SIZE CONDITIONS FOR THE EXISTENCE OF RAINBOW MATCHINGS

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Abstract. Let $f(n,r,k)$ be the minimal number such that every hypergraph larger than $f(n,r,k)$ contained in $\binom{[n]}{r}$ contains a matching of size k, and let $g(n,r,k)$ be the minimal number such that every hypergraph larger than $g(n,r,k)$ contained in the r-partite r-graph $[n]^r$ contains a matching of size k. The Erdős-Ko-Rado theorem states that $f(n,r,2)=\binom{n-1}{r-1}$ $(r \le \frac{n}{2})$ and it is easy to show that $g(n,r,k)=(k-1)n^{r-1}$.

The conjecture inspiring this paper is that if $F_1, F_2, \dots, F_k \subseteq \binom{[n]}{r}$ are of size larger than $f(n,r,k)$ or $F_1, F_2, \dots, F_k \subseteq [n]^r$ are of size larger than $g(n,r,k)$ then there exists a rainbow matching, i.e. a choice of disjoint edges $f_i \in F_i$. In this paper we deal mainly with the second part of the conjecture, and prove it for the cases $r \le 3$ and $k=2$. The proof of the $r=3$ case uses a Hall-type theorem on rainbow matchings in bipartite graphs. For the proof of the $k=2$ case we prove a Kruskal-Katona type theorem for r-partite hypergraphs.

We also prove that for every r and k there exists $n_0=n_0(r,k)$ such that the r-partite version of the conjecture is true for $n>n_0$.

1. Motivation

1.1. **The Erdős-Ko-Rado theorem and rainbow matchings.** The largest size of a matching in a hypergraph H is denoted by $\nu(H)$. The famous Erdős-Ko-Rado (EKR) theorem states that if $r \le \frac{n}{2}$ and a hypergraph $H \subseteq \binom{[n]}{r}$ has more than $\binom{n-1}{r-1}$ edges, then $\nu(H)>1$. This has been extended in more than one way to pairs of hypergraphs. For example, in [14] the following was proved:

Theorem 1.1. *If $H_1, H_2 \subseteq \binom{[n]}{r}$ satisfy $|H_1||H_2| > \binom{n-1}{r-1}^2$ (in particular if $|H_i| > \binom{n-1}{r-1}$, $i=1,2$) then there exist disjoint edges, $e_1 \in H_1$, $e_2 \in H_2$.*

It is natural to try to extend this to more than two hypergraphs. The relevant notion is that of "rainbow matchings".

Definition 1.2. Let $\mathcal{F}=(F_i \mid 1 \le i \le k)$ be a collection of hypergraphs. A choice of disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$.

Notation 1.3. For n,r,k satisfying $r \le \frac{n}{2}$ we denote by $f(n,r,k)$ the smallest number such that $\nu(H) \ge k$ for every $H \subseteq \binom{n}{r}$ larger than $f(n,r,k)$.

The value of $f(n,r,k)$ is known asymptotically:

Theorem 1.4. [8] *For every r,k there exists $n_0=n_0(r,k)$ such that for every $n \ge n_0$:*

$$f(n,r,k)=\binom{n}{r}-\binom{n-k+1}{r}$$

The following is true for all values of n:

Theorem 1.5. $f(n,r,k) \le (k-1)\binom{n-1}{r-1}$.

Here is a quick proof in the case $r|n$. Denote by P the set of perfect matchings in $\binom{[n]}{r}$, and write $p=|P|$. Form a bipartite graph Γ whose one side is P and the other side is $\binom{[n]}{r}$, and in which $e \in \binom{[n]}{r}$ is connected

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to $M \in P$ if $e \in M$. Let q be the degree of each vertex $e \in \binom{[n]}{r}$, namely the number of perfect matchings containing e. Counting the edges of Γ in two ways we get $\frac{n}{r}p = \binom{n}{r}q$, namely $p = \binom{n-1}{r-1}q$. Let $H \subseteq \binom{[n]}{r}$ be of size larger than $(k-1)\binom{n-1}{r-1}$. Then the total number of edges going out of H in Γ is larger than $(k-1)p$, and hence there exists a matching in P containing at least k edges from H, proving $\nu(H) \geq k$.

For general r the theorem can be proved using the idea from the Katona proof of the EKR theorem [12], which is of similar spirit.

It is a natural guess that Theorem 1.1 can be extended to general k, as follows.

Conjecture 1.6. *Let $\mathcal{F} = (F_1, \ldots, F_k)$ be a system of hypergraphs contained in $\binom{[n]}{r}$. If $|F_i| > f(n, r, k)$ (in particular if $|F_i| > (k-1)\binom{n-1}{r-1}$) for all $i \leq k$ then $\mathcal{F}$ has a rainbow matching.*

In Section 2.2 we shall present a proof by Meshulam for the $r = 2$ case of this conjecture.

1.2. **The r-partite case.** An r-uniform hypergraph H is called *r-partite* if $V(H)$ is partitioned into sets $V_1, \ldots, V_r$, called the *sides* of H, and each edge meets every V_i in precisely one vertex. If all sides are of the same size n, H is called *n-balanced*. The complete n-balanced r-partite hypergraph can be identified with $[n]^r$.

Note that matchability of one side V_i in an r-partite hypergraph is equivalent to the existence of a rainbow matching of the hypergraphs H_v consisting of the $r-1$-edges incident with the vertex $v \in V_i$.

Conditions of different types are known for the existence of rainbow matchings. For example, in [11] a sufficient condition was formulated in terms of domination in the line graph of $\bigcup_{i \in I} F_i$ (I ranging over all subsets of $[k]$). In [2, 3] conditions were considered in terms of lower bounds on $\nu(\bigcup_{i \in K \subseteq I} F_i)$. There are also many open conjectures on rainbow matchings, of which we mention here one, from [2], strengthening a conjecture of Ryser, Brualdi and Stein [16], [6, p.103].

Conjecture 1.7. *Any system of k matchings in a bipartite graph, each of size $k+1$, has a rainbow matching.*

Here we shall be interested in conditions formulated in terms of the sizes of the hypergraphs.

Observation 1.8. *If F is a set of edges in an n-balanced r-partite hypergraph and $|F| > (k-1)n^{r-1}$ then $\nu(F) \geq k$.*

Proof. The complete n-balanced r-partite hypergraph $[n]^r$ can be decomposed into n^{r-1} matchings M_i, each of size n. Writing $F = \bigcup_{i \leq n^{r-1}} (F \cap M_i)$ shows that one of the matchings $F \cap M_i$ has size at least k. □

The r-partite analogue of Conjecture 1.6 is:

Conjecture 1.9. *If $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ is a set of sets of edges in an n-balanced r-partite hypergraph and $|F_i| > (k-1)n^{r-1}$ for all $i \leq k$ then $\mathcal{F}$ has a rainbow matching.*

The following result, stating the case $r = 2$, will be subsumed by later results, but it is worth while to see a short proof:

Theorem 1.10. *If $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ is a set of sets of edges in an n-balanced bipartite graph and $|F_i| > (k-1)n$ for all $i \leq k$ then $\mathcal{F}$ has a rainbow matching.*

Proof. Denote the sides of the bipartite graph M and W. Since $\sum_{v \in M} d_{F_1}(v) = |F_1| > (k-1)n$, there exists a vertex $v_1 \in M$ such that $d_{F_1}(v_1) \geq k$. Write $F'_2 = F_2 - v_1$. Since $d_{F_2}(v_1) \leq n$, we have $|F'_2| > (k-2)n$, and hence there exists a vertex $v_2 \neq v_1$ such that $d_{F_2}(v_2) \geq k-1$. Continuing this way we obtain a sequence $v_1, \ldots, v_k$ of distinct vertices in M, satisfying $d_{F_i}(v_i) > k - i$. Since $d_{F_k}(v_k) > 0$ there exists an edge $e_k \in F_k$ containing v_k. Since $d_{F_{k-1}}(v_{k-1}) > 1$ there exists an edge $e_{k-1} \in F_{k-1}$ containing v_{k-1} and missing e_k. Since $d_{F_{k-2}}(v_{k-2}) > 2$ there exists an edge $e_{k-2} \in F_{k-2}$ containing v_{k-2} and missing e_k and e_{k-1}. Continuing this way, we construct a rainbow matching $e_1, \ldots, e_k$ for $\mathcal{F}$. □

We shall prove:

Theorem 1.11. *Conjecture 1.9 is true for $k = 2$.*

Theorem 1.12. *Conjecture 1.9 is true for $r = 3$.*

2. Shifting

Shifting is an operation on a hypergraph H, defined with respect to a specific linear ordering "$<$" on its vertices. For $x < y$ in $V(H)$ define $s_{xy}(e) = e \cup x \setminus \{y\}$ if $x \notin e$ and $y \in e$, provided $e \cup x \setminus \{y\} \notin H$; otherwise let $s_{xy}(e) = e$. We also write $s_{xy}(H) = \{s_{xy}(e) \mid e \in H\}$. If $s_{xy}(H) = H$ for every pair $x < y$ then H is said to be *shifted*.

Given an r-partite hypergraph G with sides M and W, and linear orders on its sides, an *r-partite shifting* is a shifting s_{xy} where x and y belong to the same side. G is said to be *r-partitely shifted* if $s_{xy}(H) = H$ for all $x < y$ on the same side.

Given a collection $\mathcal{H} = (H_i,\ i \in I)$ of hypergraphs, we write $s_{xy}(\mathcal{H})$ for $(s_{xy}(H_i),\ i \in I)$.

Observation 2.1. *Define a partial order on pairs of vertices by $(v_i, v_j) \le (v_k, v_\ell)$ if $i \le k$ and $j \le \ell$. Write $(v_i, v_j) < (v_k, v_\ell)$ if $(v_i, v_j) \le (v_k, v_\ell)$ and $(v_i, v_j) \neq (v_k, v_\ell)$. A set F being shifted is equivalent to its being closed downward in this order, which in turn is equivalent to the fact that the complement of F is closed upward.*

As observed in [8] (see also [4]) shifting does not increase the matching number of a hypergraph. This can be generalized to rainbow matchings:

Lemma 2.2. *Let $\mathcal{F} = (F_i \mid\ i \in I)$ be a collection of hypergraphs, sharing the same linearly ordered ground set V, and let $x < y$ be elements of V. If $s_{xy}(\mathcal{F})$ has a rainbow matching, then so does $\mathcal{F}$.*

Proof. Let $s_{xy}(e_i),\ i \in I$, be a rainbow matching for $s_{xy}(\mathcal{F})$. There is at most one i such that $x \in e_i$, say $e_i = a \cup \{x\}$ (where a is a set).

If there is no edge e_s containing y, then replacing e_i by $a \cup \{y\}$ as a representative of F_i, leaving all other e_s as they are, results in a rainbow matching for $\mathcal{F}$. If there is an edge e_s containing y, say $e_s = b \cup \{y\}$, then there exists an edge $b \cup \{x\} \in F_s$ (otherwise the edge e_s would have been shifted to $b \cup \{x\}$.) Replacing then e_i by $a \cup \{y\}$ and e_s by $b \cup \{x\}$ results in a rainbow matching for $\mathcal{F}$. □

3. Conjecture 1.6 for $r = 2$

In [8] the value of $f(n, 2, k)$ was determined for all k:

Theorem 3.1. $f(n, 2, k) = \max(\binom{2k-1}{2}, (k-1)(n-1) - \binom{k-1}{2})$.

In [4] this result was given a short proof, using shifting. Meshulam [15] noticed that this proof yields also Conjecture 1.6 for $r = 2$:

Theorem 3.2. *Let $\mathcal{F} = (F_i,\ 1 \le i \le k)$ be a collection of subsets of $E(K_n)$. If $|F_i| > \max(\binom{2k-1}{2}, (k-1)(n-1) - \binom{k-1}{2})$ for all $i \le k$ then $\mathcal{F}$ has a rainbow matching.*

Proof. Enumerate the vertices of K_n as $v_1, v_2, \ldots, v_n$. By Lemma 2.2 we may assume that all F_i's are shifted with respect to this enumeration. For each $i \le k$ let $e_i = (v_i, v_{2k-i+1})$. We claim that the sequence e_i is a rainbow matching for $\mathcal{F}$. Assuming negation, there exists i such that $e_i \notin F_i$. Since F_i is shifted, every edge (v_p, v_q) in F_i, where $p < q$, satisfies
(P) $p < i$ or $q < 2k - i + 1$.
The number of pairs satisfying $p < i$ is $(i-1)(n-1) - \binom{i-1}{2}$. The number of pairs satisfying $p \ge i$ and $q < 2k - i + 1$ is $\binom{2k-2i+1}{2}$, so

$$|F_i| \le (i-1)(n-1) - \binom{i-1}{2} + \binom{2k-2i+1}{2}$$

This is a convex quadratic expression in i, attaining its maximum either at $i = 1$ (in which case $|F_i| \le \binom{2k-1}{2}$) or at $i = k$ (in which case $|F_i| \le (k-1)(n-1) - \binom{k-1}{2}$). In both cases we get a contradiction to the assumption on $|F_i|$. □

4. A Kruskal-Katona type theorem for blocking pairs in r-partite hypergraphs

4.1. **Blockers.** Daykin [5] showed how the EKR theorem follows from the Kruskal-Katona theorem. His proof also yields that for $r \leq \frac{n}{2}$, if F_1, F_2 are sets in $\binom{[n]}{r}$ and $|F_1| \geq \binom{n-1}{r-1}$, $|F_2| > \binom{n-1}{r-1}$ then F_1, F_2 have a rainbow matching. The idea of the proof is that if $|F|$ is large then, by the Kruskal-Katona theorem, the r-shadow of the complements of the sets in F is large, and hence the number of the r-sets that meet all edges in F is small. In this section we use a similar idea in the case of r-partite hypergraphs. For this purpose, we shall need a Kruskal-Katona type theorem on the maximal number of edges meeting all edges in an r-partite hypergraph. The *blocker* $B(F)$ of a subset F of $[n]^r$ is the set of those edges of $[n]^r$ that meet all edges of F. For a number t we denote by $b(t)$ the maximal size of $|B(F)|$, F ranging over all sets of t edges in $[n]^r$. The theorem in question determines $b(t)$ for all $t \leq n^r$.

4.2. **A self similar sequence.** Consider an n-balanced hypergraph with sides $V_1, \ldots, V_r$, and choose one vertex v_i from each V_i. Let Ψ_r be the set of sequences σ of length $0 \leq k \leq r-1$ of $\wedge$'s and $\vee$'s, and let $\Sigma_r = \Psi_r$ together with two special elements, $\alpha = \alpha_r$ and $\omega = \omega_r$. Note that $|\Sigma_r| = 2^r + 1$. We define hypergraphs $F_r(\sigma)$ for all $\sigma \in \Sigma_r$, as follows. Let $F_r(\alpha) = \emptyset$ and $F_r(\omega) = [n]^r$. For a sequence $\sigma \in \Psi_r$ having length $m \geq 0$, and whose j-th component is denoted by σ_j $(j \leq m)$, let:

$$F_r(\sigma) = \{e \in [n]^r \mid v_1 \in e \ \sigma_1(v_2 \in e \ \sigma_2(v_3 \in e \ldots \sigma_m(v_{m+1} \in e) \ldots)\}$$

For example, $F_r(\emptyset) = \{e \in [n]^r \mid v_1 \in e\}$ and $F_r(\wedge, \wedge, \vee)$ is the set of edges $e \in [n]^r$ satisfying:

$$v_1 \in e \wedge (v_2 \in e \wedge (v_3 \in e \vee (v_4 \in e)))$$

Write $f_r(\sigma) = |F_r(\sigma)|$.

Lemma 4.1.

(1) $f_r(\sigma) = nf_{r-1}(\sigma)$
(2) $f_r(\wedge, \sigma) = f_{r-1}(\sigma)$
(3) $f_r(\vee, \sigma) = n^{r-1} + (n-1)f_{r-1}(\sigma)$

Part 1 is true since $F_r(\sigma) = F_{r-1}(\sigma) \times V_r$. Part 2 is true since an edge in $F_r(\wedge, \sigma)$ is obtained from an edge $f \in F_{r-1}(\sigma)$, with indices shifted by 1, by adding v_1. Part 3 is true since $F_r(\vee, \sigma) = \{v_1\} \times V_2 \times \ldots \times V_r \cup (V_1 \setminus \{v_1\}) \times F_{r-1}(\sigma)$ (where, again, edges in $F_{r-1}(\sigma)$ have their indices shifted by 1).

We order $f_r(\sigma)$ by size, and rename them $N(i) = N_r(i)$ $(0 \leq i \leq 2^r)$.

Example 4.2.

(1) $N(0) = f_r(\alpha) = 0$.
(2) $N(1) = f_r(\wedge, \wedge, \ldots, \wedge)$ *($r-1$ times), which is* 1.
(3) $N(2) = f_r(\wedge, \wedge, \ldots, \wedge)$ *($r-2$ times) which is* n.
(4) $N(2^{r-1}) = f_r(\emptyset) = n^{r-1}$.
(5) $N(2^r) = f_r(\omega) = n^r$.

In accord we order Σ_r as $\sigma(i)$ $(0 \leq i \leq 2^r)$. For example $\sigma(0) = \alpha$, $\sigma(2^r) = \omega$. We also define the inverse function, which we name "i": if $\sigma(q) = \tau$, then $i(\tau) = q$.

Clearly, for every $\beta \in \Psi_r$

$$i((\beta, \wedge)) < i(\beta) < i((\beta, \vee)) \tag{1}$$

The elements of Ψ_r can be viewed as the nodes of a binary tree, the depth of a node being the length of the sequence (so the root, with depth 0, is the empty sequence). The order on Ψ_r, uniquely determined by (1), is known as the "in-order depth first search" on the tree, where $\wedge$ ("left") precedes $\vee$ ("right").

This description of the order on Ψ_r entails an explicit formula for $\sigma(i)$. Represent $i \neq 0, 2^r$ in binary form: $i = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_s}$, where $k_0 > k_1 > \ldots > k_s$. Then $\sigma(i)$ is of length $r - k_s - 1$, and it consists of s $\vee$'s and $r - k_s - 1 - s$ $\wedge$'s. It starts with $r - k_0 - 1$ (possibly zero) $\wedge$'s; if $s > 0$ these are followed by a $\vee$; this is followed by $k_0 - k_1 - 1$ (possibly zero) $\wedge$'s, and if $s > 1$ this is followed by a $\vee$, followed by $k_1 - k_2 - 1$ $\wedge$'s, and so forth.

For example, $\sigma_6(13) = \sigma_6(2^3 + 2^2 + 2^0) = (\wedge, \wedge, \vee, \vee, \wedge)$.

The numbers $N(i)$ can also be written explicitly:

$$N(i) = \sum_{i \leq s} n^{k_i}(n-1)^i$$

The explicit description of $\sigma(i)$ and the formula for $N(i)$ will not be used below, and hence their proofs are omitted.

Example 4.3. *The values of N_3 are:*

$$0,\ 1,\ n,\ n + (n-1),\ n + n(n-1) = n^2,\ n^2 + (n-1),\ n^2 + (n-1)(2n-1),\ n^2 + (n-1)n^2 = n^3.$$

Lemma 4.4.

(1) For $i \leq 2^{r-1}$ we have $N_r(i) = N_{r-1}(i)$, namely the sequence $N_{r-1}(i)$ is an initial segment of $N_r(i)$.
(2) $\sigma(2^p) = (\wedge, \wedge, \ldots, \wedge)$, a sequence of $r - p - 1$ $\wedge$'s, and $N(2^p) = n^p$.
(3) For $i < 2^p$ the sequences $\sigma(i)$ are of the form $(\sigma(2^p), \wedge, \beta)$ (β being some sequence), and for $2^p < i < 2^{p+1}$ the sequences $\sigma(i)$ are of the form $(\sigma(2^p), \vee, \beta)$.
(4) For $p \leq r - 1$ and $i \leq 2^p$, we have

$$N(2^p + i) = N(2^p) + (n-1)N(i) = n^p + (n-1)N(i)$$

Part 1 is true by part 2 of Lemma 4.1, since $\sigma(1), \ldots, \sigma(2^{r-1} - 1)$ all start with a $\wedge$. Parts 2 and 3 follow from Equation (1) and the remark following it. Part 4 follows from part 3 of Lemma 4.1. Part 4 says that the numbers $N(i)$ have a fractal-like pattern, where each sequence N_r is obtained from N_{r-1} by adding on its right an $n-1$-times magnified image of itself, the first element of the right sequence being identified with the last element of the left copy, both being equal to n^{r-1}. This entails:

Lemma 4.5. *If $b, c \leq 2^p$ then $N(2^{p+1} + b) - N(2^p + c) = (n-1)(N(2^p + b) - N(c))$.*

4.3. **The size of blocking hypergraphs.** For $\sigma \in \Psi_r$ we denote by $\overline{\sigma}$ the sequence obtained by replacing each $\wedge$ by a $\vee$ and vice versa. We also define $\overline{\alpha} = \omega$ and $\overline{\omega} = \alpha$. Clearly, $i(\sigma) > i(\tau)$ if and only if $i(\overline{\sigma}) < i(\overline{\tau})$, and hence we have:

$$i(\overline{\sigma}) = 2^r - i(\sigma) \tag{2}$$

By De Morgan's law, we have:

Lemma 4.6. $B(F_r(\sigma)) = F_r(\overline{\sigma})$.

Lemma 4.7. *If $i \leq j$ then $N(j + i) - N(j) \geq (n-1)N(i)$.*

Proof. By induction on $i + j$. Assume that the lemma is true for all i', j' whose sum is less than $i + j$, and let $s < j$. By the induction hypothesis:

$$N(s+i) \geq \max(N(i) + (n-1)N(s), N(s) + (n-1)N(i)) \geq N(i) + N(s) \tag{3}$$

Let $j = 2^p + s$, where $s < 2^p$. Assume first that $j + i \leq 2^{p+1}$, and write $j + i = 2^p + t$, where $t \leq 2^p$. By part 4 of Lemma 4.4 (the part saying that N-distances beyond 2^p are $(n-1)$-magnified N-distances below 2^p) we have $N(j+i) - N(j) = (n-1)(N(t) - N(s))$. By (3), $N(t) - N(s) \geq N(t - s) = N(i)$, and thus $N(j+i) - N(j) \geq (n-1)N(i)$.

Assume next that $j + i > 2^{p+1}$ and write $j + i = 2^{p+1} + w$. Then $i = 2^p + w - s$.

By the induction hypothesis we have $N(2^p + w) - N(s) \geq N(i)$. By Lemma 4.5 $N(2^{p+1}) - N(2^p + s) = (n-1)(N(2^p) - N(s))$ and $N(2^{p+1} + w) - N(2^{p+1}) = (n-1)(N(2^p + w) - N(2^p))$. Adding the last two equalities gives $N(j+i) - N(j) = (n-1)(N(2^p + w) - N(s))$, and since by (3) $N(2^p + w) - N(s) \geq N(i)$, we are done.

$\square$

A converse inequality is also true, namely for every $k > 1$ it is true that:

$$N(k) = \max\{N(j) + (n-1)N(i) \mid j+i = k,\ i \leq j\} \tag{4}$$

Proof. Let p be maximal such that $2^p < k$, and let $k = 2^p + j$. By Lemma 4.1 (4) $N(k) = N(i) + (n-1)N(j)$. Combining this with Lemma 4.7 proves the desired equality. □

In [13] (4) was used as a defining recursion rule for the sequence $N(i)$ (which appeared there in a different context.)

For a number $t \leq n^r$ denote by $N^*(t)$ the number q such that $N(q-1) < t \leq N(q)$. This is an approximate inverse of N.

Theorem 4.8. *$b(t) = N(2^r - N^*(t))$ for every $t \leq n^r$.*

Proof. Let $F = F_r(\sigma(N^*(t))$. Then $|F| \geq t$, and since $B(F) = F_r(\bar{\sigma})$, we have $|B(F)| = N(2^r - N^*(t))$. This proves that $b(t) \geq N(2^r - N^*(t))$. To complete the proof we have to show that for every $F \subseteq [n]^r$ of size t we have $|B(F)| \leq N(2^r - N^*(t))$. Write $q = N^*(t)$. We wish to show that $|B(F)| \leq N(2^r - q)$. We do this by induction on r. The case $r = 1$ is easy, so assume that we know the result for $r-1$ and we wish to prove it for r.

Let $F^+ = \{e \setminus V_r \mid v_r \in e \in F\}$ and $F^- = \{e \setminus V_r \mid e \in F,\ v_r \notin e\}$.

By Lemma 2.2 we may assume that F is r-partitely shifted, which in particular entails $F^- \subseteq F^+$. Let $B^+ = B_{r-1}(F^+)$ and $B^- = B_{r-1}(F^-)$, and let $f^+ = |F^+|,\ f^- = |F^-|,\ b^+ = |B^+|,\ b^- = |B^-|$. Then $b^- \leq b^+$. Clearly:

$$B(F) = (B^- \times \{v_r\}) \cup (B^+ \times (V_r \setminus \{v_r\}))$$

and hence

$$|B(F)| = b^- + (n-1)b^+ \tag{5}$$

Let $i = N^*(f^-)$ and $j = N^*(f^+)$. Also let $i' = N^*(b^+),\ j' = N^*(b^-)$. By Lemma 4.7 we have:

$$|F| \leq f^+ + (n-1)f^- \leq N(i+j)$$

and hence $i + j \geq q$. By the inductive hypothesis $j' \leq 2^{r-1} - i$, and $i' \leq 2^{r-1} - j$, and hence $i' + j' \leq 2^r - (i+j) \leq 2^r - q$. By (5) and Lemma 4.7 , $|B(F)| \leq N(i' + j') \leq N(2^r - q)$, as desired.

□

Since $n^{r-1} = N(2^{r-1}) = 2^r - 2^{r-1}$, the case $k = 2$ of Conjecture 1.9 directly follows:

Corollary 4.9. *A pair F_1, F_2 of subsets of $[n]^r$ satisfying $|F_1| > n^{r-1}$ and $|F_2| \geq n^{r-1}$ has a rainbow matching.*

Alon [1] used spectral methods to prove, alongside Theorem 1.1, also an analogous result in the r-partite case, strengthening Theorem 1.11:

Theorem 4.10. *If $F_1, F_2 \subseteq [n]^r$ and $|F_1||F_2| > n^{2(r-1)}$ then the pair (F_1, F_2) has a rainbow matching.*

This follows also by the methods of the present paper.

Lemma 4.11. *$N(a)N(b) \leq N(ab)$.*

Proof. By induction on $a+b$. The case $a+b$ is trivial. By (4) $N(a) = N(c) + (n-1)N(d)$ for some $c \leq d < a$ such that $c + d = a$, and $N(b) = N(e) + (n-1)N(f)$ for some $e \leq f < b$ such that $e + f = b$. Then

$$N(a)N(b) = N(c)N(e) + (n-1)[N(d)N(e) + N(c)N(f)] + (n-1)^2 N(d)N(f)$$

Using the induction hypothesis, we get:

$$N(a)N(b) \leq N(ce) + (n-1)[N(d)N(e) + N(c)N(f)] + (n-1)^2 N(df)$$

Using Lemma 4.7 twice we get:

$$N(a)N(b) \leq N(ce+cf) + (n-1)N(de+df) \leq N(ce+cf+de+df) = N(ab).$$

$\square$

The lemma implies that $N(2^{r-1}-q)N(2^{r-1}+q) \leq N(2^{2(r-1)})$ for every $q \leq 2^{r-1}$, meaning that $tb(t) \leq n^{2(r-1)}$ for every $t \leq n^{r-1}$, which is another way of formulating Theorem 4.10.

5. A Hall-type size condition for rainbow matchings in bipartite graphs

In this section we prove a result on the existence of rainbow matchings in bipartite graphs, that will be later used for the proof of the case $r=3$ of Conjecture 1.9. This condition is not formulated in terms of sizes of individual hypergraphs, but as in Hall's theorem, in terms of sets of hypergraphs.

Theorem 5.1. *Let $F_i,\ i \leq k$ be subsets of $E(K_{n,n})$. If for every $I \subseteq [k]$ it is true that $\sum_{i\in I}|E_i| > n|I|(|I|-1)$ then the sets F_i have a rainbow matching.*

Remark 5.2. The analogous result for $r=1$ can be proved directly, or using Hall's theorem. For $r \geq 3$ the analogous result, namely that if $\sum_{i\in I}|E_i| > n^2|I|(|I|-1)$ for all I then $\mathcal{F}$ has a rainbow matching, is false. To see this, take the pair F_1, F_2 in which F_1 consists of a single edge and $F_2 = B(F_1)$.

5.1. An algorithm.

The proof of Theorem 5.1 is constructive, providing an algorithm for choosing a rainbow matching. For its description we shall use the following terminology. An edge $e=(m,w)$ in an ordered bipartite graph is said to be *left-starting* if at least one of its endpoints is first in its side. If this is the case, we choose one vertex from e that is first on its side, and denote it by $tail(e)$, and the other vertex in e is denoted by $head(e)$. The set of predecessors of $head(e)$ in its side, including $head(e)$ itself, is denoted by $SKIP(e)$. We write $\ell(e) = |SKIP(e)|$ and call it the *length* of e.

Enumerate the two sides of the bipartite graph as $M=(m_1,m_2,\ldots,m_n)$ and $W=(w_1,w_2,\ldots,w_n)$. By Lemma 2.2, we may assume that all F_i are bipartitely shifted with respect to these orders.

Order the sets F_i by their sizes,

$$|F_1| \leq |F_2| \leq \ldots \leq |F_k| \tag{6}$$

We now choose edges $e_i \in F_i$, taking at each step the longest edge available. The choice (apart from the first step, $i=1$) is done in two stages - first a "temporary" choice e'_i, that may violate the disjointness condition, and then the "real" choice e_i.

Let $e_1 = (m_{c_1}, w_{d_1})$ be the longest (i.e. having maximal ℓ value) left-starting edge in F_1. Assuming that $e_i = (m_{c_i}, w_{d_i}) \in F_i$ have been chosen for $i<t$, let $a(t)$ be the first index a such that $m_a \notin \{m_{c_i} \mid i<t\}$ and let $b(t)$ be the first index b for which $w_b \notin \{w_{d_j} \mid j<t\}$.

If one of $(m_{a(t)}, w_n)$ or $(w_{b(t)}, m_n)$ is an edge, choose one of them as e'_t. Otherwise, Let $e'_t = (m_{c_t}, w_{d_t})$ be a longest left starting edge in $\tilde{F}_t$. Let $tail(e'_t) = m_{a(j)}$ if $a(j) \in e'_t$ and $tail(e'_t) = w_{b(j)}$ otherwise.

Write $R_t = \{m_i : i < a(t)\} \cup \{w_j : j < b(t)\}$, and let $\tilde{F}_t = F_t - R_t$. Since F_t is shifted, every edge in $\tilde{F}_t$ is of the form (m_i, w_j) for $a(t) \leq i \leq a(t)+\ell_t,\ b(t) \leq j \leq b(t)+\ell(e'_t)$, and hence:

$$|\tilde{F}_t| \leq \ell(e'_t)^2 \tag{7}$$

It is possible that $head(e'_t)$ belongs to some $e_i,\ i<t$, so we choose e_t to be the longest edge e in F'_t having $tail(e'_t)$ as a vertex, which is disjoint from all $e_i,\ i<t$. Let $SKIP_t$ be the set of vertices in H_t "skipped" by e'_t, namely if $tail(e'_t) = m_{a(t)}$ then $SKIP_t = \{w_{b(t)}, w_{b(t)+1}, \ldots, w_{d_t}\}$, with a symmetrical definition if $head(e'_t) = m_{a(t)}$.

Write H_t for the side of the graph containing $head(e'_t)$ and T_t for the side containing $tail(e'_t) = tail(e_t)$.

Assume for contradiction that the process halts at some stage $p \leq k$, meaning that:

$$\tilde{F}_p = \emptyset \tag{8}$$

Let $a = a(p)$ and $b = b(p)$. The negation assumption (8), together with the shifting property, mean that $(m_a, w_b) \notin F_p$.

5.2. **Short edges.** An edge in the constructed matching having one vertex outside R_p will be called "long", and an edge contained in R_p will be called "short". If (u, v) is a long edge representing F_i, and (say) $u \in R_p$, then we have no information on $deg_{F_i}(u)$, apart from the obvious $deg_{F_i}(u) \leq n$. If, on the other hand, (u, v) is short, then (7) yields a bound on $deg_{F_i}(u)$ and $deg_{F_i}(v)$, which we shall use to get an upper bound on $\sum_{i \leq p} |F_i|$, towards the desired contradiction. Let $Q = \{q_1 < q_2 < \ldots < q_{m-1}\}$ be the list of indices q of short edges, namely satisfying $e_q < (m_a, w_b)$. Write $q_0 = 0$ and $q_m = p$.

Clearly:

$$|R_p| = p - 1 + |Q| = p - 2 + m \tag{9}$$

If $Q = \emptyset$ then by (9) we have $|R_p| = p - 1$. Since by (8) the set R_p is a cover for F_p, it follows that $|F_p| \leq (p-1)n$, implying that $\sum_{i \leq p} |F_i| \leq p(p-1)n$, contradicting the assumption of the theorem.

5.3. **A toy case - $|Q| = 1$.** To demonstrate the type of arguments involved in the general proof, let us consider separately the next simple case, $|Q| = 1$, in which there is only one index i for which $e_i = (m_{c_i}, w_{d_i}) < (a, b)$. Recall that either $c_i = a(i)$ or $d_i = b(i)$, and without loss of generality assume the first, namely $c_i = \min\{j \mid m_j \notin R_i\}$.

Write ℓ for $\ell(e_i)$. The edge e_i "skips" ℓ vertices in R_p, each being matched by some edge e_j, $i < j < p$, and hence $\ell \leq p - i$.

By (9) $|R_p| = p$, and since R_p is a cover for F_p it follows that $|F_p| \leq pn$. But in this calculation each of the ℓ edges (m_{c_i}, w_j) for $j = b(i), b(i)+1, \ldots, b(i)+\ell-1$, being contained in R_p, is counted twice, from the direction of m_{c_i} and from the direction of w_j. Thus we know that:

$$|F_p| \leq pn - \ell$$

Since no edge e_q, $q < i$, satisfies $e_q < e_i$, we have $|R_i| = i - 1$, and the number of edges in F_i incident with R_i is thus at most $(i-1)n$. By (7) we have: $|F_i| \leq (i-1)n + \ell^2$. By (6), it follows that:

$$\sum_{q \leq p} |F_q| \leq i|F_i| + (p-i)|F_p| \leq i((i-1)n + \ell^2) + (p-i)(pn - \ell).$$

Hence

$$p(p-1)n - \sum_{q \leq p} |F_q| \geq p(p-1)n - [i((i-1)n + \ell^2) + (p-i)(pn-\ell)] = (i-1)(p-i)n + (p-i)\ell - i\ell^2$$

$$= [(i-1)(p-i)n - (i-1)\ell^2] + [(p-i)\ell - \ell^2]$$

Since $\ell \leq p - i$ and $\ell \leq n$ both bracketed terms are non-negative, so $p(p-1)n - \sum_{q \leq p} |F_q| \geq 0$, reaching the desired contradiction.

5.4. **Using the short edges as landmarks.** Let us now turn to the proof of the general case. For $1 \leq j \leq m - 1$ write $s_j = q_j - q_{j-1}$ and let $S_j = \{q_{j-1}+1, q_{j-1}+2, \ldots, q_j\}$, so that $|S_j| = s_j$.

By (6) we have:

$$\sum_{i \leq p} |F_i| \leq \sum_{j \leq m} s_j |F_{q_j}| \tag{10}$$

We shall reach a contradiction by showing that this sum is not larger than $p(p-1)n$. For this purpose we shall use the fact that the edges e_{q_j} are short, yielding upper bounds on $|F_{q_j}|$.

5.5. **Three possible types of relationship between short edges.** Two short edges e_{q_i}, e_{q_j} $(i < j)$, may relate to each other in three different ways.

(1) The simplest case is in which $e_{q_i} \subseteq R_{q_j}$. In this case, of course, $e_{q_i} < e_{q_j}$, since all edges e contained in R_{q_j} satisfy $e_{q_i} < e_{q_j}$.
(2) $tail(e_{q_i}) \in T_{q_j}$ and $head(e'_{q_i}) \in SKIP_{q_j}$. In this case, again, $e_{q_i} < e_{q_j}$. We call e_{q_i} a "back edge" for e_{q_j}.
(3) The edges e_{q_i} and e_{q_j} may cross, meaning that $tail(e_{q_i}) \in R_{q_j} \cap H_{q_j}$ and $head(e_{q_i}) \in T_{q_j}$.

FIGURE 1. Relationship between short edges: Types 1 and 2

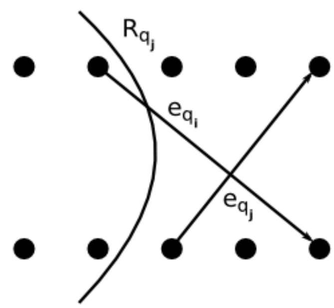

FIGURE 2. Relationship between short edges: Type 3

For $j \leq m$ let $r(j)$ be the smallest index r larger than j such that $e_{q_j} \subseteq R_{q_r}$. Thus, for $i < j$ the edges e_{q_i} and e_{q_j} bear a relationship of types 2 or 3 if $j < r(i)$.

Let $PROC_j$ be the set of indices $i < j$ satisfying 2 or 3. Namely:

$$PROC_j = \{i < m \mid i < j < r(i)\}$$

The letters $PROC$ stand for "procrastination", since $PROC_j$ consists of those indices i that in spite of being smaller than j, only one of the endpoints of e_{q_i} is in R_{q_j}. Let $BACK_j$ be the set of indices $i < j$ satisfying only 2, namely:

$$BACK_j = \{i < m \mid i < j < r(i) \text{ and } e_{q_i} < e_{q_j}\}.$$

Denote the set of back edges for q_j by $EBACK_j$, so:

$$EBACK_j = \{e_{q_i} : i \in BACK_j\}.$$

Define also:

$$\alpha_j = |BACK_j|$$

and

$$\lambda_j = \min(\ell_{q_j}^2, \ell_{q_j}(n-\alpha_j))$$

Since $|R_{q_j} \cap T_{q_j}| \geq \alpha_j$, we have $|R_{q_j} \setminus T_{q_j}| \leq n - \alpha_j$, implying $|\tilde{F}_{q_j}| \leq \ell_{q_j}(n-\alpha_j)$. Together with (7) this means:

$$|\tilde{F}_{q_j}| \leq \lambda_j \tag{11}$$

Similarly to (9), we have:

$$|R_{q_j}| = q_j - 1 + j - 1 = q_j + j - 2 - |PROC_j| \tag{12}$$

The subtraction of $|PROC_j|$ is due to the fact that for $i \in PROC_j$ the head of e_{q_i} does not belong to R_{q_j}. Writing $q_j = \sum_{i \leq j} s_i$ we can summarize:

$$|F_{q_j}| \leq (q_j + j - 2 - |PROC_j|)n + \lambda_j \tag{13}$$

But we shall do the bookkeeping a bit differently, distributing the $|PROC_j|n$ term between different stages.

5.6. **An overestimate and correction.** For $i \leq m$ write

$$Y_i = s_i(q_i + i - 2)$$

and:

$$Y = \sum_{i \leq m} Y_i = \sum_{i \leq m} s_i(q_i + i - 2) \tag{14}$$

As a first (over-) estimate to the sum $\sum_{j \leq m} s_j |F_{q_j}|$ we take the number $Yn + \sum_{j \leq m} s_j \lambda_j$, in which we assume that $|R_{q_i}| = q_i + i - 2 = (\sum_{j \leq i} s_j) + i - 2$ for each i. We shall refine this estimate in two ways:

(1) Take into account the "procrastinating" edges.
(2) Deduct the number of edges doubly counted, namely counted from both sides, in the expression Yn.

Let us denote the accumulating amount in the two types of corrections by Ω. In order to get the desired contradiction we have to show that

$$Yn + \sum_{j \leq m} s_j \lambda_j - \Omega \leq p(p-1)n \tag{15}$$

Let $p = \sum_{i \leq m} s_i$. Writing $Y = s_1(s_1 - 1) + s_2(s_1 + 1 + s_2 - 1) + s_3(s_1 + 1 + s_2 + 1 + s_3 - 1) \ldots + s_m(\sum_{i \leq m} s_i + m - 2)$, some algebraic manipulation yields:

$$p(p-1) - Y = \sum_{1 \leq i < j \leq m} (s_i s_j - s_j) \tag{16}$$

So, our aim is to show that

$$\sum_{1 \leq j \leq m} (s_j - 1)(\sum_{j < i \leq m} s_i)n - \sum_{j \leq m} s_j \lambda_j + \Omega \geq 0 \tag{17}$$

Which can be written as:

$$\sum_{1\le j\le m}(s_j-1)(n\sum_{j<i\le m}s_i-\lambda_j)-\sum_{j\le m}\lambda_j+\Omega\ge 0 \tag{18}$$

Instead we shall prove the stronger:

$$\sum_{1\le j\le m}(s_j-1)(n\sum_{j<i\le r(j)}s_i-\lambda_j)-\sum_{j\le m}\lambda_j+\Omega\ge 0 \tag{19}$$

(The difference is in the range of the second summation.)

We shall write Ω as a sum $\Omega=\sum_{j\le m}\omega_j$, where each ω_j (which is yet to be specified) is associated with a particular value of j, and prove the inequality separately for each j, namely:

$$(s_j-1)(n\sum_{j<i\le r(j)}s_i-\lambda_j)-\lambda_j+\omega_j\ge 0 \tag{20}$$

5.7. **Keeping track of the regains.** We focus on a particular index $j<m$, and collect regains associated with it. Let $r=r(j)$.

Let $\bigcup_{i=q_j}^{q_{r-1}} S_i$

The vertices of $SKIP_j$ are divided into three types:

(1) $A:=\{v\in SKIP_j \mid v\in e_k \text{ for some } q_j<k\le q_{r-1}\}$.
(2) $B:=\bigcup EBACK_j\cap H_{q_j}$, namely the set of vertices in H_{q_j} belonging to the edges in $EBACK_j$. Note that $A\cap B=\emptyset$.
(3) Vertices that are equal to $tail(e_k)$ for some $k\in S_r$.

With each type we shall associate a regain. In fact, some regains are not associated directly with these vertices: in the third type we shall consider all indices $k\in S_r$, not only those for which $tail(e_k)\in SKIP_j$.

(1) (regain on A, from procrastination):
The first type of regain associated with j is taken from the terms $Y_i,\ j\in PROC_i$.
In each term $Y_i=s_i(q_i+i-2)$ in the sum Y (see (14)) we regard each of the s_i indices k, $(q_{i-1}+1\le k\le q_i)$ as contributing (q_i+i-2) to Y_i. In this calculation, each edge e_k contributes $2n$ to Y_i, for the two vertices of e_{q_j}, while in fact only one, that of $tail(e_{q_j})$, is warranted. The contribution of $head(e'_{q_j})$ is not justified, because $j\in PROC_i$.
Thus for each index k belonging to some $S_i,\ j<i<r$, such that e_k meets $SKIP_{q_j}$, there is a regain of n. This regain we split: at the current stage we count ℓ_{q_j} as a regain, and we keep $n-\ell_{q_j}$, which we may (or may not, see below for specification) use in the i-th stage.
Note that $|A|\ge(\ell_{q_j}-1)-\alpha_j-(s_r-1)$ ($\ell_{q_j}-1$ because A does not include $head(e'_{q_j})$, and s_r-1 because S_r includes q_r, while e_{q_r} does not have a point in $SKIP_j$). Hence we regain this way at least

$$\beta_j:=(\ell_{q_j}-\alpha_j-s_r)\ell_{q_j}.$$

(2) (regain on B, from procrastination+double counting):
For pairs (i,t), where $i\in BACK_j$ and $t\in S_j\setminus\{q_j\}$ we shall consider regains of two types - procrastination and double counting. Treating them together will simplify the calculations. Let $K_i=SKIP_{q_i}\cap R_{q_j}$.

(a) Double counting: Each edge in $\{tail(e_{q_j})\}\times K_i$ was counted twice in the calculation of Y_in, and therefore we are entitled to a regain of $|K_i|$. Summing over all $t\in S_j\setminus\{q_j\}$, we have a regain of $(s_j-1)|K_i|$.

(b) Procrastination: For each $t\in S_j\setminus\{q_j\}$ there was an unwarranted contribution of n in the term q_j+j-2 appearing in Yn. Recall that for each such t we may have already used as a regain

ℓ_i (which happened if $tail(e_t) \in SKIP_i$). So we are now regaining at least $n - \ell_i$ for each such t.

(For the element q_j of S_j we haven't used ℓ_i, which is the reason that we consider it separately from the other elements of S_j.)

Combining (a) and (b), we get a regain of $n - \ell_{q_j} + |K_i|$ for each $t \in S_j \setminus \{q_j\}$.

Since $head(e'_{q_i}) \in SKIP_{q_j}$ we have $\ell_{q_i} \le |K_i| + \ell_{q_j}$, meaning that $n - \ell_{q_j} + |K_i| \ge n - \ell_{q_i}$.

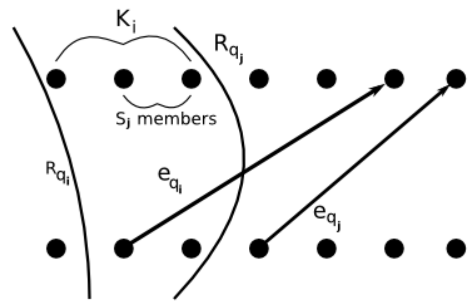

FIGURE 3. Illustration of second type of regain

The α_j indices i and $s_j - 1$ indices t thus contribute a $(s_j - 1)\alpha_j(n - \ell_j)$ regain. To this we add the regain from the index q_j, which is $\alpha_j n$, to get a regain of:

$$\gamma_j := (s_j - 1)\alpha_j(n - \ell_{q_j}) + n\alpha_j$$

(3) (S_r, double counting):
In the calculation of $Y_r = s_r(q_r + r - 2)$ all points of $SKIP_{q_j}$ were considered to be of degree n, and so was also the point $tail(e_{q_j})$, which resulted in double counting in Y all ℓ_{q_j} edges between $tail(e_{q_j})$ and $SKIP_{q_j}$. Since $|F_{q_r}|$ is multiplied in Y_r by s_r, we get a regain of

$$\delta_j := \ell_{q_j} s_r$$

Remark 5.3. In both (2) and (3) an edge doubly counted in a term Y_u is of the form (x, y), where $x = tail(e_{q_v})$ and $y \in SKIP_{q_v} \cap R_{q_u}$. In (2) $e_{q_v} \subseteq R_u$, and in (3) $e_{q_v} \not\subseteq R_u$. For this reason, we are not taking into account any double count more than once.

5.8. **Collecting all regains.** To finish the proof of the theorem note that $\sum_{t<i\le r(t)} s_i \ge \ell_{q_j} - \alpha_j$, since $SKIP_{q_j}$ consists of the α vertices of $H_{q_j} \cap \bigcup EBACK_j$ together with vertices from edges e_i, $t < i \le r(t)$. Hence

$$(s_j - 1)(n \sum_{t<i\le r(t)} s_i - \lambda_j) - \lambda_j + \omega_j \ge (s_j - 1)(n(\ell_{q_j} - \alpha_j) - \lambda_j) - \lambda_j + \beta_j + \gamma_j + \delta_j$$

$$= (s_j - 1)(n(\ell_{q_j} - \alpha_j) - \lambda_j) - \lambda_j + (\ell_{q_j} - \alpha - s_r)\ell_{q_j} + (s_j - 1)\alpha_j(n - \ell_{q_j}) + n\alpha_j + \ell_{q_j} s_r$$

$$= (s_j - 1)((n - \alpha_j)\ell_{q_j} - \lambda_j) - \lambda_j + (\ell_{q_j} - \alpha_j)\ell_{q_j} + n\alpha_j$$

The first term is nonnegative since $\lambda \le \ell_{q_j}(n - \alpha_j)$, and the sum of the other terms is nonnegative since $\lambda \le \ell_{q_j}^2$ and $\ell_{q_j} \le n$. This finishes the proof of the theorem.

6. Proof of Theorem 1.12

Let $\mathcal{F}$ be a collection of hypergraphs satisfying the condition of the theorem. Order the vertices of the first side V_1 as $v_1, \ldots, v_n$. By Lemma 2.2 we may assume that all F_i are shifted with respect to this order. Let i_1 be such that F_{i_1} has maximal degree at v_1 among all F_i's. Then we choose $i_2 \neq i_1$ for which F_{i_2} has maximal degree at v_2 among all F_i, $i \neq i_1$, and so forth. To save indices, reorder the F_i's so that $i_j = j$ for all j. Let H_j be the set of 2-edges incident with v_j in F_j. It clearly suffices to show that the collection $\mathcal{H} = (H_j : j \leq k)$ of subgraphs of $K_{n,n}$ has a rainbow matching, so it suffices to show that $\mathcal{H}$ satisfies the conditions of Theorem 5.1. Assuming it does not, $\sum_{k-t<j\leq k} |H_j| = \sum_{k-t<j\leq k} deg_{F_j}(v_j) \leq t(t-1)n$ for some $t < k$. We shall reach a contradiction to the assumption that $|F_k| > (k-1)n^2$.

Write m for $|H_k|$. Clearly

$$\sum_{j\leq k-t} deg_{F_k}(v_j) \leq (k-t)n^2$$

and by the order by which F_j were chosen

$$\sum_{k-t<j\leq k} deg_{F_k}(v_j) \leq \sum_{k-t<j\leq k} deg_{F_j}(v_j) \leq t(t-1)n$$

Since $\sum_{k-t<j\leq k} deg_{F_k}(v_j) \geq mt$, this implies that $m \leq n(t-1)$.

By the shifting property,

$$\sum_{k<j\leq n} deg_{F_k}(v_j) \leq m(n-k) \leq n(t-1)(n-k)$$

And so:

$$\sum_{j>k-t} deg_{F_k}(v_j) \leq t(t-1)n + (t-1)n(n-k) = n(t-1)(t+n-k) \leq (t-1)n^2$$

Hence

$$|F_k| = \sum_{j\leq k} deg_{F_k}(v_j) \leq (k-t)n^2 + (t-1)n^2 = (k-1)n^2$$

Which is the desired contradiction.

7. Conjecture 1.9 for large n

Theorem 7.1. *For every r and k there exists $n_0 = n_0(r,k)$ such that Conjecture 1.9 is true for all $n > n_0$.*

Proof. By Lemma 2.2 we may assume that all F_i's are shifted. Let A_i consist of the first $k-1$ vertices in V_i $(i \leq r)$, and let $A = \bigcup_{i\leq r} A_i$. Since the number of edges meeting A in two points or more is $O(n^{r-2})$, for large enough n for each i there exist at least $k-1$ points x in A such that $e \cap A = \{x\}$ for some $e \in F_i$. Hence we can choose edges $e_i \in F_i$ and distinct points $x_i \in A$ $(i \leq k-1)$ such that $e_i \cap A = \{x_i\}$. Since the number of edges going through $x_1, \ldots, x_{k-1}$ is no larger than $(k-1)n^{r-1}$, there exists an edge e_k in F_k missing $x_1, \ldots, x_{k-1}$. Using the shifting property, we can replace inductively each edge e_i , $i \leq k-1$, by an edge $e'_i \in F_i$ contained in A, missing e_k and missing all e'_j, $j < i$. This yields a rainbow matching for $F_1, \ldots, F_k$. $\square$

8. Further conjectures

Theorem 1.10 may be true also under the more general condition of degrees bounded by n.

Conjecture 8.1. *Let $d > 1$, and let $F_1, \ldots, F_k$ be bipartite graphs on the same ground set, satisfying $\Delta(F_i) \leq d$ and $|F_i| > (k-1)d$. Then the system $F_1, \ldots, F_k$ has a rainbow matching.*

For $d = 1$ this is false, since for every $k > 1$ there are matchings $F_1, \ldots, F_k$ of size k not having a rainbow matching.

Theorem 5.1 has a simpler counterpart, which we believe to be true:

Conjecture 8.2. *If $F_i,\ i \le k$ are subgraphs of $K_{n,n}$ satisfying $|F_i| \ge in$ for all $i \le k$, then they have a rainbow matching.*

We can prove this for $n \ge \binom{k}{2}$.

To formulate yet another conjecture we shall use the following notation:

Notation 8.3. (1) For a sequence $a = (a_i,\ 1 \le i \le k)$ of real numbers we denote by $\overrightarrow{a}$ the sequence rearranged in non-decreasing order.

(2) Given two sequences a and b of the same length k, we write $a \le b$ (respectively $a < b$) if $\overrightarrow{a}_i \le \overrightarrow{b}_i$ (respectively $\overrightarrow{a}_i < \overrightarrow{b}_i$) for all $i \le k$.

Given subgraphs $F_i,\ i \le k$ of $K_{n,n}$, define a $k \times n$ matrix $A = (a_{ij})$ as follows. Order one side of the bipartite graph as $v_1, v_2, \ldots, v_n$, and let $a_{ij} = deg_{F_i}(v_j)$. The i-th row sum $r_i(A)$ of A is then $|F_i|$. Thus, Theorem 5.1 can be formulated as follows:

Theorem 8.4. *If $\sum_{i \le j} \overrightarrow{r}_i > j(j-1)n$ for every $j \le k$ then there exists a permutation $\pi : [k] \to [k]$ such that $a_{i\pi(i)} \ge (1, 2, \ldots, k)$.*

We believe that the following stronger conjecture is true:

Conjecture 8.5. *If $\sum_{i \le j} \overrightarrow{r}_i > j(j-1)n$ for every $j \le k$ then there exists a permutation $\pi : [k] \to [k]$ such that $\sum_{i \le j} \overrightarrow{a}_{i\pi(i)} > j(j-1)$ for every j.*

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References

[1] N. Alon, *Private communication.*

[2] R. Aharoni and E.Berger, Rainbow matchings in hypergraphs *Electronic J. Combinatorics* (2010).

[3] R. Aharoni and P. Haxell, Hall's theorem for hypergraphs, *J. of Graph Theory* **35**(2000), 83–88.

[4] J. Akiyama and P. Frankl, On the Size of Graphs with Complete-Factors, *J. Graph Theory***9**(1),(1985)197-201.

[5] D.E Daykin, Erdős-Ko-Rado from Kruskal-Katona, *Jour. Combin. Th. Ser. A*, **17** (1974), 254–255.

[6] J. Dénes and A. D. Keedwell, Latin Squares and their Applications, Akadémiai Kiadó, Budapest, 1974.

[7] P. Erdős, A problem of independent r-tuples, *Ann. Univ. Budapest* **8** (1964), 93–95.

[8] P. Erdős and T. Gallai, On the minimal number of vertices representing the edges of a graph, *Publ . Math. Inst. Hungar . Acad. Sci.* 6(1961), 18, 1-203.

[9] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math Oxford* Ser. (2) 12(1961), 313-320.

[10] Z. Füredi, Matchings and covers in hypergraphs, *Graphs Combin.*, **4**(2) (1988), 115–206.

[11] P. E. Haxell, A condition for matchability in hypergraphs, *Graphs and Combinatorics* **11** (1995), 245–248.

[12] G. O. H. Katona, A simple proof of the Erds-Chao Ko-Rado theorem, *Jour. Combin. Th., Ser. B* **13** (1972), 183-184

[13] D.E. Knuth, a recurrence involving maxima, *American Mathematical Monthly* **114** (2007), 835; solution in **116** (2009), 649.

[14] M. Matsumoto, N. Tokushige, The exact bound in the Erdos-Ko-Rado theorem for cross-intersecting families, *J. Combin. Theory* Ser. A 52 (1989) 90-97.

[15] R. Meshulam, *Private communication.*

[16] H.J. Ryser, Neuere Probleme der Kombinatorik, *Vorträgeuber Kombinatorik, Oberwolfach, Matematisches Forschungsinstitute Oberwolfach, Germany, 24.29* July 1967, pp. 69.91.

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