Monotone Hamiltonian paths in the Boolean lattice of subsets

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Abstract

Consider the lattice whose elements are the subsets of the set of positive integers not greater than n. The Hasse diagram of this lattice is isomorphic to the n-dimensional hypercube. It is trivial that this graph is Hamiltonian. Let $\{S_1, \dots, S_{2^n}\}$ be a Hamiltonian path. We say it is monotone, if for every i, either (a) every subset of S_i appears among the sets $S_1, \dots, S_{i-1}$, or (b) only one (say S) does not, furthermore $S_{i+1} = S$.

Trotter conjectured that there if n is sufficiently large, then there are no monotone Hamiltonian paths in the n-cube. He also made a stronger conjecture, that states, that there is no path with the monotone property that covers all the sets of size at most three. In this paper we disprove this strong conjecture by explicitly constructing a monotone path covering all the 3-sets.

1 Introduction

The n-dimensional hypercube, as a graph is one of the first standard example of a Hamiltonian graph in college graph theory classes; a simple induction shows that a Hamiltonian cycle exists for every n. However, if we follow any cycle like this, we will find, that the sizes of the sets along the cycle goes up and down and does not show any kind of monotone property. The question occurs: if we give up having a cycle, and we just want a Hamiltonian path, can we find such a path, for which every time we cover a set, we will have covered all the subsets of this set? This kind of path would start from the empty set, end in the set $\{1, \dots, n\}$, and it would traverse the diagram of the subset lattice "from bottom to up".

It is immediately clear that this is impossible even for $n = 2$. So we relax the conditions even more. For this, we change our point of view, and from this point on, we consider the subset lattice of the set $\{1, \dots, n\}$. The diagram of this lattice, as a graph, is isomorphic to the n-dimensional hypercube. We are searching for a Hamiltonian path $\{S_1, \dots, S_{2^n}\}$ such that for every i, either

- every subset of S_i appears among the sets $S_1, \dots, S_{i-1}$, or

- only one (say S) does not; in this case $S_{i+1} = S$.

If a Hamiltonian path has the above properties, we call it a *monotone* Hamiltonian path. If a path is not Hamiltonian, but it starts at the empty set and has the above properties, we call it a *stub* of a monotone Hamiltonian path. If a stub covers all the sets of at most size k, we call is a k-stub. Obviously, the existence of an n-stub is equivalent to the existence of a monotone Hamiltonian path. The answer for the following questions is not known:

1. Is there a monotone Hamiltonian path for every n in the n-cube?
2. If the answer is no, what is the largest k such that k-stub of a monotone Hamiltonian path exists?

Trotter and Felsner [2] found some striking combinatorial connections related to these questions. Without going into much details, let us just mention the results. For basic definitions on interval orders and partial orders, see [4], and on graph theory, see [1].

Definition 1.1 *$C(t)$ is the largest integer h so that whenever P is an interval order of height h, the chromatic number of the diagram of P is at most t.*

Definition 1.2 *A sequence $(S_0, S_1, \ldots, S_h)$ of sets is an* F-sequence *if*

- *$S_0 \not\subseteq S_1$, and*
- *$S_j \not\subseteq S_i \cup S_{i+1}$ when $j > i + 1$*

$F(t)$ is the largest h for which there exists an F-sequence of subsets of $\{1, \ldots, t\}$.

Theorem 1.3 *For every $t \geq 1$*

$$C(t) = F(t) \leq 2^{t-1} + \left\lfloor \frac{t-1}{2} \right\rfloor,$$

with equality holding if and only there is monotone Hamiltonian path in the t-cube.

Another problem that this one may be related to is the famous "middle two level" conjecture posed by Ivan Havel. That problem is almost solved, see [3]. We do not know any specific connections, but both problems deal with the same kind of objects, so we can hope that solving one might help solving the other.

Trotter conjectured, that if n is sufficiently large, there is no monotone Hamiltonian path in the n-cube. He also conjectured that if n is sufficiently large, there is no 3-stub in the n-cube.

Then existence of a k-stub for $k = 0, 1$ is absolutely trivial, and it is really easy to show that 2-stubs exist. The number of 2-stubs is enormous, but not all of them are prefixes of 3-stubs. In our construction, we have to be really careful how we cover the 2-sets, so that we can continue it to construct a 3-stub.

2 The existence of 3-stubs

Theorem 2.1 *For every n positive integer a 3-stub exists in the subset lattice of $\{1, \ldots, n\}$.*

The existence of a monotone Hamiltonian path for $n \leq 10$ was checked with computers by Trotter and it has been rechecked by the authors. So in the following, we may assume that $n \geq 9$.

2.1 Construction of a 2-stub

Start the path with the following sequence: $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{2\}$, $\{2, 3\}$, $\{3\}$, $\ldots$, $\{n\}$. So far, this is a 1-stub.

Now we will use a table to demonstrate how to cover the remaining 2-sets. In the table, we just list the 2-sets that are not covered so far without curly braces and commas to save space. When we connect two 2-sets in this table, say $\{a, b\}$ and $\{b, c\}$ (which will be denoted by `ab` and `bc`), we mean that the path goes from $\{a, b\}$ to $\{a, b, c\}$ and then to $\{b, c\}$. Observe, that this is only possible if the set $\{a, c\}$ has been covered. We will be careful to only connect subsets, if that is allowed. Here is our table:

```
13
14   24
15   25   35
...
1n   2n   3n   ...   n-2 n
```

Later we will refer to the rows of this table. When we refer to the first, second, etc. row, we will call them row 3, row 4, etc. respectively. So row k is the row that contains the 2-sets, whose largest element is k.

Observe, that if two sets are next to each other (up-down or left-right), it is always possible to connect them, because the missing 2-set that must have been covered is of the form $\{\alpha, \alpha + 1\}$, and these were covered in the 1-stub. Additional opportunities to connect sets arise as we go along the path. It is now fairly obvious, that we can construct 2-stubs in many ways. We will do it as demonstrated in Figure 1. In this figure we used $n = 12$ and we used the letters A, B and C in place of the numbers 11, 12 and 13.

In generality, the demonstrated path is the following: after the set $\{n\}$, we cover $\{3, n\}$, $\{2, 3, n\}$, $\{2, n\}$, $\{1, 2, n\}$, $\{1, n\}$, $\{1, 3, n\}$ (it is allowed, because $\{3, n\}$ has been covered), $\{1, 3, 4\}$, $\{1, 4\}$ and so on as demonstrated. After $\{2, 7\}$, we do the following. We cover the table row-by-row, going down to the next row only after we covered every 2-set in a given row. The order of the 2-sets within a row is arbitrary, with the following restrictions ($k \geq 7$, the $*$ sign denotes an arbitrary element):

- If a row ends with the sets $\{a, k\}$, $\{*, k\}$, $\{b, k\}$, then the next row starts with the sets $\{b, k + 1\}$, $\{a, k + 1\}$.

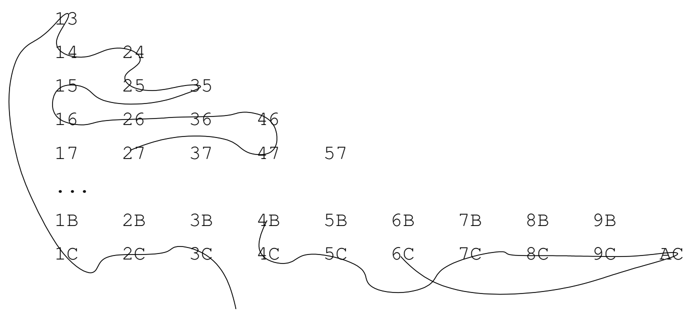

Figure 1: Construction of a 2-stub

- The row before the last one ends with the sets $\{5, n-1\}$, $\{*, n-1\}$, $\{4, n-1\}$.
- The last row ends with $\{6, n\}$.

Then, as demonstrated on the table, we cover $\{4, n\}$ (after $\{4, n-1, n\}$ of course), then $\{5, n\}$, $\{7, n\}$, ..., $\{6, n\}$ with the appropriate 3-sets.

When we cover row k, every row of index less than k has been covered, and this makes it possible that every jump within the row is allowed. So just by this fact, each row could be covered in an arbitrary order. The restrictions given above are important though in the construction of the 3-stub.

It is clear, that for large n there are still exponentially many ways to build a 2-stub that satisfies the requirements. For clarity, let us list the 2-sets covered in order for $n = 12$ in the *lexicographically first* stub. We will omit the curly braces, commas and we will not mention the implied 3-sets. We will use the letters A, B, C as before.

`3C, 2C, 1C, 13, 14, 24, 25, 35, 15, 16, 26, 36, 46, 47, 27, 17, 37, 57, 58, 18, 28, 38, 48, 68, 69, 39, 19, 29, 49, 59, 79, 7A, 4A, 1A, 2A, 3A, 6A, 5A, 8A, 8B, 6B, 1B, 2B, 3B, 6B, 5B, 7B, 4B, 4C, 5C, 7C, 8C, 9C, AC, 6C`

3 Construction of a 3-stub

The idea of this construction is the following. We will partition the non-covered 3-sets into “tables”. Table k contains the set of non-covered 3-sets, whose largest element is k. So Table 3 contains only one element: $\{1, 2, 3\}$, Table 4 contains one again ($\{2, 3, 4\}$ – the sets $\{1, 2, 4\}$ and $\{1, 3, 4\}$ were covered during the beginning of the 2-stub), Table 5 consists of $\{1, 2, 5\}$, $\{1, 4, 5\}$ and $\{3, 4, 5\}$. We will traverse the tables one by one and in an increasing order.

Of course one problem is that the last 2-set is $\{6, n\}$, so we can not make a jump to $\{1, 2, 3\}$. This is how we go back: $\{1, 6, n\}$, $\{1, 2, 6, n\}$, $\{2, 6, n\}$, $\{2, 3, 6, n\}$, $\{3, 6, n\}$, $\{1, 3, 6, n\}$, $\{1, 3, 6\}$, $\{1, 2, 3, 6\}$, $\{1, 2, 3\}$. We encourage

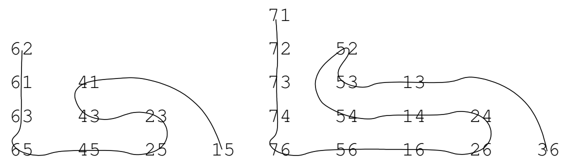

Figure 2: Table 7 on the left and Table 8 on the right

the reader to check that every step is valid. Note, that this is basically a "greedy" path; we swap out the elements of the set $\{6, n\}$ to the elements of the set $\{1, 2, 3\}$ as fast as possible.

Another important thing to observe that we only used some elements of Table n and one element of Table 6, so tables from 7 to $n-1$ are unaffected.

Here is how we cover the 3-sets from Table 3 to Table 6. We will omit the the 4-sets between each pair of 3-sets for simplicity. The reader is encouraged to check the validity.

$\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{1, 4, 5\}$, $\{1, 2, 5\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, $\{4, 5, 6\}$, $\{1, 4, 6\}$, $\{2, 4, 6\}$

Now we will show how to cover tables from 7 to $n-1$. In order to do this, we will introduce a nice, compact way to write down the element of a table in an actual table. When we consider Table k, each element will contain k, so we will omit that element. In effect we will just write down 2-sets, without curly braces and commas.

We will use custom row and column indices. An element is row i and column j will be the element `ij`, denoting the set $\{i, j, k\}$.

Consider the way we covered row k in the construction of the 2-stub. Say the order of the 2-sets in that row is $\{\alpha_1, k\}, \{\alpha_2, k\}, \ldots, \{\alpha_{k-2}, k\}$. Index the columns in order with $k-1$, α_1, α_2, $\ldots$, α_{k-4}. Index the rows in order with α_2, α_3, $\ldots$, α_{k-2}. This way we get a $k-3 \times k-3$ table. Obviously, due to symmetry, only half of the table, say the lower triangle (including the diagonal) is necessary to consider. Observe, that in the beginning, the valid moves are exactly the moves between up-down or left-right neighbors in the table. Of course while covering a table, new valid moves appear.

We need two different construction to cover the tables: one kind for even indexed tables and one for odd. Instead of writing down how to cover the tables in an abstract way, let us just show the examples of Tables 7 to 10 for the case of $n = 12$ and the lexicographically first 2-stub. The examples will clearly show the pattern and how to generalize it. See Figures 2 and 3.

The reader should carefully check that every step is valid inside a table. Also, we start to cover Table 7 with the set $\{2, 6, 7\}$, and this is possible, because the last element of Table 6 was $\{2, 4, 6\}$ and it is valid to move to $\{2, 4, 6, 7\}$ and then to $\{2, 6, 7\}$, because $\{2, 4, 7\}$ and $\{4, 6, 7\}$ were covered during the 2-stub. This is true in general, i.e. it is always possible to move from one table to the next one. We end Table k with the element $\{a, b, k\}$, where row k (covering the

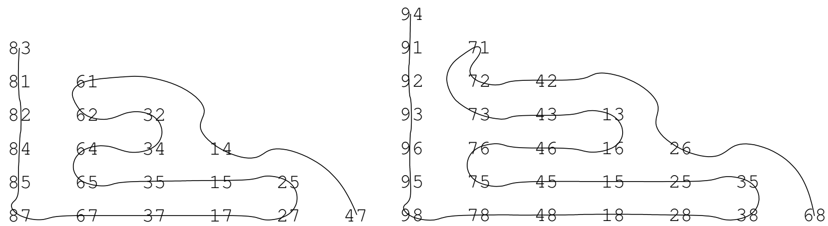

Figure 3: Table 9 on the left and Table 10 on the right

2-stub) ended with the sets $\{a, k\}$, $\{*, k\}$, $\{b, k\}$ (lower right corner of the table). We start Table $k + 1$ with the element $\{a, k, k + 1\}$ (upper left corner of the table). The implied 4-set between them is $\{a, b, k, k + 1\}$. The 3-sets that have to be covered to make this a valid step, are $\{a, b, k + 1\}$ and $\{b, k, k + 1\}$. The former was covered in the beginning of row $k + 1$, which started as $\{b, k + 1\}$, $\{a, k + 1\}$ implying the missing set. The latter was covered between row k and row $k + 1$; row k ended with $\{b, k\}$, row $k + 1$ started with $\{b, k + 1\}$, implying the missing set.

It remains to be shown how to transverse Table n. Remember that the table is special for two reasons: first row n was covered irregularly in the 2-stub, and second, because when we started to cover the 3-sets, and we had to "go back" to the set $\{1, 2, 3\}$, we used some entries of Table n.

The beauty of the construction is that these two forces neutralize each other. Remember, that the first three sets covered in row n were $\{3, n\}$, $\{2, n\}$ and $\{1, n\}$. Let us ignore these for a moment, and concentrate on the order the remaining sets were covered. Say this order is $\{\alpha_1, n\}$, ..., $\{\alpha_{n-5}, n\} = \{6, n\}$. Index the columns of Table n with $n - 1$, α_1, ...,$\alpha_{n-5} = 6$, 1, and index the rows with α_2, ..., $\alpha_{n-5} = 6$, 1, 2, 3.

When we draw the lower triangular $n-3 \times n-3$ table now, it is not completely "accurate" for the following reasons:

- The moves between column 6 and column 1 are not valid.
- The set $\{1, 6, n\}$ is missing from the table. This is not an actual difference, because this set was covered in the beginning on the 3-stub.
- The sets $\{1, 3, n\}$, $\{2, 6, n\}$ and $\{3, 6, n\}$ are present on the table, but they are in fact covered in the beginning of the 3-stub.

We can easily take care of the last problem by dropping the corresponding elements from the table. As we mentioned, the second problem is not really a problem. Now look at the first problem. After we drop the elements mentioned in the third problem, the separated columns on the right are dropped, so the problem resolved itself.

Now it is really easy to traverse Table n, because we can use the exact same techniques that we used to traverse the previous tables. Let us just illustrate

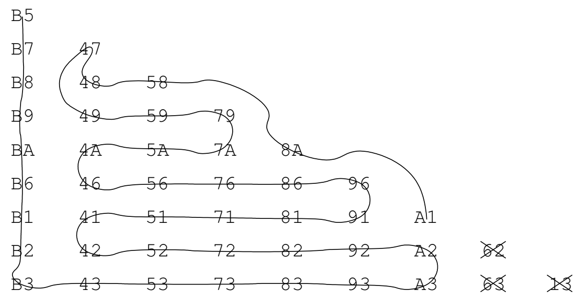

Figure 4: Table 12

how to cover the table for $n = 12$ on Figure 4. The construction needs to be changed for different parity in the same way as it was for the smaller tables. We use the letters A and B as before.

4 The general problem

Of course the immediate question arises if this technique exhibited here is useful for solving the general problem. If nothing else, maybe for constructing a 4-stub. In our opinion, with hard work, it is probably possible to construct a 4-stub using the same ideas, but it is at least questionable if the gained insight would be enough to solve the general question.

This argument has several points that are very special for 3-stubs. The basic idea, that we cover the 2-sets "row by row" and then the 3-sets "table by table" does not work without modification. This is why we needed to do special things in the beginning of covering the 2-sets and even more special things in the beginning of covering the 3-sets. The fact that the two "tweaks" fit together perfectly, almost seems to be lucky.

We hope that this construction, or a simplification of this, can provide a general pattern, that will help answer the question if there is monotone Hamiltonian path of every subset lattice.

References

[1] Béla Bollobás, *Modern graph theory*, Springer, 2002.

[2] Stephan Felsner and William T. Trotter, *Colorings of diagrams of interval orders and α-sequences of sets*, Discrete Math. **144** (1995), no. 1-3, 23–31, Combinatorics of ordered sets (Oberwolfach, 1991).

[3] Robert J. Johnson, *Long cycles in the middle two layers of the discrete cube*, J. Combin. Theory Ser. A **105** (2004), no. 2, 255–271.

[4] William T. Trotter, *Combinatorics and partially ordered sets*, The Johns Hopkins University Press, 1992.