ON A GENERALIZATION OF THE RYSER-BRUALDI-STEIN CONJECTURE

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Abstract. A *rainbow matching* for (not necessarily distinct) sets $F_1, \dots, F_k$ of hypergraph edges is a matching consisting of k edges, one from each F_i. The aim of the paper is twofold - to put order in the multitude of conjectures that relate to this concept (some first presented here), and to prove partial results on one of the central conjectures.

1. Introduction

A choice function for a family of sets $\mathcal{F} = (F_1, \dots, F_m)$ is a choice of elements $f_1 \in F_1, \dots, f_m \in F_m$. It is also called a *system of representatives* (SR) for $\mathcal{F}$. Many combinatorial questions can be formulated in terms of SRs that satisfy yet another condition. For example, in Hall's theorem [16] the extra condition is injectivity. In Rado's theorem [21] the condition is injectivity, plus the demand that the range of the choice function belongs to a given matroid. In the most general setting, a simplicial complex (closed down hypergraph) $\mathcal{C}$ is given on $\bigcup F_i$, and the condition is that the range of the function belongs to $\mathcal{C}$. The SR is then called a *$\mathcal{C}$-SR*. A *partial $\mathcal{C}$-SR* is a partial choice function satisfying the above condition. Even injectivity can be formulated in the terminology of $\mathcal{C}$-SRs, using a trick of making many copies of each element.

Given a graph G, we denote by $\mathcal{I}(G)$ the set of independent sets in G. An $\mathcal{I}(G)$-SR is also called an *ISR* (independent system of representatives). We also use this term for $\mathcal{M}$-SRs, in the case that $\mathcal{M}$ is a matroid.

If the sets F_i are hypergraphs, and the extra condition is that the edges chosen are disjoint, the system of representatives is called a *rainbow matching* for $\mathcal{F}$. Thus, a rainbow matching for sets of edges $F_1, \dots, F_m$ is an ISR in the line graph of $\bigcup F_i$.

We are motivated by the following conjecture of Aharoni and Berger (not published before):

Conjecture 1.1. *k matchings of size $k+1$ in a bipartite graph possess a rainbow matching of size k.*

This strengthens a conjecture of Brualdi and Stein, on Latin squares. A Latin square of order n is an $n \times n$ matrix whose entries are the symbols $1, \dots, n$, each appearing once in every row and once in every column. A (partial) *transversal* in a Latin square is a set of entries, each in a distinct row and a distinct column, containing distinct symbols. A well known conjecture of Ryser [22] is that for n odd every $n \times n$ Latin square contains a transversal of size n. For even n this is false, and Brualdi and Stein [11, 25] raised independently the following natural conjecture:

Conjecture 1.2. *In a Latin square of order n there exists a partial transversal of size $n-1$.*

A Latin square can be viewed as a 3-partite hypergraph, with sides (R=set of rows, C=set of columns, S=set of symbols), in which every entry e corresponds to the edge $(row(e), column(e), symbol(e))$. The 3-partite hypergraph corresponding to a Latin square satisfies stringent conditions - every pair of vertices in two different sides belongs to precisely one 3-edge. In particular, this means that for every vertex v, the set of 2-edges complementing v to a 3-edge is a matching in a bipartite graph. Hence the Brualdi-Stein conjecture would follow from:

Conjecture 1.3. *k matchings of size k in a bipartite graph possess a partial rainbow matching of size $k-1$.*

Conjecture 1.3 follows from Conjecture 1.1 by the familiar device of expanding the given matchings of size k to matchings of size $k+1$, adding the same edge to all.

Note that k matchings of size k need not have a rainbow matching of size k. This is shown by a standard example:

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Example 1.4.

$$F_1 = \ldots = F_{k-1} = \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}, \quad F_k = \{(a_1, b_2), (a_2, b_3), \ldots, (a_k, b_1)\}$$

Barát and Wanless [9] constructed clever elaborations of this basic example, of k matchings of size at least k, with sum of sizes $k^2 + \lfloor \frac{k}{2} \rfloor - 1$, that do not possess a rainbow matching.

Later on, many ramifications of Conjecture 1.1 will be mentioned. But we shall start with a natural endeavor - trying to prove lower bounds that are as small as possible on the size of the matchings, that guarantee the existence of a rainbow matching.

Definition 1.5. Let $cms(r, k)$ (standing for "critical matching size") be the least number t such that every k matchings of size t in an r-partite hypergraph have a rainbow matching. Also let $grs(r, k)$ (standing for "guaranteed rainbow matching size") be the largest number s such that every k matchings of size k in an r-partite hypergraph possess a partial rainbow matching of size s.

In this terminology, Conjecture 1.1 is that $cms(2, k) \leq k+1$, and Conjecture 1.3 is that $grs(2, k) \geq k-1$. Example 1.4 shows that:

Observation 1.6. For $k > 1$ we have $grs(2, k) \leq k - 1$ and $cms(2, k) \geq k + 1$.

Greedy arguments yield $cms(2, k) \leq 2k - 1$ and $grs(2, k) \geq \frac{k}{2}$. A proof of Woolbright [27] (using the terminology of transversals in Latin squares) can be adapted to show:

Theorem 1.7. $grs(2, k) \geq k - \sqrt{k}$.

Another simple fact, already noted above in the case $k - grs(r, k) = 1$, is:

Observation 1.8. $cms(r, k) - k \geq k - grs(r, k)$

Proof. Write p for $cms(r, k) - k$. Let $F_1, \ldots, F_k$ be k matchings of size k, in an r-partite hypergraph, we want to prove the existence of a partial rainbow matching of size $k-p$. Let $Q = \{e_1, \ldots, e_p\}$ be a matching, whose edges are disjoint from $\bigcup_{i \leq k} F_i$, and let $F'_i = F_i \cup Q$. By the definition of $cms(r, k)$, the matchings F'_i have a rainbow matching of size k, and removing the edges belonging to Q yields a partial rainbow matching of size at least $k - p$. □

The following observation shows that $k - grs(r, k)$ is not bounded by a constant:

Observation 1.9. $grs(r, 2^{r-1}) \leq 2^{r-2}$.

The proof uses:

Lemma 1.10. *For every $r > 1$ there exists a system of 2^{r-1} matchings of size 2 in an r-partite hypergraph, not possessing a partial rainbow matching of size 2.*

Proof. Let a_i, b_i be distinct elements, $1 \leq i \leq r$, and for every subset T of $[r]$ let $e_T = \{a_i : i \in T\} \cup \{b_i : i \notin T\}$ and $f_T = \{a_i : i \notin T\} \cup \{b_i : i \in T\}$, and let $M_T = \{e_T, f_T\}$. Since $M_T = M_{[r] \setminus T}$, there are 2^{r-1} such matchings, and clearly they do not possess a rainbow matching. □

Proof. (of Observation 1.9): Take 2^{r-2} disjoint copies S_i of the above construction, and let N_i be the set of 2^{r-1} matchings of size 2 in S_i from that construction. Decompose $\bigcup N_i$ into $k = 2^{r-1}$ matchings M_i of size k, each consisting of one pair e_T, f_T from each S_i. The largest partial rainbow matching of the matchings M_i is of size 2^{r-2}, obtained by choosing one edge from each S_i. □

We do not know of any examples refuting $grs(r, k) \geq k - 2^{r-2}$ or $cms(r, k) \leq k + 2^{r-2}$ (if true, this would fit in with Conjecture 1.1).

In the next two sections we shall prove the following two theorems:

Theorem 1.11. $cms(2, k) \leq \lceil \frac{7}{4} k \rceil$.

Theorem 1.12. $grs(3, k) \geq \lfloor \frac{1}{2} k \rfloor$.

2. Proof of Theorem 1.11

Let G be a bipartite graph with sides A and B, and let $F_i, i = 1, \dots, k$ be matchings of size n in G, with $n \geq 7k/4$. We have to show that they possess a rainbow matching. We assume, for contradiction, that this is not the case. We also apply an inductive hypothesis, by which we may assume that the matchings $F_2, \dots, F_k$ have a rainbow matching $M = \{f_2, f_3, \dots, f_k\}$, where $f_i \in F_i$.

By the assumption that $\mathcal{F}$ does not possess a rainbow matching, every edge in F_1 meets some vertex of $\bigcup M$. Denote by F_1' the set of edges of F_1 that are incident with exactly one vertex in $\bigcup M$. Since F_1 is a matching of size n, there are at least $(n-k+1)$ vertices in $A \setminus \bigcup M$ that are incident with an edge of F_1'. Similarly, there are at least $(n-k+1)$ vertices in $B \setminus \bigcup M$ that are incident with an edge of F_1'. Hence at least $(n-k+1)$ edges in M meet an edge in F_1', and since $n \geq 7k/4$ implies $2(n-k+1) > k$, we have that there must be some edge in M incident with two edges in F_1'. Without loss of generality, assume that f_2 is one of these edges and let f_1' and f_1'' be the two edges in F_1' meeting f_2. We may also assume that the edges of M incident with at least one edge of F_1' belong, respectively, to $F_2, F_3, \dots, F_t$ (where, as shown above, $t > n-k$).

Now choose, if possible, two edges $f_2', f_2'' \in F_2$ satisfying:

(1) both f_2' and f_2'' meet f_{i_3} for some $2 < i_3 \leq t$.
(2) both f_2' and f_2'' do not meet any other edge from M or either of the edges f_1', f_1''.

Without loss of generality, we may assume that $i_3 = 3$. Then choose, if possible, two edges $f_3', f_3'' \in F_3$ incident with f_{i_4} for some $3 < i_4 \leq t$, such that both do not meet any other edge from M or any of the edges $f_j', f_j'',\ j < 3$. Without loss of generality, we may assume that $i_4 = 4$. Continuing this way until we reach a stage p in which a choice as above is impossible, we obtain a sequence of edges f_j', f_j'' for $1 \leq j < p$, both meeting f_{j+1}, but not meeting any other edge of M or any other f_i' or f_i''.

Write $M_1 = \{f_2, \dots, f_p\}$, and let P_1 be the set of vertices $\bigcup_{1 \leq j < p}(f_j' \cup f_j'')$ (note that P_1 contains all vertices from $\bigcup M_1$). Let $M_2 = \{f_{p+1}, \dots, f_t\},\ M_3 = \{f_{t+1}, \dots, f_k\}$ and let $P_2 = \bigcup M_2,\ P_3 = \bigcup M_3$. We have: $|P_1| = 4(p-1),\ |P_2| = 2(t-p), |P_3| = 2(k-t)$. Let $S = V(G) \setminus (P_1 \cup P_2 \cup P_3)$.

Claim 2.1. *There are at most $t-p$ edges of F_p joining a vertex of P_2 with a vertex of S.*

Proof. Since the process of choosing edges $f_j',\ f_j''$ terminated at $j = p$, there do not exist $g, h \in F_p$ incident with S and incident with the same $f \in M_2$. Since M_2 contains $t-p$ edges, this proves the claim. □

Claim 2.2. *There are no edges of F_p between S and P_1 or inside S.*

Proof. If such an edge f existed, it would start an alternating path whose application to M would result in a rainbow matching for $F_1, \dots, F_k$: replace f_p by f as a representative for F_p; at least one of f_{p-1}', f_{p-1}'' does not meet f, and this edge can replace f_{p-1} as a representative of F_{p-1}, and so on..., until one of f_1', f_1'' can represent F_1. □

Claim 2.3.

(1) *An edge $f \in F_p$ contained in P_1 must meet both f_j' and f_j'' for some $j < p$.*
(2) *There exists at most one index $j < p$ for which there exists an edge in $F_p \setminus M$ that meets f_j' and f_j''.*
(3) *At most p edges of F_p are contained in P_1 (these can be the $p-1$ edges $f_2, \dots, f_p$, plus one edge connecting the non-M vertices of some $f_j' \cup f_j''$).*

Proof. Part (1) is proved as above - an edge not meeting f_j' and f_j'' for any $j < p$ would start an alternating path whose application would yield a of size k rainbow matching for $F_1, \dots, F_k$.

For the proof of part (2) of the claim, let f be an edge in $F_p \setminus M$ and let $j < p$ be such that f meets f_j' and f_j''. Recall that by the definition of the choice of the edges $f_i, i \leq p$, we know that there exists $f_1 \in F_1$ that meets f_p. The set of edges $f_1, f_2, \dots, f_{p-1}, f, f_{p+1}, \dots, f_k$ is then a rainbow matching for $F_1, \dots, F_k$, unless f_1 meets f. But since f_1 has one end meeting f_p it means its other end must meet $f_j' \cup f_j''$, which can only happen for one value of $j < p$, so this proves part (2) of the claim.

Part (3) follows from part (1) and part (2). □

Now we just have to count the number of possible edges in F_p. Denote by

- t_1 the number of edges of F_p inside P_1.
- t_2 the number of edges of F_p between P_1 and P_2,
- t_3 the number of edges of F_p between P_1 and P_3,
- t_4 the number of edges of F_p inside P_2,
- t_5 the number of edges of F_p between P_2 and P_3,
- t_6 the number of edges of F_p between P_2 and S,
- t_7 the number of edges of F_p inside P_3,
- t_8 the number of edges of F_p between P_3 and S.

We then have the following relations, the first three following from the above claims, and the others from the fact that F_p is a matching.

- $\sum_{i=1}^{8} t_i = |F_p| = n$
- $t_1 \leq p$
- $t_6 \leq t - p$
- $2t_1 + t_2 + t_3 \leq 4(p-1)$
- $t_2 + 2t_4 + t_5 + t_6 \leq 2(t-p)$
- $t_3 + t_5 + 2t_7 + t_8 \leq 2(k-t)$

Multiplying the second one by 1, the third one by 1, the fourth one by 1, the fifth one by 2, and the sixth one by 3 and adding them all gives :

$$3t_1 + 3t_2 + 4t_3 + 4t_4 + 5t_5 + 3t_6 + 6t_7 + 3t_8 \leq p + (t-p) + 4(p-1) + 4(t-p) + 6(k-t) < 6k - t$$

Now we use $n = \sum t_i$ and $t > n - k$ to get the contradiction.

$$\begin{array}{rcl} 3n & \leq & 6k - t \\ 3n & < & 6k - (n-k) \\ n & < & 7k/4, \end{array}$$

■

3. Proof of Theorem 1.12

Let $H = (V, E)$ be a 3-partite hypergraph, and let F_i, $1 \leq i \leq k$ be matchings of size k. We have to show that they possess a partial rainbow matching of size $k/2$. Let M be a maximum rainbow matching. Without loss of generality, assume that $M = \{f_1, \ldots, f_p\}$, where $f_i \in F_i$. Let $i \leq p$ and $j > p$. We say that $f_i \in M$ is a good edge for F_j if there exists two distinct edges f'_j and f''_j in F_j intersecting f_i such that $|f'_j \cap \bigcup M| = |f''_j \cap \bigcup M| = 1$.

Claim 3.1. *For any $j > p$, there are at least $(k - 2p)$ good edges for F_j.*

Proof. Since M is maximal, every edge in F_j is incident to at least one edge in M. For $f \in M$ define

$$\phi(f) = \sum_{e \in F_j} \frac{|e \cap f|}{|e \cap \bigcup M|}$$

Clearly, $\phi(f) \leq 3$. In the sum defining $\phi(f)$ there can occur the fractions $\frac{1}{1}$, $\frac{2}{2}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$. If f is not a good edge then in the sum defining $\phi(f)$ there can be at most one term $\frac{1}{1}$. Since the sum of the numerators in the non-zero terms is at most 3, this implies that if f is not good then $\phi(f) \leq 2$.

Note also that for each edge e in F_j, we have that $\sum_{f \in M} |f \cap e| = |e \cap \bigcup M|$. Therefore

$$k = \sum_{e \in F_j} \frac{1}{|e \cap \bigcup M|} \sum_{f \in M} |f \cap e| = \sum_{f \in M} \phi(f) \leq 2(p-g) + 3g = g + 2p$$

where g denotes the number of good edges, and this gives the desired inequality. □

Claim 3.2. *No edge in M is good for 3 distinct matchings not represented in M.*

Proof. Denote by A, B, C the three sides of the hypergraph. In the following a vertex denoted by a_i (resp b_i,c_i) will always belong to A (resp. B,C). Moreover a_i and a_j for distinct i and j will always denote distinct vertices of the hypergraph. Assume by contradiction that such an edge e exists. Its vertices are (a_1, b_1, c_1), and it is a good edge for three distinct indices j_1, j_2, j_3.

Therefore for $s = 1, 2, 3$, there exist in F_{j_s} two edges e^s and f^s meeting e in *exactly* one vertex. Note that for $s \neq s'$ it is not possible that e^s and $e^{s'}$ are disjoint. Indeed in that case one could replace e in the matching by these two edges, contradicting the maximality of M. Of course this is also true for e^s and $f^{s'}$, so that amongst these 6 edges all pairs intersect except for the pairs (e^s, f^s) (which are disjoint since they come from the same matching).

Without loss of generality we can assume that $e^1 = (a_1, b_2, c_2)$ and $f^1 = (a_2, b_1, c_3)$. Again without loss of generality we can assume that e^2 meets e in a_1. Since e^2 has to intersect f^1, and since e^2 does not contain b_1, this implies that e^2 contains c_3.

Consider now e^3 and f^3 : if one of these edges contains b_1, since it cannot contain a_1, it will fail to meet both e^1 and e^2. Without loss of generality we can therefore assume that e^3 contains a_1 and f^3 contains c_1. But then as before, since e^3 meets f^1, it has to contain c_3. But now f^2 is subject to the same constraints as e^3 and f^3 just before, it cannot contain b_1 or else it will fail to intersect e^1 and e^3. Hence f^2 contains c_1.

Now we have that e^2 and e^3 both contains a_1 and c_3 and f^2 and f^3 both contain c_1. But since f^2 and f^3 must intersect e^1 it implies that both need to contain b_2. But now we get a contradiction because e^2 cannot contain b_2 and since f^3 cannot contain a_1, these two edges do not meet. □

By Claims 3.1 and 3.2 we have:

$$2p \geq \sum_{j>p} |\{e \in M : e \text{ is good for } F_j\}| \geq (k-p)(k-2p)$$

namely

$$2p^2 - (3k+2)p + k^2 \leq 0$$

which in turn implies that p is larger than the smallest root of the quadratic expression:

$$p \geq \frac{3k+2-\sqrt{(3k+2)^2-8k^2}}{4} > \frac{3k+2-\sqrt{(k+6)^2}}{4} = \frac{k}{2} - 1$$

■

4. A topological method

Hypergraph matching theory abounds with conjectures and is meager with results. In such a field putting order to the conjectures is of value. One of the aims of this paper is to place Conjecture 1.1 in a general setting, and relate it to other conjectures, some known and some new. For this purpose, we first need to present a few basic theorems on $\mathcal{C}$-SRs, that use algebraic topology.

As already mentioned, a hypergraph that is closed down (namely, a subset of an edge is necessarily an edge) is called in topology a "simplicial complex", or simply a "complex". The edges are called "simplices". Complexes have geometric realizations. For example, every graph can be realized geometrically in 3-dimensional space, by placing the vertices in points of general position, and connecting by straight segments pairs of vertices that are connected by an edge of the graph. The general position of the points guarantees that there are no fortuitous intersections of edges. In general, if the maximal size of an edge of the complex is k, placing the vertices in points of general position in $\mathbb{R}^{2k-1}$ and realizing every edge by the convex hull of the points corresponding to its vertices yields the desired realization of the complex. It is easy to see that the realization is unique, up to isomorphism.

The topological connectivity $\eta(\mathcal{C})$ of a complex $\mathcal{C}$ is defined as the minimal dimension of a "hole" in the geometric realization. Namely, it is the minimum over all k such that there exists a continuous function from

S^k to $\mathcal{C}$ that is not extendable to a continuous function from B^{k+1} to $\mathcal{C}$, plus 1. An old result of Whitney is that for a matroid η is either the rank, or infinity.

In [7] a topological version of Hall's theorem was proved:

Theorem 4.1. *If V_i, $i \in I$ are subsets of the vertex set of $\mathcal{C}$ and $\eta(\bigcup_{j\in J} V_j) \geq |J|$ for all $J \subseteq I$, then there exists a simplex meeting all V_is.*

This theorem becomes useful in combinatorics, when combined with combinatorially formulated lower bounds on η. For a graph G denoted by $\gamma^i(G)$ the maximum, over all independent sets Y in G, of the minimal size of a set of vertices dominating Y.

Theorem 4.2. [7] $\eta(\mathcal{I}(G)) \geq \gamma^i(G)$.

For a hypergraph H denote by $L(H)$ the line graph of H, namely the graph whose vertices are the edges of H, two vertices being connected if the corresponding edges meet. Since in the graph $L(H)$ of an r-uniform hypergraph a vertex (edge of H) can dominate (meet) at most r disjoint edges, we get:

Corollary 4.3. *For an r-homogeneous hypergraph H we have $\eta(L(H)) \geq \frac{\nu(H)}{r}$.*

This was strengthened in [5]:

Theorem 4.4. *For an r-homogeneous hypergraph H we have $\eta(L(H)) \geq \frac{\nu^*(H)}{r}$.*

Combining Theorems 4.4 and 4.1 yields:

Corollary 4.5. [5] *Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a set of r-uniform hypergraphs on the same vertex set. If $\nu^*(\bigcup_{i\in I} F_i) > r(|I|-1)$ for every $I \subseteq [m]$ then $\mathcal{F}$ has a rainbow matching.*

In [4] a lower bound was proved on the connectivity of the intersection of r matroids. For a hypergraph H denote by $rank(H)$ the maximal size of an edge in it.

Theorem 4.6. *Given matroids $\mathcal{M}_1, \dots, \mathcal{M}_r$ on the same ground set, $\eta(\bigcap \mathcal{M}_i) \geq \frac{rank(\bigcap \mathcal{M}_i)}{r}$.*

5. The covering number of a complex

A parameter that will play a crucial role in the results and conjectures below is the *covering number*.

Definition 5.1. The *covering number* (of vertices by edges) of a complex $\mathcal{C}$, denoted by $\beta(\mathcal{C})$, is the minimal number of edges of $\mathcal{C}$ needed to cover $V(\mathcal{C})$.

This parameter is sometimes denoted by ρ, but since we shall apply it also to matroids, and there ρ may be interpreted as rank, we prefer the β notation. For example, for a graph G we have $\beta(\mathcal{I}(G)) = \chi(G)$ (for which reason the covering number is denoted in [4] by χ).

The fractional counterpart $\beta^*(\mathcal{C})$ is the minimal sum of weights on edges from $\mathcal{C}$, such that every vertex belongs to edges whose sum of weights is at least 1.

In [4] the following was proved:

Theorem 5.2. $\beta(\mathcal{C}) \leq \max_{I\subseteq V(\mathcal{C})} \frac{|I|}{\eta(\mathcal{C}[I])}$.

For $\mathcal{C}$ a matroid, this is a classical theorem of Edmonds. Remember that for matroids η is the rank.

6. Putting the main conjecture in context

While Conjecture 1.1 generalizes the Brualdi-Ryser-Stein conjecture, its most natural background is probably the following observation, proved by a greedy argument:

Observation 6.1. Any set $\mathcal{F} = (F_1, \dots, F_k)$ of independent sets in a matroid $\mathcal{M}$, where $|F_i| = k$ for all i, has an $\mathcal{M}$-SR.

As often happens, when a fact is true for a very simple reason, it can be strengthened, and acquire depth by the addition of other ingredients. In this particular case, we are aware of three possible such ingredients:

(1) Adding another matroid, namely replacing $\mathcal{M}$ by the intersection of two matroids.

A special case is that of rainbow matchings in bipartite graphs, and as noted, this renders Observation 6.1 false: a price of 1 has to be paid. The conjecture becomes:

Conjecture 6.2. [3] *Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same vertex set. If $F_1, \dots, F_k$ are sets of size $k+1$ belonging to $\mathcal{M} \cap \mathcal{N}$, then they have an $\mathcal{M} \cap \mathcal{N}$-SR.*

(2) Decomposability, meaning requiring the existence of "many" $\mathcal{M}$-SR's, in the sense that $\bigcup \mathcal{F}$ is the union of k $\mathcal{M}$-SR's.

In this case Observation 6.1 becomes a famous conjecture of Rota:

Conjecture 6.3. [19] *Given a set $\mathcal{F} = (F_1, \dots, F_k)$ of independent sets in a matroid $\mathcal{M}$, where $|F_i| = k$ for all i, the multiset union $\bigcup \mathcal{F}$ can be decomposed into k $\mathcal{M}$-SRs.*

(3) A "scrambled" version, obtained by scrambling the F_is, resulting in another family of k sets of size k. The resulting family, call it $G_1, \dots, G_k$, has the property that $\bigcup G_i$, considered as a multiset, is decomposable into k independent sets. In this case, the observation remains true. In fact, a more general fact is true, making the scrambled version more interesting: the number of G_is can be general.

Theorem 6.4. *If $\mathcal{G} = (G_1, \dots, G_m)$ is a set of disjoint, but not necessarily independent, sets of size k in a matroid $\mathcal{M}$, and if $\beta(\mathcal{M}) \le k$, then $\mathcal{G}$ has an $\mathcal{M}$-SR.*

Proof. Rado's theorem [21] states that a necessary and sufficient condition for $\mathcal{G}$ to have an $\mathcal{M}$-SR is that for every set of indices $i_1, \dots, i_k$ the rank of $G := \bigcup_{1 \le j \le k} G_{i_j}$ is at least k. Since $|G| = kn$, there exists some i_j such that G contains at least k elements from G_{i_j}, and since G_{i_j} is independent, it follows that $rank(G) \ge k$. □

Things become more complicated, and more interesting, when two of the ingredients are added together, or even all three. These combinations we study below.

7. Scrambling and covering numbers

We shall need the following easy corollary of Edmonds' matroids intersection theorem [14]:

Theorem 7.1. *If $\mathcal{M}, \mathcal{N}$ are matroids on the same ground set V, then $rank(\mathcal{M} \cap \mathcal{N}) \ge \frac{|V|}{\max(\beta(\mathcal{M}), \beta(\mathcal{N}))}$.*

In the following theorem a matroid is added, and at the same time the covering number of the complex is bounded.

Theorem 7.2. *If $\mathcal{M}$, $\mathcal{N}$ are matroids on the same vertex set satisfying $\max(\beta(\mathcal{M}), \beta(\mathcal{N})) \le k$, and $\mathcal{F} = (F_1, \dots, F_m)$ is a family of disjoint sets belonging to $\mathcal{M} \cap \mathcal{N}$, all of size $2k$, then $\mathcal{F}$ has a $\mathcal{M} \cap \mathcal{N}$-SR.*

Proof. By Theorem 4.1 it suffices to show that for $J \subseteq [m]$ it is true that $\eta((\mathcal{M} \cap \mathcal{N})[\bigcup_{j \in J} F_j]) \ge |J|$. By Theorem 7.1 we have $rank((\mathcal{M} \cap \mathcal{N})[\bigcup_{j \in J} F_j]) \ge \frac{|\bigcup_{j \in J} F_j|}{k} \ge \frac{2k|J|}{k} = 2|J|$. By Theorem 4.6 it follows that $\eta((\mathcal{M} \cap \mathcal{N})[\bigcup_{j \in J} F_j]) \ge |J|$, as required. □

When $\mathcal{M}$ and $\mathcal{N}$ are partition matroids, $\mathcal{M} \cap \mathcal{N}$ is the complex of matchings in a bipartite graph G. By König's edge coloring theorem, the condition $\beta(\mathcal{M} \cap \mathcal{N}) \le k$ is equivalent to $\Delta(G) \le k$ (as usual, $\Delta(G)$ denotes the maximal degree of a vertex). This case of the observation can be generalized:

Theorem 7.3. *Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a set of r-uniform hypergraphs on the same vertex set. If $|F_i| \ge r\Delta(\bigcup_{j \le m} F_j)$ for all i (here the union is taken as a multiset, namely degrees are counted with multiplicity), then $\mathcal{F}$ has a rainbow matching.*

Proof. For every subset I of $[m]$ the constant function $f(e) = \frac{1}{\Delta(\bigcup_{i \in I} F_i)}$ is a fractional matching of $\bigcup_{i \in I} F_i$. This shows that $\nu^*(\bigcup_{i \in I} F_i) \ge r|I|$, which, by Theorem 4.4, implies the existence of a rainbow matching. □

Theorem 7.3 is tight even when the hypergraphs F_i are r-partite. In the next example, provided to us by Philipp Sprüssel and Shira Zerbib [24], the hypergraphs F_i are multihypergraphs, meaning that they contain repeated edges.

Example 7.4. For $i = 1, \dots, k$ let F_i be a matching M_i of size r, repeated k times (each edge of M_i is of size r). Let F_{k+1} consist of k matchings N_i, each of size r, such that each edge in N_i meets each edge in M_i. Then $|F_i| = kr$ for all $i \le k+1$, the degree of every vertex in $\bigcup F_i$ is $kr+1$, and there is no rainbow matching.

With Eli Berger we found also examples of simple hypergraphs showing tightness. We do not describe them here.
In the r-partite case, possibly something stronger is true.

Conjecture 7.5. *There exists a function $d(r)$ such that if the F_is consist of edges in an r-partite hypergraph and $|F_i| \geq (r-1)\Delta(\bigcup_{j\leq m} F_j) + d(r)$ for all i then there exists a rainbow matching.*

For r such that there exists a projective plane of edge size r this conjecture is sharp (namely, $r-1$ cannot be replaced by a smaller function of r).

Here is a conjectured matroidal version of Theorem 7.3:

Conjecture 7.6. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_r$ be r matroids on the same vertex set, and let $\mathcal{F} = (F_i,\ i \leq m)$ be a family of disjoint sets, all belonging to $\bigcup_{i\leq m} \mathcal{M}_i$. If $\beta(\mathcal{M}_i) \leq k$ for all $i \leq r$ and $|F_i| \geq rk$ then $\mathcal{F}$ has a $\bigcup_{i\leq m} \mathcal{M}_i$-SR.*

It may well be the case that the conclusion of Theorem 7.3 can be strengthened, in the spirit of Rota's conjecture. Namely, that there exists not merely a single rainbow matching, but many.

Conjecture 7.7. [2]
Let $F_1, \ldots, F_m$ be r-uniform hypergraphs satisfying $|F_i| \geq r\Delta(\bigcup_{j\leq m} F_j)$ for all i. Then $\bigcup_{i\leq m} F_i$ can be decomposed into $\max_{i\leq m} |F_i|$ *rainbow matchings.*

For $r = 2$ Conjecture 7.7 is a generalization of a conjecture of Hilton [18]:

Conjecture 7.8. *An $n \times 2n$ Latin rectangle can be decomposed into $2n$ transversals.*

In [17] Hilton's conjecture was proved for $n \times (1 + o(1))n$ Latin rectangles.

In [2] Conjecture 7.7 was proved for $|S| = 2$. Another result there was that the conjecture is true up to a factor of 2, namely under the stronger condition $deg(u) \geq 2(r-1)deg(v)$ for every $u \in S$ and $v \in V \setminus S$.

8. Scrambling and decomposing into rainbow matchings

What happens in Rota's conjecture if we first scramble the elements? That is, if the sets $F_i,\ i = 1, \ldots, k$ are not necessarily bases, but $\bigcup_{i\leq k} F_i$ is the (multiset) union of k bases? In [8] it was shown that for all $k > 2$ there exist examples of F_is as above, with no decomposition into k $\mathcal{M}$-SRs. However, the following may be true:

Conjecture 8.1. [8] *If $F_1, \ldots, F_m$ are sets of size k in a matroid $\mathcal{M}$ satisfying $\beta(\mathcal{M}) \leq k$, then there exist $k-1$ disjoint $\mathcal{M}$-SRs.*

A dual conjecture relates to covering, rather than packing:

Conjecture 8.2. *Given sets $F_1, \ldots, F_m$ of size k in a matroid $\mathcal{M}$ satisfying $\beta(\mathcal{M}) \leq k$, there exist $k+1$ $\mathcal{M}$-SRs whose union is $\bigcup_{i\leq m} F_i$.*

The following is a far reaching generalization, made by the first author and Eli Berger:

Conjecture 8.3. [4] *For any pair of matroids $\mathcal{M}$, $\mathcal{N}$ on the same ground set,*

$$\beta(\mathcal{M} \cap \mathcal{N}) \leq \max(\beta(\mathcal{M}), \beta(\mathcal{N})) + 1.$$

In Rota's conjecture one of the matroids is the partition matroid whose parts are the sets F_i. Conjecture 8.3 is close in spirit to a well known conjecture of Goldberg and Seymour [15, 23]:

Conjecture 8.4. *In any multigraph $\chi' \leq \chi'^* + 1$.*

The kinship between the two conjectures was given a precise formulation in [6], where a common generalization of the two was suggested, in terms of 2-polymatroids.

As often happens in this field, Conjecture 8.3 is known up to a factor of 2:

Theorem 8.5. [4] *For $\mathcal{M}$, $\mathcal{N}$ as above, $\beta(\mathcal{M} \cap \mathcal{N}) \leq 2\max(\beta(\mathcal{M}), \beta(\mathcal{N}))$.*

This follows from Theorems 4.6, 5.2 and 7.1.

Corollary 8.6.
If $F_1, \ldots, F_m$ are disjoint sets of size k in a matroid $\mathcal{M}$ satisfying $\beta(\mathcal{M}) \leq k$ then there exist $2k$ $\mathcal{M}$-SRs whose union is $\bigcup_{i\leq m} F_i$.

9. Distinct edges

It is an intriguing fact that in some theorems and conjectures on hypergraph matchings the only known examples showing sharpness use repeated edges. It is tempting to conjecture that under an assumption of distinctness (of either edges or sets) the conditions can be weakened. Here are two examples:

Theorem 9.1. [13, 3]
$2k-1$ matchings of size k in a bipartite graph have a partial rainbow matching of size k.

The example showing sharpness is $F_1 = \ldots = F_{k-1} = \{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$, and $F_k = F_{k+1} = \ldots = F_{2k-2} = \{(a_1,b_2),(a_2,b_3),\ldots,(a_k,b_1)\}$.

Conjecture 9.2. *$k+1$ disjoint matchings of size k in a bipartite graph have a partial rainbow matching of size k.*

Here is yet another generalization of the Ryser-Brualdi-Stein conjecture:

Conjecture 9.3. *In a d-regular $n \times n \times n$ 3-partite simple (i.e., not containing repeated edges) hypergraph there exists a matching of size at least $\lceil \frac{d-1}{d} n \rceil$.*

For $d = 2$ the conjecture is true, by the inequality $\nu \geq \frac{1}{2}\tau$ proved in [1] (in a regular $n \times n \times n$ 3-partite hypergraph $\tau = n$, because $\tau^* = \nu^* = n$). The Ryser-Brualdi-Stein conjecture is obtained by taking $d = n$. If true, the conjecture is sharp for all n and d. To see this, write $n = kd + \ell$, where $\ell < d$, and take k disjoint copies of a 3-partite d-regular $d \times d \times d$ hypergraph with $\nu = d-1$, together with a disjoint $\ell \times \ell \times \ell$ d-regular 3-partite hypergraph. To see that the non-repetition of edges is essential, take the Fano plane with a vertex deleted (containing 4 edges: $(a_1,b_1,c_1),(a_1,b_2,c_2),(a_2,b_1,c_2),(a_2,b_2,c_1)$), and repeat every edge $d/2$ times for any d even. Then the degree of every vertex is d, and the matching number is 1.

In [12] some progress on the conjecture was made:

Theorem 9.4. *In a d-regular $n \times n \times n$ 3-partite hypergraph with no pair of edges intersecting in more than one vertex, there exists a matching of size at least* $\max(d(1-\frac{1}{\sqrt{n}}),\ d-\frac{n}{n-d},\ \frac{5n}{9} - O(\frac{n}{\sqrt{d}}))$.

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