

A proof of a Parabolic Maximum Principle

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Define the following parabolic operator in divergence form

$$Lu \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_j} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_i} \right)$$

for $(x, t) \in \Omega_T = \Omega \times (0, T]$, where Ω is a bounded simply connected subset of \mathbb{R}^m for $m \geq 2$ and $T > 0$ is a fixed but arbitrary positive number. We also assume that the a_{ij} are bounded and measurable in $\bar{\Omega}_T$ and satisfy the uniform parabolicity condition:

$$1/\nu |z|^2 \leq \sum_{i,j=1}^m a_{ij}(x, t) z_i z_j \leq \nu |z|^2 \quad (1)$$

for some $\nu > 0$, almost everywhere in $\bar{\Omega}_T$ and all $z \in \mathbb{R}^m$. Let $\Gamma = (\partial\Omega \times [0, T]) \cap (\Omega \times \{t = 0\})$. We consider the following problem

$$\left. \begin{aligned} Lu &= \nabla \cdot \mathbf{F} - f, & (x, t) \in \Omega_T \\ u &= \phi(x, t), & \text{on } \Gamma \end{aligned} \right\} \quad (2)$$

Our goal here is to furnish a proof of the following maximum principle whose statement can be found in [1]

Theorem 1 (Maximum Principle). *Let u be a smooth solution of (2) in $\bar{\Omega}_T$ where $f \in L^s([0, T]; L^r(\Omega))$ for $m/2r + 1/s < 1$ and $\mathbf{F} = (f_1, \dots, f_m)$ with $f_i \in L^q([0, T]; L^p(\Omega))$ for $m/2p + 1/q < 1/2$. There exists a constant $K > 0$ which depends only on m, p, q, r, s, ν and $|\Omega|$ such that*

$$|u| \leq \sup_{\Gamma} \phi + KM \quad (3)$$

in $\bar{\Omega}_T$, where $M = \left[\|f\|_{r,s}^2 + \sum_{i=1}^m \|f_i\|_{p,q}^2 \right]^{1/2}$

The idea for the proof is old and I think was first used by S.N. Bernstein at least in the elliptic case. The proof idea was outlined in the quoted article by Aronson and we fill in the details here:

Proof. Let $k \geq k_0 := \max_{\Gamma} \phi$ and define the sets $A_k(t) = \{x \in \Omega | (u(x, t) - k) > 0\}$. Let $v(x, t) = \max(u(x, t) - k, 0)$. That is

$$v(x, t) = \begin{cases} 0, & u \leq k \\ u - k, & \text{otherwise,} \end{cases}$$

and

$$Dv(x, t) = \begin{cases} 0, & u \leq k \\ Du, & \text{otherwise} \end{cases}$$

It follows that support of $v \equiv A_k(t)$. We will use v as a test function in the weak formulation of the PDE. We multiply (2) by v and integrating over the domain Ω to obtain:

$$\int_{\Omega} v \frac{\partial u}{\partial t} dx - \int_{\Omega} v \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) dx = \int_{\Omega} v \nabla \cdot \mathbf{F} dx - \int_{\Omega} f v dx. \quad (4)$$

We handle each term separately and for convenience we drop the differential dx from the integrand. For the first term we note that

$$\frac{d}{dt} \int_{\Omega} v^2 = \frac{d}{dt} \int_{\Omega} (u - k)_+^2 = 2 \int_{\Omega} (u - k)_+ \frac{\partial u}{\partial t},$$

which then implies that for the first term we have

$$\int_{\Omega} v \frac{\partial u}{\partial t} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 \equiv \frac{1}{2} \frac{d}{dt} \int_{A_k(t)} v^2.$$

For the second term we integrate by parts

$$- \int_{\Omega} v \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) = - \left[\int_{\partial\Omega} v (a_{ij} D_j u \cdot \hat{n}) d\sigma - \int_{\Omega} a_{ij} D_j v D_i u \right] = \int_{\Omega} a_{ij} D_j v D_i u.$$

Here we have uses the fact that $k \geq \max_{\Gamma} \phi$ so that v vanishes on Γ and note that $D_j \equiv \frac{\partial}{\partial x_j}$. By the uniform parabolicity (1) and the fact that $D_i v = D_i u$ on the set $A_k(t)$, we can write

$$\frac{1}{2} \frac{d}{dt} \int_{A_k(t)} v^2 + \frac{1}{\nu} \int_{A_k(t)} |Dv|^2 \leq \int_{\Omega} v \nabla \cdot \mathbf{F} - \int_{\Omega} f v. \quad (5)$$

We now estimate the RHS term by term from above. Using Cauchy's inequality with ϵ and Hölder's inequality we have

$$\begin{aligned} - \int_{\Omega} f v &\leq \frac{1}{2\epsilon} \int_{A_k(t)} f^2 + \frac{\epsilon}{2} \int_{A_k(t)} v^2 \\ &\leq \frac{\epsilon}{2} \int_{A_k} v^2 + \frac{1}{2\epsilon} \left[\left(\int_{A_k} (f^2)^{r/2} \right)^{2/r} \left(\int_{A_k} 1 \right)^{1-2/r} \right] \\ &\leq \frac{1}{2\epsilon} |A_k|^{1-\frac{2}{r}} \|f\|_r^2 + \frac{\epsilon}{2} \int_{A_k(t)} v^2, \end{aligned}$$

which holds for arbitrary $\epsilon > 0$. We use the notation $|A_k|$ to denote the Lebesgue measure of the set A_k . Similarly for the other term we have, using the divergence theorem, Cauchy's inequality and Hölder's inequality:

$$\int_{\Omega} v \nabla \cdot \mathbf{F} \leq \frac{1}{2\theta} |A_k|^{1-\frac{2}{p}} \sum_i \|f_i\|_p^2 + \frac{\theta}{2} \int_{A_k(t)} |Dv|^2.$$

Combining the computations we have

$$\frac{1}{2} \frac{d}{dt} \int_{A_k(t)} v^2 + \frac{1}{\nu} \int_{A_k(t)} |Dv|^2 \leq \frac{1}{2\theta} |A_k|^{1-\frac{2}{p}} \sum_i \|f_i\|_p^2 + \frac{\theta}{2} \int_{A_k(t)} |Dv|^2 + \frac{1}{2\epsilon} |A_k|^{1-\frac{2}{r}} \|f\|_r^2 + \frac{\epsilon}{2} \int_{A_k(t)} v^2. \quad (6)$$

We can simplify the inequality (6) by combining terms. Since we have Dv on both sides we begin by choosing $\theta = \frac{1}{\nu}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{A_k(t)} v^2 + \frac{1}{2\nu} \int_{A_k(t)} |Dv|^2 \leq \frac{\nu}{2} |A_k|^{1-\frac{2}{p}} \sum_i \|f_i\|_p^2 + \frac{1}{2\epsilon} |A_k|^{1-\frac{2}{r}} \|f\|_r^2 + \frac{\epsilon}{2} \int_{A_k(t)} v^2. \quad (7)$$

Now we try to get rid of the Dv integral. To do this we write, assuming $m \geq 3$, and using Hölder's inequality (again)

$$\int_{A_k(t)} v^2 \leq \left(\int_{A_k(t)} (v^2)^{\frac{m}{m-2}} \right)^{\frac{m-2}{m}} \left(\int_{A_k(t)} 1 \right)^{\frac{2}{m}} = |A_k(t)|^{\frac{2}{m}} \|v\|_{\frac{2m}{m-2}}^2.$$

But v vanishes on the boundary so that by the Gagliardo-Sobolev-Nirenberg inequality we have

$$\|v\|_{\frac{2m}{m-2}}^2 \leq C \|Dv\|_2^2,$$

for which the two preceding inequalities then imply

$$\frac{|A_k|^{-\frac{2}{m}}}{C} \|v\|_2^2 \leq \|Dv\|_2^2,$$

where C is the optimal Sobolev constant. Thus by choosing $\epsilon = \frac{|A_k|^{-\frac{2}{m}}}{2\nu C}$, (7) becomes

$$\frac{1}{2} \frac{d}{dt} \int_{A_k(t)} v^2 + \frac{|A_k|^{-\frac{2}{m}}}{4\nu C} \int_{A_k(t)} v^2 \leq \frac{\nu}{2} |A_k|^{1-\frac{2}{p}} \sum_i \|f_i\|_p^2 + C\nu |A_k|^{\frac{2}{m}} |A_k|^{1-\frac{2}{r}} \|f\|_r^2.$$

We now let $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{m}$ and define $I_k(t) = \int_{A_k(t)} v^2$ so that the above equation becomes the differential inequality

$$I_k'(t) + C_1 |A_k|^{-\frac{2}{m}} I_k(t) \leq C_2 \left[|A_k|^{1-\frac{2}{p}} \sum_i \|f_i\|_p^2 + |A_k|^{1-\frac{2}{r^*}} \|f\|_r^2 \right] \quad (8)$$

The differential inequality holds for t which $A_k(t) \neq 0$. By Gronwall's Lemma we have

$$\begin{aligned} I_k(t) &\leq C_2 \int_0^t \exp \left[-C_1 \int_\eta^t |A_k(\tau)|^{-\frac{2}{m}} d\tau \right] \left[|A_k(\eta)|^{1-\frac{2}{p}} \sum_i \|f_i\|_p^2(\eta) + |A_k(\eta)|^{1-\frac{2}{r^*}} \|f\|_r^2(\eta) \right] d\eta \\ &\leq C_2 [I_k^a(t) + I_k^b(t)] \end{aligned} \quad (9)$$

Let $L := \sup_{t \in [0, T]} |A_k(t)|$ and note that $0 \leq L \leq |\Omega|$. Using Hölder's inequality (again?) we get

$$\begin{aligned} I_k^a(t) &\leq \left(\int_0^t \left(\sum_i \|f_i\|_p^2(\eta) \right)^{\frac{q}{2}} d\eta \right)^{\frac{2}{q}} \left(\int_0^t \left(\exp \left[-C_1 \int_\eta^t |A_k(\tau)|^{-\frac{2}{m}} d\tau \right] |A_k(\eta)|^{1-\frac{2}{p}} \right)^{\frac{q}{q-2}} d\eta \right)^{\frac{q-2}{q}} \\ &\leq m^{\frac{q-2}{q}} L^{1-\frac{2}{p}} \left(\sum_i \|f_i\|_{p,q}^q \right)^{\frac{2}{q}} \left(\int_0^t \exp \left[-\frac{q}{q-2} C_1 L^{-\frac{2}{m}} (t-\eta) \right] d\eta \right)^{\frac{q-2}{q}} \\ &\leq m^{\frac{q-2}{q}} \left(\sum_i \|f_i\|_{p,q}^2 \right) L^{1-\frac{2}{p}+\frac{2}{m}(1-\frac{2}{q})} \left(\frac{q-2}{C_1 q} \right)^{\frac{q-2}{q}}. \end{aligned}$$

Similarly we obtain that

$$I_k^b(t) \leq \|f\|_{r,s}^2 L^{1-\frac{2}{r^*}+\frac{2}{m}(1-\frac{2}{s})} \left(\frac{s-2}{C_1 s} \right)^{\frac{s-2}{s}},$$

which in turn implies that

$$I_k(t) \leq \widetilde{K} \widetilde{M} L^\alpha, \quad (10)$$

where $\widetilde{K} = \widetilde{K}(\nu, |\Omega|, m, p, q, r, s)$ and

$$\alpha = \min \left(1 - \frac{2}{p} + \frac{2}{m} \left(1 - \frac{2}{q} \right), 1 - \frac{2}{r^*} + \frac{2}{m} \left(1 - \frac{2}{s} \right) \right) > 1. \quad (11)$$

To finish up the proof, we use an idea (probably not originally) from [2]. Define $g(k) := \int_{A_k(t)} (u-k) dx$ and note that by Hölder's inequality and (10) we have

$$g(k) \leq \left(\int_{A_k(t)} (u-k)^2 dx \right)^{1/2} \left(\int_{A_k(t)} dx \right)^{1/2} \leq \sqrt{\widetilde{K} \widetilde{M}} |A_k(t)|^{\frac{\alpha+1}{2}}.$$

It's not hard to see that $g'(k) = -|A_k(t)|$. It now follows that:

$$\left(\widetilde{K} \widetilde{M} \right)^{-\frac{1}{\alpha+1}} \leq -(g(k))^{-\frac{2}{\alpha+1}} g'(k).$$

Now we integrate with respect to k from k_0 to $h > k_0$ to obtain:

$$\left(\widetilde{K} \widetilde{M} \right)^{-\frac{1}{\alpha+1}} (h - k_0) \leq (g(k_0))^{\frac{\alpha-1}{\alpha+1}} - (g(h))^{\frac{\alpha-1}{\alpha+1}}. \quad (12)$$

We are almost done. We only need to let $h = \max_{\Omega_T} |u|$ and we can then simplify the above estimate to get:

$$\max_{\Omega_T} |u| - \max_{\Gamma} |u| \leq \sqrt{\widetilde{K} \widetilde{M}} |\Omega|^{\frac{\alpha-1}{2}} := KM,$$

which proves the theorem. □

References

- [1] D.G Aronson, *On the Green's function for second order parabolic differential equations with discontinuous coefficients*, Bull. Amer. Math. Soc **69** (1963), 841–847.
- [2] O.A Ladyzhenskaya and N.N Ural'tseva, *Linear and quasilinear elliptic equations*, vol. 46, Academic Press, 1968.