1a. Are the pairs of vectors below parallel, orthogonal or neither?

(i) $\langle -5, 15, 10\rangle$ and $\langle 3, -9, -6\rangle$
Parallel: $\langle -5, 15, 10\rangle = -\frac{5}{3}\langle 3, -9, -6\rangle$. (Alternately, the vectors are parallel because the cross product is zero.)

(ii) $\langle 1, 0, 7\rangle$ and $\langle 3, 6, 2\rangle$
Neither: Both the cross and dot products are nonzero.

(iii) $\langle \sin\theta, \cos\theta, e^{-\theta}\rangle$ and $\langle \sin\theta, \cos\theta, -e^{\theta}\rangle$
Orthogonal: The dot product is zero.

1b. Find a unit vector perpendicular to $\langle 0, 1, -1\rangle$ and $\langle 3, 2, 1\rangle$.

Solution. The cross product

$$V = \langle 0, 1, -1\rangle \times \langle 3, 2, 1\rangle = \langle 3, -3, -3\rangle$$

is perpendicular. Divide V by it's length to get a unit vector

$$\boldsymbol{U = \frac{V}{|V|} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle}\,.$$

(Notice that $-U = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$ is also a correct answer.)

2. The planes $x - z = 4$ and $-2y + z = -3$ intersect in a line. Write down the symmetric equations of this line.

Solution. Any point (x, y, z) must satisfy *both* equations. This means that *both* $z = x - 4$ *and* $z = 2y - 3$. So the symmetric equations are

$$\boldsymbol{x - 4 = 2y - 3 = z}\,.$$

Remark. Compare this with the generic formula for symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

to see that the line passes through the point $(x_0, y_0, z_0) = (4, \frac{3}{2}, 0)$ and points in the dirction of $\langle a, b, c\rangle = \langle 1, \frac{1}{2}, 1\rangle$.

Alternate Solution. The normal vectors to the planes are $N_1 = \langle 1, 0, -1\rangle$ and $N_2 = \langle 0, -2, 1\rangle$. Since the line lies in *both* planes, it will be perpendicular to *both* normal vectors. This means the line points in the direction of

$$V = \langle a, b, c\rangle = N_1 \times N_2 = \langle 1, 0, -1\rangle \times \langle 0, -2, 1\rangle = \langle -2, -1, -2\rangle\,.$$

Also, the point $(x_0, y_0, z_0) = (4, \frac{3}{2}, 0)$ lies in both planes (because it satisfies both equations). So the line of intersection must pass through this point. We may write the symmetric equations as

$$\frac{x - 4}{-2} = \frac{y - \frac{3}{2}}{-1} = \frac{z}{-2}\,.$$

This answer is correct! Just multiply each term by -2 to see that we have the answer in the first solution.

3. Compute the linearization of $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$, and use it to approximate the number $\sqrt{(2.8)^2 + (4.1)^2}$.

Solution. Use the formula $L(x, y) = f(3, 4) + f_x(3, 4)(x - 3) + f_y(3, 4)(y - 4)$ to obtain

$$\boldsymbol{L(x, y) = \frac{3}{5}x + \frac{4}{5}y\,.}$$

Then

$$\boldsymbol{\sqrt{(2.8)^2 + (4.1)^2} = f(2.8, 4.1) \approx L(2.8, 4.1) = \frac{124}{25} = 4 + \frac{24}{25}\,.}$$

4. Consider the hyperboloid of one-sheet $x^2 + 4y^2 - 9z^2 = 16$.

4(a). Find the tangent plane at $(5, 0, -1)$.

Solution. This surface is a *level set* of the function $f(x, y, z) = x^2 + 4y^2 - 9z^2$. So the gradient

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, 8y, -18z \rangle = \langle 10, 0, 18 \rangle\,.$$

is perpendicular to the surface. This is our normal vector $N = \langle a, b, c \rangle$. So the generic formula for a plane $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ gives us

$$0 = 10(x - 5) + 18(z + 1)$$

or

$$\boldsymbol{0 = 5x + 9z - 16\,.}$$

4(b). At which points of the hyperboloid is the tangent plane perpendicular to the line $L(t) = \langle -10, 10t, 1 + 9t \rangle$?

Solution. The tangent plane will be perpendicular when its normal vector $N = \nabla f = \langle 2x, 8y, -18z \rangle$ points in the same direction as the line $V = \langle 0, 10, 9 \rangle$. In order for these two vectors to be parallel we must have

$$x = 0 \qquad \text{and} \qquad \frac{10}{9} = \frac{8y}{-18z}\,.$$

Equivalently,

$$x = 0 \qquad \text{and} \qquad y = -\frac{5}{2}z\,.$$

Plug this into the equation of the surface to obtain $z = \pm 1$. This gives us $y = \mp\frac{5}{2}$. So there are two points on the hyperboloid where the tangent is perpendicular to the line

$$\boldsymbol{\left(0, -\frac{5}{2}, 1\right) \qquad \text{and} \qquad \left(0, \frac{5}{2}, -1\right)\,.}$$

5. The table to the right lists some of the values for a function $f(x,y)$, and its derivatives, at the points $(2,0)$ and $(0,\frac{\pi}{2})$.

	f	f_x	f_y
$(2,0)$	10	5	-1
$(0,\frac{\pi}{2})$	-2	3	1

Use $f(x,y)$ to define a new function

$$g(\theta,\varphi) = f(\cos\theta + \sin\varphi, \tan(\theta\varphi))\,.$$

Compute $\dfrac{\partial g}{\partial \theta}$ and $\dfrac{\partial g}{\partial \varphi}$ at $(0,\frac{\pi}{2})$.

Solution. We have $(\theta,\varphi) = (0,\frac{\pi}{2})$ and $(x,y) = (\cos\theta+\sin\varphi, \tan(\theta\varphi))$, so that $(x,y) = (2,0)$. This give us

$$\begin{aligned}\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -5\sin 0 + (-1)\frac{\pi}{2}\sec^2 0 = \boldsymbol{-\frac{\pi}{2}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial \varphi} = \frac{\partial f}{\partial \varphi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} \\ &= 5\cos\frac{\pi}{2} - 1\cdot 0\sec^2 0 = \mathbf{0}\,.\end{aligned}$$

6. Use the table above to find the maximum rate of change in $f(x,y)$ at $(2,0)$ and the direction in which it occurs.

Solution. The direction is

$$\boldsymbol{\nabla f = \langle 5, -1\rangle}\,,$$

and the rate is

$$\boldsymbol{|\nabla f| = \sqrt{26}}\,.$$

7. Suppose $f_x(-1,1) = 0 = f_y(-1,1)$. In each case, what can you say about $z = f(x,y)$?

(a) $f_{xx}(-1,1) = -100$, $f_{xy}(-1,1) = 5$, $f_{yy}(-1,1) = 0$

(b) $f_{xx}(-1,1) = 4$, $f_{xy}(-1,1) = -6$, $f_{yy}(0,2) = 9$

(c) $f_{xx}(-1,1) = -1$, $f_{xy}(-1,1) = 2$, $f_{yy}(-1,1) = -8$

Solution. The second derivative test tells us that $(-1,1)$ is a **saddle point** in part (a), and that $f(-1,1)$ is a **local maximum** in (c). The second derivative test fails ($D=0$) in part (b). So we may only say that the **tangent plane is horizontal** here.

8. Find the area of the largest rectangle that can be inscribed in the ellipse

$$a^2x^2 + b^2y^2 = 1\,.$$

Solution. This is a Lagrange multiplier problem. We want to maximize area $f(x, y) = (2x)(2y) = 4xy$ subject to the constraint $g(x, y) = a^2x^2 + b^2y^2 = 1$. Start by computing the gradients

$$\nabla f = 4\langle y, x\rangle \qquad \text{and} \qquad \nabla g = 2\langle a^2x, b^2y\rangle\,.$$

These two vectors will be parallel when f is optimized:

$$y = \lambda a^2 x \qquad \text{and} \qquad x = \lambda b^2 y\,.$$

Multiply the first equation by x and the second by y to see that

$$\lambda a^2x^2 = \lambda b^2y^2\,.$$

We would live to divide both sides by λ. To do so we must rule out the possibility that $\lambda = 0$. (Notice that $0 = \lambda = x = y$ *is* a solution to the equations above.) But if λ is zero, then the two Langrange equations above say $x = 0 = y$. This will give us Area $= 0$, which is certainly not the maximum. So it is safe for us to assume that $\lambda \neq 0$. Hence

$$a^2x^2 = b^2y^2\,.$$

Now the equation of the ellipse tells us that $a^2x^2 = \frac{1}{2}$, so that

$$x = \frac{1}{a\sqrt{2}} \qquad \text{and} \qquad y = \frac{1}{b\sqrt{2}}\,.$$

We conclude that the largest rectangle that can be inscribed in the ellipse has area

$$\boldsymbol{4xy = \frac{2}{ab}\,.}$$

9a. Given only the information that

$$f_{yx} = x^2 + \cos y\,,$$

determine f_{xy}.

Solution. Use the equality of mixed partial derivatives: $f_{xy} = f_{yx}$.

9b. Compute the partial derivatives of $z = \tan(xy) + e^{xy^2} + \dfrac{\ln y}{x}$.

Solution.

$$\begin{aligned}\boldsymbol{\frac{\partial z}{\partial x}} &= \boldsymbol{y\sec^2(xy) + y^2e^{xy^2} - \frac{\ln y}{x^2}}\\ \boldsymbol{\frac{\partial z}{\partial y}} &= \boldsymbol{x\sec^2(xy) + 2xye^{xy^2} + \frac{1}{xy}\,.}\end{aligned}$$

10. Compute the volume enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$ and $z = 0$.

Solution. We are integrating over the triangle in the xy-plane enclosed by the lines $x = 0$ (the y-axis), $y = 1$ (horizontal line), and $y = x$ (diagonal line). So the volume will be

$$\int_0^1 \int_0^y (x^2 + 3y^2)\, dx\, dy \;=\; \mathbf{\frac{5}{6}}\,.$$