mathcal{W}_d$ is defined as the set of solutions to a collection of transitively oriented sandwich problems [5] where the order among some pairs of elements are defined and other pairs of points are defined to be incomparable. However, we can exploit the structure of elements of $\mathcal{W}_d$ to provide a more natural description of the class as interval orders. This characterization of $\mathcal{W}_d$ as a collection of interval orders joins with results such as the forbidden subposet characterization of posets with linear discrepancy at most two [6, 7], the NP-completeness of linear discrepancy [3], and the behavior of online algorithms for linear discrepancy [8] in emphasizing the centrality of interval orders in the study of linear and weak discrepancy.

Let $W \in \mathcal{W}_d$ and let $C = a_0 < c_1^1 < c_1^2 < \cdots < c_1^{d-1} < c_2^1 < \cdots < c_2^{d-1} < \cdots < c_t^1 < \cdots < c_t^{d-1} < a_t \parallel a_{t-1} \parallel a_{t-2} \parallel \cdots \parallel a_1$ be an optimal forcing cycle of W. We first note that if $a_i < a_j$, then $a_i < c_{i+2}^1$ and $c_{j-1}^{d-1} < a_j$. But then, since $j \geq i+2$, this implies that every element of the chain $a_0 < c_1^1 < \cdots < c_t^{d-1} < a_t$ is comparable to either a_i or a_j. Thus W does not contain a $\mathbf{2}+\mathbf{2}$ and hence is an interval order. Now in order to characterize the elements of $\mathcal{W}_d$ it suffices to provide a collection of intervals or rules for generating the intervals that will realize every element of $\mathcal{W}_d$. We note that since $a_i < a_j$ implies that every element of the chain $a_o < c_1^1 < c_1^2 < \cdots < c_1^{d-1} < c_2^1 < \cdots < c_2^{d-1} < \cdots < c_t^1 < \cdots < c_t^{d-1} < a_t$ is comparable to either a_i or a_j, we may assume that the intervals associated with the long chain are degenerate. In particular, we assume that the interval for c_i^j is $\{(i-1)d+j\}$ and that the intervals for a_0 and a_t are $\{0\}$ and $\{dt\}$, respectively.

Now for $1 \leq i \leq t-1$ let the endpoints of the interval associated with a_i be ℓ_i and r_i. Using that c_i^j is assigned to the degenerate interval $\{(i-1)d+j\}$ it is clear that we may assume for $1 \leq i \leq t-1$, $[\ell_i, r_i] \subseteq (d(i-1)-1, d(i+1)+1)$. The constraint that $a_i \parallel a_{i+1}$ and that $a_i < a_{i+2}$ imposes that $\ell_{i+1} < r_i < \ell_{i+2}$. In fact, any interlaced sequence $-1 < \ell_2 < r_1 < \ell_3 \cdots < \ell_t < r_{t-1} < dt+1$ such that $r_i < d(i+1)+1$ for $1 \leq i < t-1$ and $d(j-1)-1 < \ell_j$ for $1 < j \leq t$ will yield an interval representation of an element of $\mathcal{W}_d$. For example see Figure 1.

Acknowledgements. The authors would like to thank Mitchel T. Keller for his helpful comments on an earlier version of this manuscript which helped improve the clarity and exposition of the current manuscript.



(a) Hasse Diagram (b) Interval Representation

Figure 1: A element of $\mathcal{W}_3$ on 13 points.

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