1. **(9pts)** Find an equation of the sphere if one of its diameters has endpoints $(2,1,4)$ and $(4,3,10)$.

Center $C(3,2,7)$

Radius $r = \frac{1}{2}| < 2,2,6 > | = \sqrt{11}$

Answer: $(x-3)^2 + (y-2)^2 + (z-7)^2 = 11.$

2. **(9pts)** For what values of c is the angle between the vectors $< 1,2,1 >$ and $< 1,0,c >$ equal to $60°$?

$$1 + c = \sqrt{6}\sqrt{1+c^2}\cos\theta$$
$$(1+c)^2 = 6(1+c^2)\frac{1}{4}$$
$$c^2 - 4c + 1 = 0$$

Answer: $c = 2 \pm \sqrt{3}.$

3. **(9pts)** Find the volume of the parallelepiped determined by the vectors $\mathbf{a} =< 1,0,6 >$, $\mathbf{b} =< 2,3,-8 >$, and $\mathbf{c} =< 8,-5,6 >$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & -8 \\ 8 & -5 & 6 \end{vmatrix} = -226$$

Answer: 226.

4. **(9pts)** Find parametric equations of the line passing through $(-1,0,5)$ and $(4,-3,3)$.

$\mathbf{v} =< 5,-3,-2 >$

Answer: $\begin{aligned} x &= -1 + 5t \\ y &= 0 - 3t \\ z &= 5 - 2t \end{aligned}$

5. **(9pts)** Find an equation of the plane that contains the point $(1, 2, 3)$ as well as the line $x = 3t$, $y = 1 + t$, $z = 2 - t$.

$P(1, 2, 3) \quad Q(0, 1, 2)$,

$$\vec{QP} =< 1, 1, 1 > .$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{vmatrix} =< -2, 4, -2 >$$

Answer: $2(x-1) - 4(y-2) + 2(z-3) = 0.$

6. **(9pts)** What are the spherical coordinates of the point with cylindrical coordinates $(4, \pi/3, 4)$?

Answer: $\left(4\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right).$

7. **(9pts)** Find parametric equations for the tangent line to the curve given by $\mathbf{r}(t) =< t^2 - 1, t^2 + 1, t + 1 >$ at the point $(-1, 1, 1)$.

$$\mathbf{r}'(t) =< 2t, 2t, 1 >$$

$$\mathbf{r}'(0) =< 0, 0, 1 >$$

Answer:

$$\begin{aligned} x &= -1 \\ y &= 1 \\ z &= 1 + t \end{aligned}$$

8. **(9pts)** Reparametrize the curve $\mathbf{r}(t) = (1 + 2t)\ \mathbf{i} + (3 + t)\ \mathbf{j} - 5t\ \mathbf{k}$ with respect to arc length starting from the point $(1, 3, 0)$.

$$\mathbf{r}'(t) =< 2, 1, -5 >$$

$$|\mathbf{r}'(t)| = \sqrt{30}$$

$$s = \sqrt{30}t$$

Answer: $\mathbf{r}(s) = \left(1 + \frac{2}{\sqrt{30}}s\right)\mathbf{i} + \left(3 + \frac{1}{\sqrt{30}}s\right)\mathbf{j} - \left(\frac{5}{\sqrt{30}}s\right)\mathbf{k}.$

9. **(9pts)** Find the curvature of the curve given by $\mathbf{r}(t) = t\ \mathbf{i} + t\ \mathbf{j} - (1+t^2)\ \mathbf{k}$.

$$\mathbf{r}'(t) =< 1, 1, -2t >$$

$$\mathbf{r}''(t) =< 0, 0, -2 >$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = -2\mathbf{i} + 2\mathbf{j}$$

Answer: $\kappa(t) = \dfrac{2\sqrt{2}}{(2+4t^2)^{\frac{3}{2}}} = \dfrac{1}{(1+2t^2)^{\frac{2}{3}}}$.

10. **(9pts)** Find the position vector of a particle that has acceleration $\mathbf{a}(t) = t^2\ \mathbf{i} + \mathbf{j} - \cos 2t\ \mathbf{k}$, initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$, and initial position $\mathbf{r}(0) = \mathbf{j}$.

$$\mathbf{v}(t) = \left\langle \frac{t^3}{3} + 1, t, -\frac{\sin 2t}{2} + 1 \right\rangle$$

Answer: $\mathbf{r}(t) = \left\langle \dfrac{t^4}{12} + t, \dfrac{t^2}{2} + 1, \dfrac{\cos 2t}{4} + t - \dfrac{1}{4} \right\rangle$.

11. **(10pts)** Let $\mathbf{r}(t) = e^t\ \mathbf{i} + e^t \sin t\ \mathbf{j} - e^t \cos t\ \mathbf{k}$.

(a) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

$$\mathbf{r}'(t) = e^t < 1, \sin t + \cos t, \sin t - \cos t >$$

$$|\mathbf{r}'(t)| = e^t\sqrt{3}$$

$$\boxed{\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{3}}, \frac{\sin t + \cos t}{\sqrt{3}}, \frac{\sin t - \cos t}{\sqrt{3}} \right\rangle.}$$

$$\mathbf{T}'(t) = \left\langle 0, \frac{\cos t - \sin t}{\sqrt{3}}, \frac{\cos t + \sin t}{\sqrt{3}} \right\rangle$$

$$\boxed{\mathbf{N}(t) = \left\langle 0, \frac{\cos t - \sin t}{\sqrt{2}}, \frac{\cos t + \sin t}{\sqrt{2}} \right\rangle.}$$

$$\boxed{\mathbf{B}(t) = \left\langle \frac{2}{\sqrt{6}}, -\frac{\sin t + \cos t}{\sqrt{6}}, \frac{\cos t - \sin t}{\sqrt{6}} \right\rangle.}$$

(b) Calculate both the normal and the tangential components of the acceleration.

$$\mathbf{r}''(t) = e^t < 1, 2\cos t, 2\sin t >$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3e^{2t}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = e^{2t} < 2, -\sin t - \cos t, -\sin t + \cos t >$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = e^{2t}\sqrt{6}$$

$$\boxed{\textit{Answer:}\quad a_T = \sqrt{3}e^t, \quad a_N = \sqrt{2}e^t.}$$