The t-Discrepancy of a Poset

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ABSTRACT

Linear discrepancy and weak discrepancy have been studied as a measure of fairness in giving integer ranks to points of a poset. In linear discrepancy, the points are totally ordered, while in weak discrepancy, ties in rank are permitted. In this paper we study the t-discrepancy of a poset, a hybrid between linear and weak discrepancy, in which at most t points can receive the same rank. Interestingly, t-discrepancy is not a comparability invariant while both linear and weak discrepancy are. We show that computing the t-discrepancy of a poset is NP-complete in general but give a polynomial time algorithm for computing the t-discrepancy of a semiorder. We also find the t-discrepancy for posets that are the sum of chains and for the standard example of an n-dimensional poset.

1 Introduction

In this paper we consider only finite posets. We begin with some definitions and notation. We denote the cardinality of set S by $|S|$. A poset $P = (X, \prec)$ consists of a ground set X together with an order relation $\prec$. If there are several posets under consideration, we write $\prec_P$. When points $x, y \in X$ are incomparable we write $x \parallel_P y$ or just $x \parallel y$. If there are no incomparabilities then P is a *linear order* or *chain*. A *linear extension* L of a poset P is a linear order that respects the relation of P, that is, $x \prec_L y$ whenever $x \prec_P y$. The *height* of a point x in a linear order L, denoted by $h_L(x)$, is the greatest cardinality of a chain whose maximum point is x. The poset $\mathbf{r} + \mathbf{s}$ consists of the disjoint union of a chain of r points with a chain of s points, and more generally, the poset $\mathbf{r_1} + \mathbf{r_2} + \cdots + \mathbf{r_p}$ consists of the disjoint union of chains of cardinalities $r_1, r_2 \ldots, r_p$.

For all terminology and notation not defined here, we refer the reader to [12].

In this paper we explore assigning integer ranks to the points of a poset so that two fairness conditions are satisfied and at most t points get the same rank. The first fairness condition ensures that for a pair of comparable points, the higher point gets a higher rank. The second ensures that the ranks assigned to an incomparable pair of elements are not too far apart. Limiting the number of points so that at most t have the same rank can allow us to model situations where resources are limited. These ideas are made formal in the definitions below.

Definition 1 Let $P = (V, \prec)$ be a poset and t be a positive integer. An integer-valued function $f : V \to \mathbf{Z}$ is called a *(k,t)-labeling for P* if it satisfies the following three conditions for all $x, y, \in V$:
(i) if $x \prec y$ then $f(x) < f(y)$,
(ii) if $x \parallel y$ then $|f(x) - f(y)| \leq k$,
(iii) $|f^{-1}(i)| \leq t$ for all $i \in \mathbf{Z}$.

Definition 2 If k is the least integer for which poset P has a (k,t)-labeling, then we write $d_t(P) = k$ and say that P has *t-discrepancy* equal to k. A (k,t)-labeling f for which $k = d_t(P)$ is called a *t-optimal* labeling function (or just an *optimal* labeling function).

Definitions 1 and 2 can be combined to define the t-discrepancy directly as

$$d_t(P) = \min_f \max_{x \parallel y} |f(x) - f(y)|$$

where $f : V \to \mathbf{Z}$ satisfies (i) and (iii) above.

The above definitions are inspired by questions in which a ranking of points in a poset is required and it is desirable to choose one that minimizes the difference in rank between incomparable points. For example, a poset can represent a set of projects on a professor's desk, ordered by urgency (or perhaps importance). Suppose the professor can work on at most t projects at a time (condition iii) and wishes to rank them so that more urgent projects are done before less urgent ones (condition i). In addition, if two projects are incomparable, the professor would not want to complete them at widely different times (condition ii), for this could be viewed as unfair by the person awaiting the completion of the second project. Thus the professor seeks a t-optimal labeling function for this poset.

The t-discrepancy represents a hybrid between linear discrepancy and weak discrepancy as we next describe. When $t = 1$, condition (iii) of Definition 1 means that each rank can appear at most once in a (k,t)-labeling. In fact, as we will see in Lemma 6 with $m = 1$, a 1-optimal labeling function can always be viewed as a linear extension L of the poset in which the value $f(x)$ corresponds to the height of x in L. The resulting 1-discrepancy of P, introduced in [10], is called *linear discrepancy* and denoted by $ld(P)$. Linear discrepancy has been studied by other authors, for example, see [1], [2], [7], [9] and [8]. In the other extreme, if we allow $t = \infty$, condition (iii) is always satisfied. In this case, a (k, t)-labeling of P is called a *k-weak labeling of P* and corresponds to a weak extension of P. The resulting t-discrepancy is called the *weak discrepancy* of P and is denoted by $wd(P)$. Weak discrepancy was introduced in [5] and [11] and studied further in [10]. Additional examples of real world problems that motivated the definitions of linear and weak discrepancy appear in [10] and several of these have analogues for t-discrepancy.

The following remark follows from the definitions of linear discrepancy, t-discrepancy and weak discrepancy.

Remark 3 For any poset P and any integer $t \geq 2$ we have $ld(P) \geq d_t(P) \geq wd(P)$.

Given a particular (k, t)-labeling, it will sometimes be useful to refer to the maximum difference in the function values between pairs of incomparable points.

Definition 4 Let P be a poset and f a (k, t)-labeling function for P. The *t-discrepancy of P in f*, denoted by $d_t(P_f)$, is

$$d_t(P_f) = \max_{x \| y} |f(x) - f(y)|.$$

Thus $d_t(P) = \min_f d_t(P_f)$ where the minimum is taken over all functions $f : V \to \mathbf{Z}$ satisfying (i) and (iii) of Definition 1. For example, the $(2, 2)$-labeling f of $P = S_3$ shown in Figure 2 is optimal and $d_2(P_f) = d_2(P) = 2$. However, if we change the label 4 to a 5, the resulting labeling g has $d_2(P_g) = 3$.

In Section 4 we will need the following result from [10] about weak discrepancy.

Theorem 5 *If $P = \mathbf{r_1} + \mathbf{r_2} + \cdots + \mathbf{r_p}$ is the disjoint sum of p chains and $r_1 \geq r_2 \geq \cdots \geq r_p$ then $wd(P) = \lceil \frac{r_1 + r_2}{2} \rceil - 1$.*

2 Elementary Results

It is often convenient to have a (k, t)-labeling of poset P that satisfies one or both of the following properties.

(a) The minimum value of f is a specified integer m.

(b) There are no gaps in the set of integers that appear as function values.

We say that a (k, t)-labeling of P is *gap-free* if it satisfies property (b) and prove in Lemma 6 that every (k, t)-labeling of P can be transformed into a gap-free (k, t)-labeling of P with any specified minimum value.

Lemma 6 *Let m be any integer. If poset $P = (V, \prec)$ has a (k, t)-labeling then it has a gap-free (k, t)-labeling with minimum value m.*

Proof. Suppose f is a (k,t)-labeling of P. Adding a constant to every function value yields another (k,t)-labeling of P. Thus we may assume the minimum value of a (k,t)-labeling is m. Let M be the maximum value attained, and suppose there exists a gap in the labeling, that is, an integer i with $m < i < M$ so that $f(v) \neq i$ for all $v \in V$. Define a new function $g : V \to \mathbf{Z}$ with $g(v) = f(v)$ for all v with $f(v) < i$ and $g(v) = f(v) - 1$ for all v with $f(v) > i$.

Then $|g(x) - g(y)| \leq |f(x) - f(y)|$ for any pair of points $x, y \in V$ and it is easy to check that g is also (k,t)-labeling of P. Repeat this process if necessary until no gaps remain. $\Box$

Any gap-free (k,t)-labeling of a poset $P = (V, \prec)$ with minimum value $m = 1$ has maximum value $M \leq |V|$, and thus we have the following remark.

Remark 7 *For any poset $P = (V, \prec)$ and any integer $t \geq 1$ we have $d_t(P) \leq |V| - 1$.*

We collect several additional elementary results which we will need later. The next remark follows because any (k,t)-labeling of a poset induces a (k,t)-labeling on any subposet.

Remark 8 If P is an induced subposet of Q then $d_t(P) \leq d_t(Q)$.

Lemma 9 *If A is an antichain then $d_t(A) = \left\lceil \frac{|A|}{t} \right\rceil - 1$.*

Proof. The labeling function in which t points get label 1, t points get label 2, and so on is t-optimal. The largest label used is $\left\lceil \frac{|A|}{t} \right\rceil$, the smallest is 1, and thus $d_t(A) = \left\lceil \frac{|A|}{t} \right\rceil - 1$. $\Box$

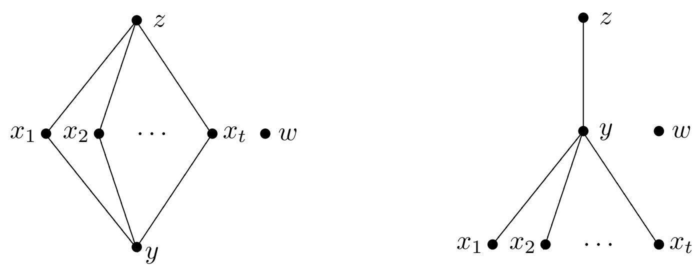
3 Comparability Invariance

The *comparability graph* of a poset $P = (V, \prec)$ is the graph $G = (V, E)$ where $xy \in E$ if and only if x and y are comparable in P. A parameter π defined for posets is said to be a *comparability invariant* if for all posets P and Q, we have $\pi(P) = \pi(Q)$ whenever the comparablity graphs of P and Q are isomorphic. Some well-known poset parameters, such as dimension, are known to be comparability invariants (see [12]). Weak discrepancy is shown to be a comparability invariant in [5] and linear discrepancy is shown to be a comparability invariant in [10]. The latter also follows from the main result in [3] that all posets P satisfy

$$ld(P) = bandwidth(\overline{G}) \tag{1}$$

where $\overline{G}$ is the incomparabilty graph of P, that is, the complement of the comparablity graph of P.

Surprisingly, for all integers $t > 1$, t-discrepancy is *not* a comparability invariant even though we think of t-discrepancy as lying between linear and weak discrepancy. This is proven below in Theorem 10.



P Q

Figure 1: Two posets with the same comparability graph but with different t-discrepancy.

Theorem 10 *For any integer $t > 1$ there exist posets P and Q that have the same comparability graph, but for which $d_t(P) \neq d_t(Q)$. Thus t-discrepancy is not a comparability invariant.*

Proof. Fix an integer $t > 1$ and consider the posets P and Q shown in Figure 1. It is easy to check that P and Q have the same set of comparabilities, thus they have the same comparability graph. We next show $d_t(Q) = 1$ and $d_t(P) = 2$.

Since Q has an antichain of size $t+1$, Remark 8 and Lemma 9 imply that $d_t(Q) \geq 1$. The function f defined by $f(x_i) = 1$ for $1 \leq i \leq t$, $f(y) = f(w) = 2$, $f(z) = 3$ is a $(1,t)$-labeling for Q, thus $d_t(Q) \leq 1$. Together these imply that $d_t(Q) = 1$.

Next we show $d_t(P) > 1$. Suppose, for a contradiction, that $d_t(P) \leq 1$ and let f be a $(1,t)$-labeling for P. Without loss of generality, we may assume $f(y) = 0$, and since $w \parallel y$ we have $f(w) \leq 1$ by (ii) of Definition 1.

If there exist $1 \leq i, j \leq t$ with $f(x_i) < f(x_j)$ then since $y \prec x_i$ and $x_j \prec z$, using Definition 1 we have $f(x_i) \geq 1$, $f(x_j) \geq 2$ and $f(z) \geq 3$. However, $w \parallel z$ and $|f(z) - f(w)| \geq f(z) - f(w) \geq 3 - 1 = 2$ which contradicts (ii) of Definition 1. Otherwise, $f(x_i) = c$ for all $1 \leq i \leq t$, where $c \geq 1$ because $y \prec x_i$. Since $y \prec x_i \prec z$ and $f(y) = 0$, we know $f(z) \geq 2$ by (i) of Definition 1. If $f(w) \leq 0$ then $|f(z) - f(w)| \geq f(z) - f(w) \geq 2 - 0 = 2$ violating (ii) of Definition 1. If $f(w) = 1$ then $c \geq 2$ and $f(z) \geq 3$, so $|f(z) - f(w)| \geq f(z) - f(w) \geq 3 - 1 = 2$ again violating (ii) of Definition 1. Thus $d_t(P) > 1$. Indeed, the function f defined by $f(y) = 0$, $f(x_i) = 1$ for $1 \leq i \leq t$, $f(w) = 2$, and $f(z) = 3$ is a $(2,t)$-labeling function, thus $d_t(P) = 2$. $\square$

As a consequence of Theorem 10, we know there is no result analogous to equation (1) that relates the t-discrepancy of a poset to a parameter of its incomparability graph.

Figure 2: The posets S_3 and S_4 together with 2-optimal labelings of them.

4 Special Classes of Posets

In this section we find $d_t(P)$ for special classes of posets, in particular the standard example of a poset of dimension n and the disjoint sum of chains.

Let $n \geq 3$ be an integer. The poset $S_n = (X, \prec)$ is called the *standard example* of a poset of dimension n. It has as its ground set $X = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$, and the only comparabilities are $a_i \prec b_j$ for $i \neq j$. Figure 2 shows the posets S_3 and S_4 together with 2-optimal labelings of them. According to Theorem 11, we have $d_2(S_3) = 2$ and $d_2(S_4) = 2$.

Theorem 11 *Let S_n be the standard example of a poset of dimension n and let t be an integer $t \geq 2$, then $d_t(S_n) = \lceil n/t \rceil$.*

Proof. Let S_n have ground set $\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n\}$ where the a_i are minimal, the b_i are maximal, $a_i \parallel b_i$ for each i and $a_i \prec b_j$ for each $i \neq j$. Write $n = qt + a$ where q is an integer and $0 < a \leq t$. Thus we seek to show $d_t(S_n) = q + 1$.

First we construct a $(q+1, t)$-labeling f of S_n, showing $d_t(S_n) \leq q + 1$. Label the minimal elements using the labels $1, 2, 3, \ldots, q + 1$. There will be sufficient labels since $n \leq (q + 1)t$. Let $f(b_i) = f(a_i) + q + 1$. This function f is a $(q + 1, t)$-labeling of S_n, thus $d_t(S_n) \leq q + 1 = \lceil n/t \rceil$.

For the reverse inequality, let f be a t-optimal labeling of S_n. By Lemma 9. the antichain A of minimal elements has $d_t(A) = \lceil n/t \rceil - 1 = q$. If $d_t(A_f) \geq q+1$ then $d_t(S_n) \geq q + 1$ and we are done. So instead we may assume $d_t(A_f) = q$. The antichain A of minimal elements requires at least $q + 1$ labels and they must be consecutive to achieve $d_t(A_f) = q$, so without loss of generality we may assume $f(a_1) = 1$ and $f(a_n) = q + 1$. Since $a_n \prec b_1$ we must have $f(b_1) \geq q + 2$ and then $|f(b_1) - f(a_1)| \geq q + 1$. Since f was assumed to be t-optimal, we know $d_t(S_n) \geq q + 1$. $\square$

Next we consider the t-discrepancy of the disjoint sum of chains. Figure 3 illustrates three examples. In each, we think of the points as fitting on a rectangular grid where the height (i.e., y-coordinate) of a point is the value of the label assigned to it. The width of the grid is t since each label can occur at most t times. The grid must be tall enough to accommodate both the biggest chain in the poset and also the total number of points in the poset. Whichever of these factors is more limiting determines the part of the formula for $d_t(P)$ in Theorem 12 that applies.

Theorem 12 *Let* $P = \mathbf{r_1} + \mathbf{r_2} + \cdots + \mathbf{r_p}$ *where* $r_1 \geq r_2 \geq \cdots \geq r_p$*, let* $P' = \mathbf{r_2} + \mathbf{r_3} + \cdots + \mathbf{r_p}$ *and let* $t \geq 2$ *be an integer. Furthermore, let* $s = r_1 + r_2 + \cdots + r_p$*, let* $s' = r_2 + \cdots + r_p$*, let* $q = \lceil \frac{s}{t} \rceil$*, let* $q' = \left\lceil \frac{s'}{t-1} \right\rceil$ *and let* $M = \max\{r_2, q'\}$*. Then*

$$d_t(P) = \begin{cases} q-1, & \text{if } q > r_1 \\ \\ \lceil (r_1 + M)/2 \rceil - 1, & \text{if } q \leq r_1. \end{cases}$$

Before presenting the proof of Theorem 12, we illustrate the upper bound by giving a (k,t)-labeling for three examples.

Example 13 For all three posets in this example, we use $t = 3$, and thus the grids of points each have three columns.

For poset $P_1 = \mathbf{4} + \mathbf{4} + \mathbf{3} + \mathbf{3} + \mathbf{3}$ we have $s = 17$, $s' = 13$, $q = 6$, $q' = 7$, and $M = 7$, which falls in case 1 in the proof of Theorem 12. Here the height of the grid is determined by $q = 6 > 4 = r_1$. The labels assigned to $\mathbf{r_1}$ are $1, 2, 3, 4$; to $\mathbf{r_2}$ are $5, 6, 1, 2$; to $\mathbf{r_3}$ are $3, 4, 5$; to $\mathbf{r_4}$ are $6, 1, 2$; and to $\mathbf{r_5}$ are $3, 4, 5$ as illustrated in Figure 3. Thus $d_t(P_1) \leq 5$.

For poset $P_2 = \mathbf{5} + \mathbf{4} + \mathbf{2} + \mathbf{2}$ we have $s = 13$, $s' = 8$, $q = 5$, $q' = 4$, and $M = 4$, which falls in case 2a in the proof of Theorem 12. The height of the grid is determined by $r_1 = 5 \geq 5 = q$. The labels assigned to $\mathbf{r_1}$ are $1, 2, 3, 4, 5$; to $\mathbf{r_2}$ are $1, 2, 3, 4$; to $\mathbf{r_3}$ are $5, 1$; to $\mathbf{r_4}$ are $2, 3$ as illustrated in Figure 3. Thus $d_t(P_2) \leq 4$.

For poset $P_3 = \mathbf{5} + \mathbf{2} + \mathbf{2} + \mathbf{2}$ we have $s = 11$, $s' = 6$, $q = 4$, $q' = 3$, and $M = 3$, which falls in case 2b in the proof of Theorem 12. The height of the grid is determined by $r_1 = 5 \geq 4 = q$. The labels assigned to $\mathbf{r_1}$ are $1, 2, 3, 4, 5$; to $\mathbf{r_2}$ are $2, 3$; to $\mathbf{r_3}$ are $4, 2$; to $\mathbf{r_4}$ are $3, 4$ as illustrated in Figure 3. Thus $d_t(P_3) \leq 3$.

Proof of Theorem 12. We consider two cases depending on whether the range of labels needed for P will be determined by the size of the largest chain or by the total number of points in P.

Case 1: $q > r_1$.

First we show the upper bound $d_t(P) \leq q - 1$. Form a sequence of qt labels consisting of the sequence $1, 2, 3, \ldots, q$ repeated t times. Assign the first r_1 numbers in the sequence to be labels for the points in $\mathbf{r_1}$, the next r_2 numbers in the sequence to be labels for the points in $\mathbf{r_2}$, etc. In assigning labels to the points in $\mathbf{r_i}$ follow rule (i) of Definition 1. This is illustrated in the labeling of poset P_1 in Figure 3 as detailed in Example 13. By the definition of $q = \lceil s/t \rceil$, there are sufficient labels. In this case, $r_i \leq r_1 < q$ for each i, so each chain $\mathbf{r_i}$ is assigned r_i distinct labels, so the labeling is valid. The largest possible difference in label is $q - 1$, so $d_t(P) \leq q - 1$.

Next we show the lower bound $d_t(P) \geq q - 1$. For a contradiction, assume $d_t(P) \leq q - 2$ and using Lemma 6, let f be a gap-free, t-optimal labeling of P with minimum value $m = 1$. If $f(x) \leq q - 1$ for all points x in P, then there

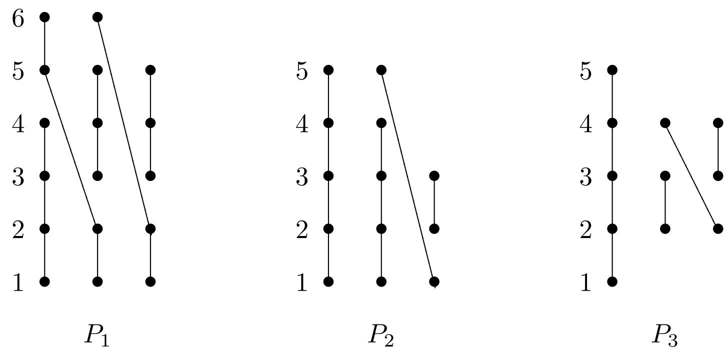


Figure 3: Optimal labeling functions for posets P_1, P_2, P_3 when $t = 3$. In each, the height of a point is the value of its label.

are at most $q-1$ labels, each appearing at most t times, for a total of at most $(q-1)t < s$ labels available. Thus there are not enough labels for all the points in P. Hence there must be a point of P with label at least q. For each label $\ell \geq q$ that appears, we can only have one point labeled $\ell - (q-1)$, because two such points would be incomparable to each other and hence one of them would be incomparable to the point labeled ℓ, contradicting $d_t(P) \leq q-2$. So for each point with label q or bigger we lose $t-1 \geq 1$ potential labels for points. Thus there will not be sufficient labels to label the s points of P, a contradiction.

Case 2: $q \leq r_1$.

We begin by showing that

$$M \leq r_1. \tag{2}$$

Since $M = \max\{r_2, q'\}$ and we already know $r_2 \leq r_1$, it suffices to show $q' \leq r_1$. We know $r_1 \geq q = \lceil s/t \rceil \geq s/t$ and thus $r_1 t \geq s = s' + r_1$. Subtracting r_1 from both sides yields $r_1(t-1) \geq s'$ or equivalently $r_1 \geq s'/(t-1)$. Since r_1 is an integer we have, $r_1 \geq \lceil s'/(t-1) \rceil = q'$ as desired.

Next we establish the upper bound $d_t(P) \leq \lceil (r_1+M)/2 \rceil - 1$. We give a labeling of P as follows. Label the chain $\mathbf{r_1}$ using labels $1, 2, 3, \ldots, r_1$. For the s' elements in the other chains, make $t-1$ copies of the sequence

$$\lceil (r_1+M)/2 \rceil - M + 1, \lceil (r_1+M)/2 \rceil - M + 2, \ldots, \lceil (r_1+M)/2 \rceil$$

for a total of $M(t-1)$ labels. Since $s' \leq q'(t-1) \leq M(t-1)$ we have sufficient labels. As before, assign the first r_2 elements of this sequence to the chain $\mathbf{r_2}$, the next r_3 elements to be the labels for $\mathbf{r_3}$, etc. This is illustrated in the labeling of posets P_2 and P_3 of Figure 3 as detailed in Example 13. For P_2 we have $M = r_2 = 4$ and for P_3 we have $M = q' = 3$. Since $r_i \leq r_2$ for each $i \geq 2$, each sequence $\mathbf{r_i}$ is assigned r_i distinct labels, so the labeling is valid. Any two elements of $\mathbf{r_2} + \mathbf{r_3} + \cdots + \mathbf{r_p}$ have labels that differ by at most $M-1$, and using equation (2) we have $M - 1 \leq \lceil (r_1+M)/2 \rceil - 1$. The largest difference in label between a point in $\mathbf{r_1}$ and a point in $\mathbf{r_2} + \mathbf{r_3} + \cdots + \mathbf{r_p}$ will occur between the highest label in one and the lowest in the other, thus will be either $\lceil (r_1 + M)/2 \rceil - 1$ or $r_1 - (\lceil (r_1 + M)/2 \rceil - M + 1) \leq r_1 - (r_1 + M)/2 + M - 1 \leq \lceil (r_1 + M)/2 \rceil - 1$. Hence $d_t(P) \leq \lceil (r_1 + M)/2 \rceil - 1$ as desired.

For the lower bound $d_t(P) \geq \lceil (r_1 + M)/2 \rceil - 1$, we consider two subcases depending on whether $M = r_2$ or $M = q'$.

Subcase 2a: $q' \leq r_2 = M$.

In this instance we use Remark 3 and Theorem 5 to conclude

$$d_t(P) \geq wd(P) = \lceil (r_1 + r_2)/2 \rceil - 1 = \lceil (r_1 + M)/2 \rceil - 1.$$

Subcase 2b: $r_2 < q' = M$.

We have already shown the upper bound $d_t(P) \leq \lceil (r_1 + M)/2 \rceil - 1$. Combining this with equation (2) yields

$$d_t(P) \leq r_1 - 1. \tag{3}$$

Let the points of the chain $\mathbf{r_1}$ be $b_1 \prec b_2 \prec \cdots \prec b_{r_1}$. Let f be an t-optimal labeling of P with $f(b_1) = 1$ and $h = f(b_{r_1})$ as small as possible.

Claim: $h = r_1$.

We know $h \geq r_1$ to accomodate the r_1 points of the chain, so for a contradiction, assume $h \geq r_1 + 1$. Since every point in P' is incomparable to both b_1 and b_{r_1}, by equation (3), the labels 1 and h can not appear on points in P'. We will apply the following algorithm to point x with label $f(x) = c$. Initially, let $x = b_{r_1}$, thus $f(x) = c = h$. Lower x's label by 1, that is, set $f(x) := c - 1$. Since we wish the resulting labeling to be t-optimal, three potential problems could arise, (i) a comparability problem − there is a point w with $w \prec x$ and $f(w) = c - 1$, (ii) an overcrowding problem − there are already t points with label $c - 1$, and (iii) an incomparability problem − there is a point z with $x \parallel z$ and $f(z) - c = d_t(P)$. We will show below that (iii) never occurs. In case (i), there can only be one such w since P is the sum of chains, and we then apply the algorithm to w. Since w's label will be lowered by 1, this also resolves any overcrowding problem at label $c - 1$ which may arise simultaneously. If there is no comparability problem, but there is an overcrowding problem, we find another point y in P' with $f(y) = c - 1$ and apply the algorithm to y. Note that such a y will exist since $t \geq 2$. When no problems occur, the algorithm terminates.

Next we describe how the algorithm progresses and show it will terminate with all labels between 1 and $h-1$. The algorithm will stop at or before reaching a point with label 1, since we've already shown that there is exactly one point (b_1) with label 1 in P. By our assumption that $f(b_1) = 1$ and $f(b_{r_1}) = h \geq r_1+1$, we know the labels of points in $\mathbf{r_1}$ are not consecutive and hence there are one or more gaps. The algorithm starts at b_{r_1} and continues considering points down $\mathbf{r_1}$ resolving comparability problems (i) until the first gap in labels is reached. If there is no overcrowding problem, the algorithm terminates. If there is an overcrowding problem, then from this point on, the algorithm is only applied to points in P', each of which has label at least 2. Either the algorithm stops before reaching a point in P' with label 2 or if one is reached, its label can be lowered to 1 without causing any problems since b_1 is the only point with label 1 and no points have label 0.

Finally, we show that an incomparability problem never occurs. It can not occur when considering x in $\mathbf{r_1}$ since such an x has $f(x) \geq 2$ and has the same incomparabilities as b_1 with $f(b_1) = 1$. Likewise, it will not occur when considering x in P'. Any $x \in P'$ is incomparable to b_{r_1}, and so $f(b_{r_1}) - f(x) = h - f(x) \leq d_t(P)$. Initially, b_{r_1} is the only point in P with label h, and after the first pass of the algorithm, all points in P have labels at most $h-1$. Thus for any point z with $x \parallel z$ we have $f(z) - f(x) \leq h - 1 - f(x) \leq d_t(P) - 1$ and so an incomparability problem never occurs for $x \in P'$.

When the algorithm terminates, none of the potential problems (i), (ii), (iii) occur and thus the resulting labeling is still t-optimal. However, we have contradicted the minimality of h. This justifies our claim that $h = r_1$.

Now we know there exists a t-optimal labeling of P in which the points in the chain $\mathbf{r_1}$ are labeled $1, 2, 3, \ldots, r_1$. Let m_2 be the largest label that appears in P' and m_1 be the minimum such label. We know $1 \leq m_1$ and $m_2 \leq r_1$ by equation (3). We also know $q' - 1 \leq m_2 - m_1$ in order to have enough labels to accommodate the points in P'. Thus $d_t(P) = \max\{r_1 - m_1, m_2 - 1\}$. If these two quantities differ by 2 or more, we could add one to each label in P' (if the first is larger) or subtract one from each label in P' (if the second is larger) to get a smaller value of $d_t(P)$. Thus $|(r_1 - m_1) - (m_2 - 1)| \leq 1$ and

$$d_t(P) = \max\{r_1 - m_1, m_2 - 1\} = \left\lceil \frac{(r_1 - m_1) + (m_2 - 1)}{2} \right\rceil \geq \left\lceil \frac{r_1 + q' - 2}{2} \right\rceil$$

and so $d_t(P) \geq \left\lceil \frac{r_1 + q'}{2} \right\rceil - 1$ as desired. □

5 Computing $d_t(P)$ is NP-Complete

In this section we show that the problem of deciding whether a poset $P = (V, \prec)$ has a (k,t)-labeling is NP-complete. We accomplish this by constructing its *t-duplicated poset* $P' = (V', \prec')$ as follows. Let V' consist of t points $v_1, v_2, \ldots, v_t$ for each $v \in V$. For each $x, y \in V$ and each $i, j \in \{1, 2, \ldots, t\}$ we have $x_i \prec' y_j$ if and only if $x \prec y$ in P. Thus each point of P is replaced by an antichain of t points in P'. We call this antichain the *cluster* corresponding to the point $v \in V$. Figure 4 shows a poset P and its 3-duplicated poset P'.

Suppose g is a (k,t)-labeling function for the *t-duplicated poset* poset P'. If C_v is the cluster of points in P' corresponding to $v \in V$ we define $\min(C_v) = \min\{g(v_i) : v_i \in C_v\}$ and $\max(C_v) = \max\{g(v_i) : v_i \in C_v\}$. A cluster C_v is *uniform* if $\max(C_v) = \min(C_v)$.

The next remark follows because two points in a cluster together have the same comparabilities and incomparabilities.

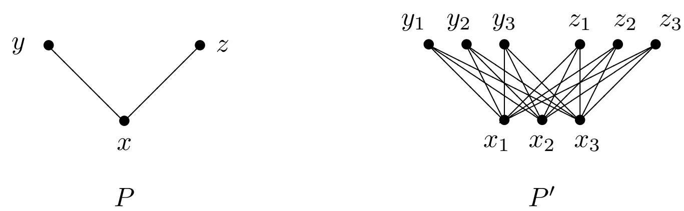


Figure 4: A poset P and its 3-duplicated poset P'.

Remark 14 *Let g be a (k,t)-labeling of the* t-duplicated poset P' *in which points v_i and v_j are in the same cluster. If there are fewer than t points with label $g(v_j)$ then setting the value of $g(v_i)$ to equal that of $g(v_j)$ also results in a (k,t)-labeling of P'.*

Theorem 15 *Let $P = (V, \prec)$ be a poset and t, k be positive integers. The decision problem $d_t(P) \leq k$ is NP-complete.*

Proof. Construct the t-duplicated poset P' from P. We will show that $d_t(P') = ld(P)$. The result in Theorem 15 then follows since the decision problem $ld(P) \leq k$ is NP-complete [3] and constructing P' from P can be accomplished in polynomial time.

Recall from Definition 4 that $d_t(P_f)$ measures the maximum difference in function values between pairs of incomparable points for the (k,t)-labeling f. In the case $t = 1$, t-discrepancy is linear discrepancy and we denote $d_1(P_f)$ by $ld(P_f)$.

First we show $d_t(P') \leq ld(P)$. Let $k = ld(P)$ and take an optimal 1-labeling of P. We obtain a (k,t)-labeling g of P' by setting $g(x_i) = f(x)$ for each $x \in V$ and $i = 1, 2, \ldots, t$.

Next we show the reverse inequality $ld(P) \leq d_t(P')$. If there exists a (k,t)-labeling g of P' in which all clusters are uniform, we immediately obtain a 1-labeling f of P, namely $f(v) = g(v_1)$ for each $v \in V$, with $d_t(P'_g) = ld(P_f)$. When g is t-optimal, we have $ld(P) \leq ld(P_f) = d_t(P'_g) = d_t(P')$ as desired.

Otherwise, let j be the maximum so that P' has an t-optimal labeling g in which the points labeled i are in a cluster together for $i = 1, 2, 3, \ldots, j-1$. By our assumption, $j \leq |V|$. We will show that we can swap some labels to arrive at a (k,t)-labeling g' of P' so that points labeled i are in a cluster together for $i = 1, 2, 3, \ldots, j$, contradicting the maximality of j.

By Remark 14, we can make the clusters containing the points with labels less than j into uniform clusters and still have a t-optimal labeling. Since all points with labels less than j are in uniform clusters, these clusters include all such points. Thus any point with label j is in a cluster with other points whose labels are at least j. If, in fact, the points labeled j are now all in a cluster together, we violate the maximality of j. Thus we may assume there are at least two clusters containing points with label j. Among all such clusters choose one

C_v for which $\max(C_v)$ is largest and another, C_w for which $\max(C_w)$ is smallest. If there are r points in C_v with label j then at most $t-r$ points in C_w have label j, and thus at least r points in C_w have a label greater than j. For each point in C_v with label j, switch its label with that of a point in C_w whose label is greater than j. Note that this new labeling of P' is still t-optimal since all points in a cluster have the same set of incomparabilities. Yet the new labeling has one fewer cluster containing a point with label j. Continue this process until all the points labeled j are in the same cluster. The resulting labeling function is t-optimal, contradicting the maximality of j. $\square$

6 Semiorders

There are several equivalent definitions of a semiorder. One involves forbidden posets: P is a semiorder if and only if it does not contain a $\mathbf{2}+\mathbf{2}$ or a $\mathbf{3}+\mathbf{1}$ as a subposet. Alternatively, semiorders are also known as unit interval orders: $P=(V,\prec)$ is a semiorder if we can assign a unit interval $I(v)$ in the real line to each $v \in V$ so that $x \prec y$ in P precisely when $I(x)$ is completely to the left of $I(y)$. Such unit interval representations can always be found so that the interval endpoints are distinct (see, for example, [6]). In what follows, we will always choose interval representations with distinct endpoints.

Since Theorem 15 shows that computing $d_t(P)$ is NP-complete in general, we seek special classes of posets for which $d_t(P)$ can be computed in polynomial time. A natural class to consider are the semiorders since both the linear discrepancy and the weak discrepancy can be computed efficiently for semiorders (see Tanenbaum, Trenk and Fishburn [10]).

Theorem 16 ([10]) *Let P be a semiorder, Then*
(a) $ld(P) = width(P) - 1$ *and*
(b) $wd(P) \leq 1$.
In particular, $wd(P)=1$ if P contains a $\mathbf{2}+\mathbf{1}$ and $wd(P)=0$ otherwise.

The following result gives bounds on the t-discrepancy of a semiorder $P=(V,\prec)$. The upper and lower bounds differ by less than two, so Theorem 17 restricts the value of $d_t(P)$ to at most two integers. Note that if $t=1$, then $d_t(P) = ld(P)$ and indeed Theorem 17 reduces to Theorem 16(a). Similarly, if $t \geq |V|$, then $d_t(P) = wd(P)$ and in this case Theorem 17 reduces to Theorem 16(b).

Theorem 17 *If $P=(V,\prec)$ is a semiorder and t is a positive integer, then*

$$\left\lceil \frac{width(P)}{t} - 1 \right\rceil \leq d_t(P) \leq \left\lfloor \frac{width(P)}{t} + 1 - \frac{2}{t} \right\rfloor.$$

Proof. First we establish the lower bound. Let A be an antichain in P of size $width(P)$, thus $|A| = width(P)$. Using Remark 8 and Lemma 9 we have $d_t(P) \geq d_t(A) = \left\lceil \frac{|A|}{t} \right\rceil - 1 = \left\lceil \frac{|width(P)|}{t} \right\rceil - 1 = \left\lceil \frac{|width(P)|}{t} - 1 \right\rceil$.

Next we establish the upper bound. If P is a chain then $width(P) = 1$ and $d_t(P) = 0$ so the result holds. Otherwise, P has at least one pair of incomparable elements. We label the elements of V according to the following greedy algorithm.

Greedy Algorithm for labeling semiorders:

Fix a unit interval representation of P in which endpoints are distinct. Consider the elements of $V = \{x_1, x_2, \dots, x_n\}$ indexed by their left endpoint (and therefore also by their right endpoint) in this representation. Initialize: Let $f(x_1) = 0$.
Iterate for $i = 1, 2, \dots, n-1$: Assume that $x_1, x_2, \dots, x_i$ have been labeled and let $j = f(x_i)$. If there are fewer than t elements labeled j and x_{i+1} is incomparable to all of them, then let $f(x_{i+1}) = j$. Otherwise, let $f(x_{i+1}) = j+1$.

Example 18 shows the greedy algorithm applied to the representation of the semiorder given in Figure 5.

Note that by construction, the function f satisfies (i) and (iii) of Definition 1. Let k be the largest value for which there exists $x, y \in V$ with $x \parallel y$ and $|f(x) - f(y)| = k$. Then by construction, the function f is a (k,t)-labeling of P and hence $d_t(P) \leq k$. Choose integers r, s with $1 \leq r < s \leq n$ so that $x_r \parallel x_s$ and $f(x_s) - f(x_r) = k$. Thus the intervals assigned to x_r and x_s intersect in an interval we call $\mathcal{I}$. Since our representation of P is a unit interval representation with points indexed by left endpoints, the intervals assigned to $x_r, x_{r+1}, \dots, x_s$ all intersect the interval $\mathcal{I}$ and thus the points $x_r, x_{r+1}, \dots x_s$ form an antichain A.

By the definition of f, we know there are t points in A that received the label $f(x_r) + i$ for $i = 1, 2, \dots, k-1$ and two additional points, x_r and x_s in A. Thus $width(P) \geq |A| \geq t(k-1) + 2 \geq t(d_t(P) - 1) + 2$. Isolating the term $d_t(P)$ yields the inequality $d_t(P) \leq \frac{width(P)}{t} + 1 - \frac{2}{t}$, and because $d_t(P)$ is an integer, we may take the floor of the right hand side to achieve the desired inequality.$\square$

Example 18 Let $t = 2$ and consider the semiorder P and its representation from Figure 5. The semiorder P has $width(P) = 6$ and Theorem 17 gives the inequalities $2 \leq d_2(P) \leq 3$. The greedy algorithm assigns the labeling $f(x_1) = 0$, $f(x_2) = 0$, $f(x_3) = 1$, $f(x_4) = 1$, $f(x_5) = 2$, $f(x_6) = 2$, $f(x_7) = 3$. This is a $(3,2)$-labeling of P, so $d_2(P) \leq 3$. The value $k = 3$ is attained uniquely at $x = x_r = x_2$, $y = x_s = x_7$ and the antichain formed is $A = \{x_2, x_3, x_4, x_5, x_6, x_7\}$ with $|A| = 6$. We will see in Section 6.1 that $d_2(P) = 2$.

We next develop a polynomial-time algorithm for finding the t-discrepancy of a semiorder. Given any poset $P = (V, \prec)$, a linear extension $L = (x_1 \prec x_2 \prec \cdots \prec x_n)$ of P and an integer-valued function f defined on V, we get a sequence of integers $s(L,f) : f(x_1), f(x_2), \cdots, f(x_n)$. If the sequence $s(L,f)$ is non-decreasing, we say f *is nondecreasing on* L. If not, we say $s(L,f)$ *first fails at position* r if there exists $s > r$ so that $f(x_r) > f(x_s)$ but whenever $i < r$, we have $i < j$ implies $f(x_i) \leq f(x_j)$. For example, the sequence $1, 1, 2, 3, 3, 4, 5, 2$ first fails at position 4 with $f(x_4) = 3 > 2 = f(x_8)$.

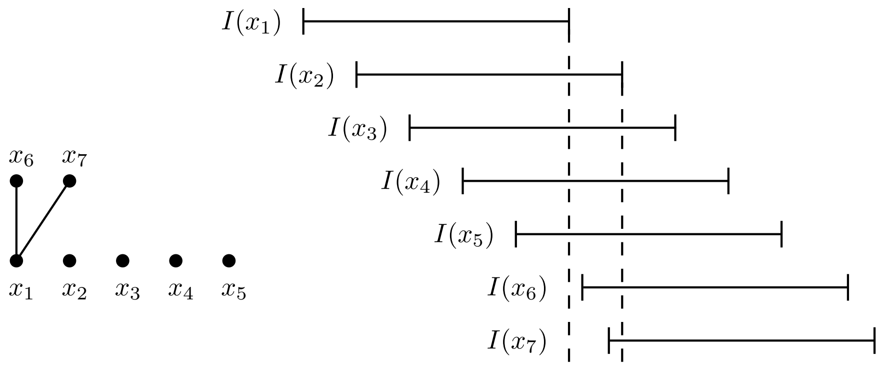


Figure 5: A semiorder P and a representation of it as a unit interval order.

In Example 18 with $L = (x_1 \prec x_2 \prec \cdots \prec x_7)$, the sequence $s(L, f)$ is $0, 0, 1, 1, 2, 2, 3$ which is non-decreasing. More generally, any sequence $s(L, f)$ arising from the greedy algorithm for labeling semiorders will be non-decreasing by construction. However, as we saw in Example 18, a labeling arising from the greedy algorithm is not always optimal. The next lemma shows that for any semiorder P and the linear extension L given by the left endpoint ordering of any unit interval representation of P, there exists an optimal labeling function f for which $s(L, f)$ is non-decreasing. This lemma is crucial in proving the correctness of our algorithm for computing the t-discrepancy of a semiorder.

Lemma 19 *Let $P = (V, \prec)$ be a semiorder and fix a unit interval representation of P with distinct endpoints. Let L be the linear extension of P given by the left endpoint ordering of this representation. Then there exists a labeling f of P that is t-optimal and is non-decreasing on L.*

Proof. Let $n = |V|$ and $I(v)$ be the unit interval assigned to v in the representation. Let L be the linear extension $x_1 \prec_L x_2 \prec_L \cdots \prec_L x_n$ of P given by the left endpoint ordering in this representation. We wish to show a t-optimal labeling function f of P exists that is non-decreasing on L. For a contradiction, assume no such t-optimal labeling exists and let f be a t-optimal labeling that first fails at position r where r is maximum. By assumption, $r \leq n-1$. To reach a contradiction, we will construct a labeling function g of P that is t-optimal and first fails at position $\ell > r$.

Let $f(x_r) = b$, let $a = \min\{f(x_i) : i \geq r+1\}$, and let x_s be any point with $f(x_s) = a$ and $s \geq r+1$. Since $s(L, f)$ first fails at position r, we know $f(x_s) = a < b = f(x_r)$. Create a new labeling function g by swapping the labels of x_r and x_s, that is, $g(x_i) = f(x_i)$ for $i \notin \{r, s\}$ and $g(x_r) = a$ and $g(x_s) = b$. We next show that g is a t-optimal labeling function of P.

First note that g satisfies condition (iii) of Definition 1 because f is a (k, t)-labeling for P and g simply swaps two of these labels. Similarly, for any points

$x_i, x_j \notin \{x_r, x_s\}$, we know that (i) and (ii) are satisfied for g because they are satisfied for f. Thus we need only show that conditions (i) and (ii) are satisfied when one or both of x_i, x_j are in the set $\{x_r, x_s\}$.

Since $f(x_r) > f(x_s)$, by condition (i) of Definition 1 we know that $x_r \not\prec x_s$. In addition, $x_s \not\prec x_r$ because $x_r \prec_L x_s$ and L is a linear extension of P. Thus

$$x_r \parallel x_s \text{ and } |g(x_r) - g(x_s)| = |f(x_s) - f(x_r)| \leq k. \tag{4}$$

It remains to consider a point $x_i \in V - \{x_r, x_s\}$ and to check that the pairs x_i, x_r and x_i, x_s satisfy (i) and (ii) of Definition 1 for the function g.

First consider x_i with $x_i \prec x_r$. By the definition of L, the left endpoint of $I(x_r)$ comes before the left endpoint of $I(x_s)$, so $x_i \prec x_r$ implies $x_i \prec x_s$. Since f is a (k, t)-labeling for P, we know $g(x_i) = f(x_i) < f(x_s) = g(x_r)$ and $g(x_i) = f(x_i) < f(x_r) = g(x_s)$ as desired.

Next consider x_i with $x_r \prec x_i$. In this case, $b = f(x_r) < f(x_i) = g(x_i)$ and thus $g(x_r) = a < b < g(x_i)$ so the pair x_i, x_r satisfies condition (i) of Definition 1 for g. We next consider the pair x_s, x_i. If $x_s \prec x_i$ then $g(x_s) = b < g(x_i)$ as desired. If $x_i \prec x_s$ then transitivity yields $x_r \prec x_s$, contradicting $x_r \parallel x_s$ from (4). Otherwise, $x_s \parallel x_i$ and since f satisfies (ii) of Definition 1 we have $|f(x_i) - a| \leq k$. Now $|g(x_i) - g(x_s)| = |f(x_i) - b| \leq |f(x_i) - a| \leq k$ where the inequality follows because $a < b < f(x_i)$.

Finally, consider x_i with $x_r \parallel x_i$. If $x_s \prec x_i$ we show a contradiction arises. Given that $r < s$ and our representation is unit, we know the right endpoint of $I(x_r)$ is smaller than the right endpoint of $I(x_s)$. Then $x_s \prec x_i$ would imply $x_r \prec x_i$, a contradiction. If $x_s \parallel x_i$ then $|g(x_r) - g(x_i)| = |f(x_s) - f(x_i)| \leq k$ and $|g(x_s) - g(x_i)| = |f(x_r) - f(x_i)| \leq k$ because f is a (k, t)-labeling of P. Lastly consider $x_i \prec x_s$. In this instance, $f(x_i) < f(x_s) = a < b$ so $g(x_i) = f(x_i) < b = g(x_s)$ and g satisfies (i) for the pair x_i, x_s. For the pair x_i, x_r we show $|g(x_i) - g(x_r)| \leq k$. Since this is a unit representation and $x_i \prec x_s$, $x_i \parallel x_r$ and $x_r \parallel x_s$, we know the left endpoint of $I(x_i)$ comes before the left endpoint of $I(x_r)$, thus $i < r$. By our assumption that f first fails at position r, $f(x_i) \leq f(x_s) = a$. Now $|g(x_i) - g(x_r)| = |f(x_i) - a| < |f(x_i) - b| = |f(x_i) - f(x_r)| \leq k$ with the inequality following from $f(x_i) \leq a < b$. $\square$

We next present an algorithm that determines whether a semiorder has a (k, t)-labeling and in the affirmative case, constructs such a labeling. This algorithm is a modification of the algorithm for determining whether a poset has weak discrepancy at most k in [11]. As we will see in Corollary 24, this can be used to calculate $d_t(P)$. We discuss correctness and complexity afterwards.

Algorithm (k, t)-Labeling for Semiorders

Input: An ordered set $P = (V, \prec)$, integers $k \geq 0$ and $t \geq 1$.

Output: A (k, t)-labeling function $f : V \to \mathbf{Z}$ of P, or the statement that no such labeling exists.

The algorithm:

Step 1: Construct a unit interval representation of P with distinct endpoints in which $x_i \in V$ is assigned the unit interval $I(x_i)$. This can be accomplished in linear time (see [4]).

Consider the elements of $V = \{x_1, x_2, \ldots, x_n\}$ indexed by their left endpoint in the representation.

Step 2: [Initialization Step] Let $f(x_1) = 0$ and let $U = \{2, 3, \ldots, n\}$.

Form a $\{0, 1\}$-matrix M whose rows and columns are indexed by U. Initialize:

$$M_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Step 3: [Assign Initial Ranges] Assign the range for x_1 as $R(x_1) = [\ell(x_1), u(x_1)] = [0, 0]$ and the range $R(x_i) = [\ell(x_i), u(x_i)]$ for each $i \in U$ as follows:

- If $x_1 \prec x_i$ set $R(x_i) = [1, n-1]$.
- If $x_1 \parallel x_i$ set $R(x_i) = [0, k]$.

Since we indexed the elements of V by left endpoints in the unit interval representation of P, we can not have $x_i \prec x_1$.

Step 4: [Narrowing the Ranges]
Narrowing Steps (NS): Pick two distinct indices $2 \leq i < j \leq n$ with $M_{ij} = 0$. Thus either $x_i \prec x_j$ or $x_i \parallel x_j$.

(a) If $x_i \prec x_j$ and $\ell(x_j) \leq \ell(x_i)$, increase $\ell(x_j)$ to $\ell(x_i) + 1$.

(b) If $x_i \prec x_j$ and $u(x_i) \geq u(x_j)$, decrease $u(x_i)$ to $u(x_j) - 1$.

(c) If $x_i \parallel x_j$ and $u(x_j) \geq u(x_i) + k + 1$, decrease $u(x_j)$ to $u(x_i) + k$.

(d) If $x_i \parallel x_j$ and $u(x_i) \geq u(x_j) + k + 1$, decrease $u(x_i)$ to $u(x_j) + k$.

(e) If $x_i \parallel x_j$ and $\ell(x_j) \leq \ell(x_i) - k - 1$, increase $\ell(x_j)$ to $\ell(x_i) - k$.

(f) If $x_i \parallel x_j$ and $\ell(x_i) \leq \ell(x_j) - k - 1$, increase $\ell(x_i)$ to $\ell(x_j) - k$.

If $\ell(x_i) > u(x_i)$ or $\ell(x_j) > u(x_j)$, STOP. There is no (k, t)-labeling of P.

If $R(x_i)$ was narrowed in this pass of the narrowing steps, set $M_{ir} = M_{ri} = 0$ for all r other than i and j. Likewise, if $R(x_j)$ was narrowed in this pass of the narrowing steps, set $M_{jr} = M_{rj} = 0$ for all r other than i and j.

In any event, set $M_{ij} = M_{ji} = 1$.

If all entries of M are 1's, continue to Step 5. Otherwise, begin Step 4 again.

Step 5: [Sweeping steps]
(a) Left to right sweep: For $i = 1$ to $n - t$,

- if $\ell(x_{i+t}) \leq \ell(x_i)$, increase $\ell(x_{i+t})$ to $\ell(x_i) + 1$.
- If $\ell(x_{i+t}) > u(x_{i+t})$, STOP. There is no (k, t)-labeling of P.

	i	1	2	3	4	5	6	7
Step 3		[0,0]	[0,2]	[0,2]	[0,2]	[0,2]	[1,6]	[1,6]
Step 4(c)	$x_2 \parallel x_6$	[0,0]	[0,2]	[0,2]	[0,2]	[0,2]	[1,4]	[1,6]
Step 4(c)	$x_2 \parallel x_7$	[0,0]	[0,2]	[0,2]	[0,2]	[0,2]	[1,4]	[1,4]
Step 5a		[0,0]	[0,2]	[1,2]	[1,2]	[2,2]	[2,4]	[3,4]
Step 5b		[0,0]	[0,1]	[1,1]	[1,2]	[2,2]	[2,4]	[3,4]
Step 4(c)	$x_2 \parallel x_6$	[0,0]	[0,1]	[1,1]	[1,2]	[2,2]	[2,3]	[3,4]
Step 4(c)	$x_2 \parallel x_7$	[0,0]	[0,1]	[1,1]	[1,2]	[2,2]	[2,3]	[3,3]
Step 4(f)	$x_2 \parallel x_7$	[0,0]	[1,1]	[1,1]	[1,2]	[2,2]	[2,3]	[3,3]
Step 5(a)		[0,0]	[1,1]	[1,1]	[2,2]	[2,2]	[3,3]	[3,3]

Table 1: The ranges $R(x_i)$ when $k = 2$, $t = 2$, $n = 7$, and the algorithm is applied to the semiorder P with representation shown in Figure 5.

(b) Right to left sweep: For $i = n$ down to $t + 1$,

- if $u(x_{i-t}) \geq u(x_i)$, decrease $u(x_{i-t})$ to $u(x_i) - 1$.
- If $u(x_{i-t}) < \ell(x_{i-t})$, STOP. There is no (k, t)-labeling of P.

If no values were changed in Step 5, then continue to Step 6. Otherwise, begin Step 4 again.

Step 6: Set $f(x_i) = \ell(x)$ for $i = 2, 3, 4, \ldots, n - 1$.
(End of Algorithm (k, t)-Labeling for Semiorders)

6.1 Illustration of Algorithm (k, t)-Labeling for Semiorders

In Table 1 we illustrate how the ranges change when Algorithm (k, t)-Labeling for Semiorders is applied to the semiorder P shown in Figure 5 in the instance of $t = 2$ and $k = 2$. In this example, at the end, each range set consists of a single integer, and we obtain the $(2, 2)$-labeling function $f(x_1) = 0$, $f(x_2) = 1$, $f(x_3) = 1$, $f(x_4) = 2$, $f(x_5) = 2$, $f(x_6) = 3$, $f(x_7) = 3$. Thus $d_t(P) \leq 2$. We observe that this function is an improvement over the one constructed using the greedy algorithm in Example 18.

Next we consider this same semiorder P, the same unit interval representation, and continue to consider $t = 2$ but change the value of k.

When $k = 1$ the initial range values assigned in Step 3 are $[0, 0]$ for x_1, $[0, 1]$ for x_2, x_3, x_4, x_5 and $[1, 6]$ for x_6, x_7. In Step 4, comparing x_2 with x_6 and x_7 results in narrowing $R(x_6)$ and $R(x_7)$ to $[1, 2]$ and these are the only changes that occur. In the left to right sweep of Step 5, we get $\ell(x_5) = 2$ and $u(x_5) = 1$, and the algorithm stops with the conclusion that $d_2(P) > 1$. Combining this with $d_t(P) \leq 2$ from above, we conclude $d_t(P) = 2$.

When $k = 3$ the initial range values assigned in Step 3 are $[0, 0]$ for x_1, $[0, 3]$ for x_2, x_3, x_4, x_5 and $[1, 6]$ for x_6, x_7. No changes occur as a result of applying Step 4. After both sweeping passes are made in Step 5, the ranges are $R(x_1) = [0,0]$, $R(x_2) = [0,2]$, $R(x_3) = [1,2]$, $R(x_4) = [1,3]$, $R(x_5) = [2,3]$, $R(x_6) = [2,6]$, $R(x_7) = [3,6]$. No further modifications occur in the range sets, and thus the resulting labeling is indeed a $(3,2)$-labeling of P, but is not 2-optimal. Indeed, it is the same labeling found by the greedy algorithm in Example 18.

6.2 Correctness and Complexity of Algorithm (k,t)-Labeling for Semiorders

We establish the correctness of Algorithm (k,t)-labeling for Semiorders using Lemma 19 and two propositions. After this we consider the complexity of the algorithm.

Proposition 20 *If Algorithm (k,t)-labeling for Semiorders terminates with all ranges non-empty, then $d_t(P) \le k$ and picking the smallest element in each range set is a valid (k,t)-labeling.*

Proof. Suppose that $R(v) = [\ell(v), u(v)]$ is the range assigned to point v when the algorithm terminates. Let $f(v) = \ell(v)$ for each $v \in V$. It suffices to show that f is a valid (k,t)-labeling for P. We consider any pair of distinct points x_i, x_j in P and show that conditions (i), (ii) and (iii) of Definition 1 are satisfied. Without loss of generality, we may assume $i < j$ and thus either $x_i \prec x_j$ or $x_i \parallel x_j$. If $x_i \prec x_j$ then by Step 4(a) of the algorithm, $\ell(x_j) \ge \ell(x_i)+1$ thus $f(x_i) < f(x_j)$ as required by (i). If $x_i \parallel x_j$ then by Steps 4(e) and 4(f) of the algorithm, $\ell(x_i) - k \le \ell(x_j) \le \ell(x_i) + k$ thus $|f(x_i) - f(x_j)| \le k$ as required by (ii). As a result of the left to right sweep in Step 5, if $\ell(x_i) = r$ then $\ell(x_{i+t}) \ge r + 1$, thus at most t points can receive the label $f(x) = \ell(x) = r$ for each r, establishing (iii). $\square$

Proposition 21 *If P is a semiorder with $d_t(P) \le k$ then Algorithm (k,t)-labeling for Semiorders terminates with each range set non-empty.*

Proof. In Step 1 of the algorithm, a unit interval representation of P is constructed in which all endpoints of intervals are distinct. As in the algorithm, we consider the points of $V = \{x_1, x_2, \ldots, x_n\}$ indexed by their left endpoint in this representation. Let L be the linear extension $x_1 \prec_L x_2 \prec_L \cdots \prec_L x_n$ of P. By Lemma 19, there exists a labeling function f that is t-optimal and non-decreasing on L. Thus f is a (k,t)-labeling of P and $f(x_i) \le f(x_j)$ whenever $i < j$.

As in the proof of Lemma 6(a), we may add a constant to each function value so that $f(x_1) = 0$ and the resulting function is still non-decreasing on L. Similarly, following the proof of Lemma 6, we may assume that $f(x_i) \le n-1$ for each i. Therefore, the initial ranges assigned in Step 3 of the algorithm satisfy $f(x_i) \in R(x_i)$ for $i = 2, 3, 4, \ldots, n$.

Indeed, we will see that as we continue through the algorithm, we maintain the invariant:

$(*)$ $f(x_i) \in R(x_i)$, or equivalently, $\ell(x_i) \le f(x_i) \le u(x_i)$ for $i = 2, 3, \ldots, n$.

In Step 4 of the algorithm, we apply (i) of Definition 1 (in Steps 4(a) and 4(b)) or (ii) of Definition 1 (in Steps 4(c) – 4(f)) to the pair (x_i, x_j). For example, if $(*)$ holds true at the start of Step 4(a), and if $x_i \prec x_j$ then $f(x_i) < f(x_j)$ hence $\ell(x_i) \leq f(x_i) < f(x_j)$. Since f is an integer-valued function we know $f(x_j) \geq \ell(x_i) + 1$ and we can narrow the range of possible values for $f(x_j)$ to $[\ell(x_i) + 1, u(x_j)]$. Thus $(*)$ holds true at the end of Step 4(a). Similarly, we maintain the invariant $(*)$ when the other parts of Step 4 are applied.

The sweeping steps (step 5) of the algorithm proceed by applying Lemma 19 and (iii) of Definition 1. Since f is non-decreasing on L and there are at most t occurrences of the function value $f(x_i)$, we know $f(x_{i+t}) > f(x_i) \geq \ell(x_i)$ so we can increase $\ell(x_{i+t})$ to $\ell(x_i) + 1$.

Thus the algorithm maintains the invariant $(*)$. Since the function f exists, each range $R(x_i)$ must contain the value $f(x_i)$ and thus be non-empty when the algorithm terminates. □

Theorem 22 *Algorithm (k, t)-labeling for Semiorders correctly determines whether a semiorder P has $d_t(P) \leq k$ and in the affirmative case, it produces a (k, t)-labeling for P.*

Proof. There are two ways in which the algorithm can terminate: either in Step 4 when a range is narrowed to the empty set, or in Step 6 when all ranges stabilize, are non-empty and can not be narrowed further. In the former case, we conclude $d_t(P) > k$ by the contrapositive of Proposition 21. In the latter case, Proposition 20 implies that $d_t(P) \leq k$ and picking the smallest element in each range set is a valid (k, t)-labeling for P. □

Theorem 23 *With input $P = (V, \prec)$, and $n = |V|$, Algorithm (k, t)-labeling for Semiorders runs in time $O(n^4)$.*

Proof. Step 1 can be accomplished in time $O(n)$ as shown by Gardi in [4]. Clearly Step 2 runs in time $O(n^2)$ and Steps 5 and 6 in time $O(n)$, so we focus on Step 4.

The initial ranges have length at most $n-2$, where the length of range $R(x_i)$ is defined as $u(x_i) - \ell(x_i)$. When a range is narrowed, its length decreases by at least 1, hence each range is narrowed at most $n - 1$ times. Thus at most n^2 narrowings occur during Step 4 over the course of the whole algorithm.

Furthermore, after all $\binom{n-1}{2} < n^2$ pairs of points are considered, either a narrowing occurs or the matrix M fills with 1's and the algorithm proceeds to Step 5. Thus the total amount of time spent in Step 4 is $O(n^4)$. □

Finally, Algorithm (k, t)-labeling for Semiorders can be applied repeatedly with different values of k to determine $d_t(P)$.

Corollary 24 *Given $P = (V, \prec)$ with $n = |V|$, we can determine $d_t(P)$ in time $O(n^4 \log n)$.*

Proof. Use Algorithm (k, t)-labeling for Semiorders to determine if $d_t(P) \leq k$ for $k = 0, 1, 2, \ldots$ and stop as soon as a value of k is found for which an affirmative answer is reached. That value of k is $d_t(P)$. By Remark 7 we know $d_t(P) \leq n-1$, so we would need to run the algorithm a maximum of $n-1$ times giving a total running time of $O(n^5)$. The running time can be shortened to $O(n^4 \log n)$ if we instead use a binary search to choose the appropriate values of k.

7 Conclusion

We conclude with an open question and acknowledgements. We show in Theorems 15 that the decision problem $d_t(P) \leq k$ is NP-complete for general posets P. However, the problem is polynomial for semiorders P as shown in Theorems 22 and 23. A natural class to consider next is interval orders.

Question: Is there a polynomial-time algorithm for determining if $d_t(P) \leq k$ when P is an interval order? Is the decision problem $d_t(P) \leq k$ NP-complete when P is an interval order?

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