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IMAGES OF *C*-SETS AND RELATED LARGE SETS UNDER NONHOMOGENEOUS SPECTRA

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Abstract

Let $\alpha > 0$ and $0 < \gamma < 1$. Define $g_{\alpha,\gamma} : \mathbb{N} \to \mathbb{N}$ by $g_{\alpha,\gamma}(n) = \lfloor \alpha n + \gamma \rfloor$. The set $\{g_{\alpha,\gamma}(n) : n \in \mathbb{N}\}$ is called the *nonhomogeneous spectrum of* α *and* γ. By extension, we refer to the maps $g_{\alpha,\gamma}$ as spectra. Bergelson, Hindman, and Kra showed that if A is an IP-set, a central set, an IP^*- set, or a central*-set, then $g_{\alpha,\gamma}[A]$ is the corresponding object. We extend this result to include several other notions of largeness: C-sets, J-sets, strongly central sets, and piecewise syndetic sets. Of these, C-sets are particularly interesting because they are the sets which satisfy the conclusion of the Central Sets Theorem (so have many of the strong combinatorial properties of central sets) but have a much simpler elementary description than do central sets.

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1. Introduction

Given a positive real number α, the set $\{\lfloor n\alpha\rfloor : n \in \mathbb{N}\}$ is called the *spectrum* of α. (We write $\mathbb{N}$ for the set of positive integers and ω for the set of non-negative integers.) Numerous results about spectra were derived by Skolem [18] and Bang [1]. (For a nice presentation of these results see [16].) In [19] Skolem introduced the more general sets $\{\lfloor n\alpha + \gamma\rfloor : n \in \mathbb{N}\}$, determining for example when two such sets can be disjoint. In terminology introduced by Graham, Lin, and Lin [9], the set $\{\lfloor n\alpha + \gamma\rfloor : n \in \mathbb{N}\}$ is called a *nonhomogeneous* spectrum. By extension, we refer to the function $g_{\alpha,\gamma} : \mathbb{N} \to \mathbb{N}$ defined by $g_{\alpha,\gamma}(n) = \lfloor n\alpha + \gamma\rfloor$ as a nonhomgeneous spectrum.

In [4], V. Bergelson, B. Kra, and the first author of this paper showed that if $\alpha > 0$, $0 < \gamma < 1$, and a subset A of $\mathbb{N}$ is large in any of several senses, then so is $g_{\alpha,\gamma}[A]$. The main reason anyone should care about this fact is that it provides explicit nontrivial examples of sets with these largeness properties. For example, a set A is an IP^*-set if and only if whenever $\langle x_n\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$, $FS(\langle x_n\rangle_{n=1}^{\infty}) \cap A \neq \emptyset$, where $FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. Given any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. It is trivial that for each $n \in \mathbb{N}$, $n\mathbb{N}$ is an IP^*-set — just take n terms of any sequence which are congruent mod n and add them. It is not so trivial that $\big\{\big\lfloor\sqrt{7}\cdot\lfloor n\cdot\pi + \frac{1}{e}\rfloor + \frac{2}{3}\big\rfloor : n \in \mathbb{N}\big\}$ is an IP^*-set. It is because it is $g_{\sqrt{7},2/3}\big[g_{\pi,1/e}[\mathbb{N}]\big]$.

The notions of largeness which were shown in [4] to be preserved by spectra were, in addition to the IP^*-sets mentioned above, IP-sets, central sets, central* sets, Δ-sets, and Δ^*-sets. The set A is an IP-set if and only if it contains $FS(\langle x_n\rangle_{n=1}^{\infty})$ for some sequence $\langle x_n\rangle_{n=1}^{\infty}$. (So a set is an IP^*-set if and only if it has nontrivial intersection with each IP-set.) The set A is a Δ-set if and only if it contains $\{y - x : x, y \in B \text{ and } x < y\}$ for some infinite set $B \subseteq \mathbb{N}$. And a Δ^*-set is one which has nontrivial intersection with each Δ-set.

Central sets in $\mathbb{N}$ were introduced by Furstenberg in [8] and were defined using notions from topological dynamics. They have a much simpler characterization in terms of the algebra of $\beta\mathbb{N}$. This characterization was obtained in [2] with the assistance of B. Weiss. (It was shown to be equivalent in arbitrary semigroups by H. Shi and H. Yang [17].) In order to discuss this we give a very brief overview of that algebraic structure. The reader is referred to [13] for an elementary introduction to the subject. Detailed historical references can also be found in that book. Most of the basic facts mentioned in the next two paragraphs are not due to either of the authors of [13].

Let $(S,+)$ be a semigroup, not necessarily commutative, with the discrete topology. (We denote the operation by + because we are primarily interested in the semigroup $(\mathbb{N},+)$.) The Stone-Čech compactification βS of S can be taken to be the set of ultrafilters on S, with the points of S identified with the principal ultrafilters. The topology on βS has a basis consisting of $\{\overline{A} : A \subseteq S\}$ where $\overline{A} = \{p \in \beta S : A \in p\}$. The operation extends to βS making $(\beta S,+)$ a right topological semigroup (so for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous) with S contained in its topological center (so for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$. The reader should be cautioned that $(\beta S,+)$ is very unlikely to be commutative; the center of $(\beta\mathbb{N},+)$ is $\mathbb{N}$.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal, $K(T)$, which is the union of all of the minimal right ideals of T and is also the union of all of the minimal left ideals of T. The intersection of any minimal right ideal and any minimal left ideal is a group, and any two such groups are isomorphic. (In $(\beta\mathbb{N},+)$ there are $2^{\mathfrak{c}}$ minimal left ideals and $2^{\mathfrak{c}}$ minimal right ideals.) In particular there are idempotents in $K(T)$. An idempotent in T is in $K(T)$ if and only if it is minimal with respect to the ordering of idempotents, wherein $p \leq q$ if and only if $p = p + q = q + p$. Idempotents in $K(T)$ are referred to as *minimal* idempotents.

A subset A of S is *central* if and only if A is a member of a minimal idempotent of βS. And A is *central** if and only if it has nontrivial intersection with each central set; equivalently, A is central* if and only if it is a member of every minimal idempotent in βS. It is a trivial consequence of the definition that central sets are partition regular. That is, if A is central and A is partitioned into finitely many cells, then one of these cells is central. In particular, whenever S is partitioned into finitely many parts, one of these must be central. From a combinatorial point of view, what is probably the most important fact about central sets is that they satisfy the *Central Sets Theorem*. We state here the commutative version thereof because it is much simpler to state and we are primarily concerned with $(\mathbb{N},+)$. We write ${}^{\mathbb{N}}S$ for the set of sequences in S.

Theorem 1.1. *Let $(S,+)$ be a commutative semigroup and let A be a central subset of S. There exist functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f({}^{\mathbb{N}}S) \to$*

$\mathcal{P}_f(\mathbb{N})$ *such that*

(1) *if* $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ *and* $F \subsetneq G$, *then* $\max H(F) < \min H(G)$ *and*

(2) *whenever* $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, *and for each* $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, *one has* $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.

Proof. [7, Theorem 2.2]. □

The following is the original Central Sets Theorem as proved by Furstenberg [8, Proposition 8.21].

Corollary 1.2. *Let* $l \in \mathbb{N}$ *and for each* $i \in \{1, 2, \ldots, l\}$, *let* $\langle y_{i,n} \rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{Z}$. *Let* A *be a central subset of* $\mathbb{N}$. *Then there exist sequences* $\langle a_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{N}$ *and* $\langle H_n \rangle_{n=1}^{\infty}$ *in* $\mathcal{P}_f(\mathbb{N})$ *such that*

(1) *for all* n, $\max H_n < \min H_{n+1}$ *and*

(2) *for all* $F \in \mathcal{P}_f(\mathbb{N})$ *and all* $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} (a_n + \sum_{t \in H_n} y_{i,t}) \in A$.

Proof. We first note that A is central in $\mathbb{Z}$ because by [13, Theorem 1.65 and Exercise 4.3.5], $K(\beta\mathbb{N}) = K(\beta\mathbb{Z}) \cap \beta\mathbb{N}$. Pick $\alpha : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z}) \to \mathbb{Z}$ and $H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z}) \to \mathcal{P}_f(\mathbb{N})$ as guaranteed by Theorem 1.1.

Let $G_1 = \{\overline{0}, \langle y_{1,n} \rangle_{n=1}^{\infty}, \langle y_{2,n} \rangle_{n=1}^{\infty}, \ldots, \langle y_{l,n} \rangle_{n=1}^{\infty}\}$, where $\overline{0}$ is the function constantly equal to 0. For $n \in \mathbb{N}$, pick $f \in {}^{\mathbb{N}}\mathbb{N} \setminus G_n$ and let $G_{n+1} = G_n \cup \{f\}$.

For $n \in \mathbb{N}$, let $H_n = H(G_n)$ and let $a_n = \alpha(G_n)$. To see that each $a_n \in \mathbb{N}$, note that $\overline{0} \in G_n$ so $a_n = \alpha(G_n) + \sum_{t \in H(G_n)} \overline{0}(t) \in A \subseteq \mathbb{N}$. Conclusion (1) holds directly. To verify conclusion (2), let $F \in \mathcal{P}_f(\mathbb{N})$ and $i \in \{1, 2, \ldots, l\}$. Then for each $n \in F$, $\langle y_{i,n} \rangle_{n=1}^{\infty} \in G_n$ so $\sum_{n \in F} (a_n + \sum_{t \in H_n} y_{i,t}) = \sum_{n \in F} \big(\alpha(G_n) + \sum_{t \in H(G_n)} y_{i,t}\big) \in A$. □

From Corollary 1.2 Furstenberg deduced several facts about central subsets of $\mathbb{N}$, including the fact that any partition regular system of homogeneous linear equations with coefficients from $\mathbb{Q}$ has solutions in any central set. See [13, Chapters 14 through 16] for many other examples of the strong properties enjoyed by central subsets of $\mathbb{N}$.

The algebraic description of central sets is very simple and quite easy to work with. There is an elementary description of central sets which was produced in [12] (or see [13, Theorem 14.25]). However, that description is based on the notion of a *collectionwise piecewise syndetic* family of sets — a notion which is extremely complicated. The masochistic reader is referred to [13, Definition 14.19].

For many of the facts about central sets, what is important is that they satisfy the Central Sets Theorem. We define a set to be a C-set if and only if it satisfies the conclusion of the Central Sets Theorem. We have not stated the general version of the Central Sets Theorem. In the case of commutative semigroups, the definition becomes as follows.

Definition 1.3. Let $(S,+)$ be a commutative semigroup. A set $A \subseteq S$ is a *C-set* if and only if there exist functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that

(1) if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(2) whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1,2,\ldots,m\}$, $f_i \in G_i$, one has $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.

There is an elementary characterization of C-sets [14] which is very similar in form to the elementary description of central sets. The crucial distinction is that the notion of a family of sets being collectionwise piecewise syndetic is replaced by the notion of a single set being a J-set.

Definition 1.4. Let $(S,+)$ be a commutative semigroup. A set $A \subseteq S$ is a *J-set* if and only if whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in A$.

Definition 1.5. Let S be a semigroup. Then $J(S) = \{p \in \beta S : (\forall A \in p)(A$ is a J-set$)\}$.

Of course one defines C^*-set and J^*-sets analogously to IP^*-sets and central* sets. That is a set is a C^*-set if and only if it has nonempty intersection with every C-set and a set is a J^*-set if and only if it has nonempty intersection with every J-set. Other notions of size with which we will be concerned in this paper include strongly central sets (abbreviated by SC), piecewise syndetic sets (abbreviated by PS), AP-sets, syndetic sets, and thick sets. A subset of $\mathbb{N}$ is *strongly central* if and only if it is a member of some idempotent in every minimal left ideal of $\beta\mathbb{N}$. (Since every left ideal contains a minimal left ideal, this is equivalent to being a member of an idempotent in every left ideal of $\beta\mathbb{N}$.) In $(\mathbb{N},+)$, a set A is *piecewise syndetic* if and only if there is a bound b such that there are arbitrarily long blocks of $\mathbb{N}$ in which A has no gaps longer than b. That is $(\exists b \in \mathbb{N})(\forall k \in \mathbb{N})(\exists x \in \mathbb{N})(\{x+1, x+2, \ldots, x+k\} \subseteq \bigcup_{t=1}^{b} -t + A)$. In $(\mathbb{N},+)$, a set A is an *AP-set* if and only if it contains arbitrarily long arithmetic progressions, is *syndetic* if and only if it has bounded gaps, and is *thick* if and only if it contains arbitrarily long integer intervals. We say that a set is PS^* if and only if it has nonempty intersection with each piecewise syndetic set and is AP^* if and only if it has nonempty intersection with each AP-set.

Of the notions of size that we are considering, IP-sets, central sets, C-sets, J-sets, piecewise syndetic sets, AP-sets, and Δ-sets are all partition regular.

Notice that any J-set in $\mathbb{N}$ is an AP-set. In fact, it must contain arbitrarily long arithmetic progressions with increment taken from the finite sums of any prespecified sequence in $\mathbb{N}$. To see this, let a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ be given and let $l \in \mathbb{N}$. Let

$$F = \left\{ \langle tx_n \rangle_{n=1}^{\infty} : t \in \{1, 2, \ldots, l\} \right\}.$$

Pick $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $t \in \{1, 2, \ldots, l\}$, $a + \sum_{t \in H} tx_n \in A$. If $d = \sum_{t \in H} x_t$, then

$$\left\{ a + td : t \in \{1, 2, \ldots, l\} \right\} \subseteq A.$$

Definition 1.6. $\mathcal{AP} = \{p \in \beta\mathbb{N} : (\forall A \in p)(A \text{ is an } AP\text{-set})\}$.

In Section 2 we will describe the relationship among the various notions of size that we are considering and also present elementary proofs regarding the preservation of these notions by $g_{\alpha,\gamma}$ for those notions for which we have such elementary proofs.

In Section 3 we introduce the functions $f_\alpha : \mathbb{N} \to \mathbb{T}$ and $h_\alpha = g_{\alpha,1/2}$ and present some of the basic facts about these functions.

In Section 4 we present some background material on C-sets and J-sets, including the fact in Theorem 4.1 that in a commutative semigroup, the translate a may be taken to be a member of the J-set itself. We then present positive results about the preservation of J-sets, C-sets, and C^*-sets under spectra. We then show that if P is any property which is preserved by all spectra (including any of the properties in the left two columns of Figure 1), $\alpha > 1$, and $0 < \gamma < 1$, then a set A has property P if and only if $g_{\alpha,\gamma}[A]$ has property P. Consequently, if A is an example of something with one of those properties but not others, so is $g_{\alpha,\gamma}[A]$.

In Section 5 we show the extent to which $\widetilde{g_{\alpha,\gamma}}$ preserves certain ideals of $\beta\mathbb{N}$, where $\widetilde{g_{\alpha,\gamma}} : \beta\mathbb{N} \to \beta\mathbb{N}$ is the continuous extension of $g_{\alpha,\gamma}$.

2. The Various Notions of Size

We begin with a theorem providing characterizations in terms of $(\beta\mathbb{N},+)$ of each of the notions of size that we are considering. (Most of these characterizations hold for arbitrary semigroups.)

We let $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ and given $T \subseteq \beta\mathbb{N}$, $E(T) = \{p \in T : p+p = p\}$. Given $p \in \beta\mathbb{N}$, $-p$ is the member of $\beta\mathbb{Z}$ generated by $\{-A : A \in p\}$.

Theorem 2.1. *Let $A \subseteq \mathbb{N}$.*

(a) A is a Δ-set if and only if there is some $p \in \mathbb{N}^*$ such that $A \in -p+p$.
(b) A is an AP-set if and only if $\overline{A} \cap \mathcal{AP} \neq \emptyset$.
(c) A is an IP-set if and only if $\overline{A} \cap E(\beta\mathbb{N}) \neq \emptyset$.
(d) A is a J-set if and only if $\overline{A} \cap J(\mathbb{N}) \neq \emptyset$.
(e) A is a C-set if and only if $\overline{A} \cap E\big(J(\mathbb{N})\big) \neq \emptyset$.
(f) A is a PS-set if and only if $\overline{A} \cap K(\beta\mathbb{N}) \neq \emptyset$.
(g) A is a central set if and only if $\overline{A} \cap E\big(K(\beta\mathbb{N})\big) \neq \emptyset$.
(h) A is a syndetic set if and only if for every left ideal L of $\beta\mathbb{N}$, $\overline{A} \cap L \neq \emptyset$.
(i) A is a SC-set if and only if for every left ideal L of $\beta\mathbb{N}$, $\overline{A} \cap E(L) \neq \emptyset$.
(j) A is a thick set if and only if there exists a left ideal L of $\beta\mathbb{N}$ such that $L \subseteq \overline{A}$.
(k) A is a central set if and only if $E\big(K(\beta\mathbb{N})\big) \subseteq \overline{A}$.*
(l) A is a PS^*-set if and only if $K(\beta\mathbb{N}) \subseteq \overline{A}$.
(m) A is a C^*-set if and only if $E\big(J(\mathbb{N})\big) \subseteq \overline{A}$.
(n) A is a J^*-set if and only if $J(\mathbb{N}) \subseteq \overline{A}$.
(o) A is an IP^* set if and only if $E(\beta\mathbb{N}) \subseteq \overline{A}$.
(p) A is an AP^* set if and only if $\mathcal{AP} \subseteq \overline{A}$.
(q) A is a Δ^*-set if and only if for all $p \in \mathbb{N}^*$, $A \in -p+p$.

Proof. Statements (g) and (i) are the definitions of central and SC respectively.

Statements (a) and (q) follow from [3, Lemma 1.9(a) and (j)] respectively.

Statement (b) follows from [13, Theorem 3.11] and the fact that AP-sets are partition regular.

Statements (c), (d), (e), and (f) follow from [13, Theorems 5.12, 14.14.7, 14.15.1, and 4.40] respectively.

Statements (h) and (j) follow from [5, Theorem 2.9(d) and (c)] respectively.

Statements (k), (l), (m), (n), (o), and (p) follow from statements (g), (f), (e), (d), (c), and (b) respectively. □

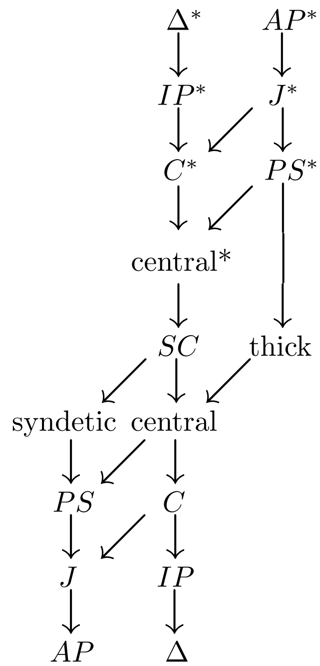

Figure 1

The diagram above shows the relations that hold among the notions of size that we are considering. After noting that, by [13, Theorems 14.4.4 and 14.5], $J(\mathbb{N})$ and $\mathcal{AP}$ are ideals of $(\beta\mathbb{N},+)$ and, as we have already observed, each J-set is an AP-set so $J(\mathbb{N}) \subseteq \mathcal{AP}$, each of the implications follows quickly from the characterizations in Theorem 2.1 except for the fact that $IP \Rightarrow \Delta$ and the corresponding fact that $\Delta^* \Rightarrow IP^*$. (To see that central* $\Rightarrow SC$ and thick $\Rightarrow$ central, one needs to note that by [13, Corollary 2.6], each left ideal of $\beta\mathbb{N}$ contains a minimal left ideal with an idempotent, which is therefore an idempotent in $K(\beta\mathbb{N})$.)

The easiest way to see that any IP-set is a Δ-set (and thus any Δ^*-set is an IP^*-set) is to note that if $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A$ and for each $n \in \mathbb{N}$, $y_n = \sum_{t=1}^n x_t$, then $\{y_n - y_m : n,m \in \mathbb{N} \text{ and } m < n\} \subseteq A$.

That none of the missing implications is valid is established by the following table which lists for each property, a subset of $\mathbb{N}$ with that property that does not have any of the other properties except those that it is forced to have because of the implications in the diagram. In that table, the set D is the set produced in [10] which is a C-set but has Banach density 0. In particular $\overline{D} \cap K(\beta\mathbb{N}) = \emptyset$. Since $D \subseteq 2\mathbb{N}$, $D+1$ is not a Δ-set.

Property	Set With No Extra Properties
Δ	$\{2^n - 2^m : n, m \in \mathbb{N} \text{ and } m < n\}$
AP	$\{2^{2n} + m2^n + 1 : n, m \in \mathbb{N} \text{ and } m < n\}$
IP	$\{\Sigma_{n\in F}\ 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$
J	$D+1$
C	D
PS	$\{2^n + 2m - 1 : n, m \in \mathbb{N} \text{ and } m < n\}$
central	$\{2^n + 2m : n, m \in \mathbb{N} \text{ and } m < n\}$
syndetic	$2\mathbb{N}+1$
SC	$2\mathbb{N} \cap \big\{x \in \mathbb{N} : (\sqrt{2}x - \lfloor\sqrt{2}x + 1/2\rfloor) \in \bigcup_{n=1}^{\infty} \big(1/(2n+1), 1/(2n)\big)\big\}$
thick	$\{2^n + m : n, m \in \mathbb{N} \text{ and } m < n\}$
central*	$2\mathbb{N} \setminus D$
PS^*	$\mathbb{N} \setminus D$
C^*	$2\mathbb{N} \setminus \{\Sigma_{n\in F}\ 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$
J^*	$\mathbb{N} \setminus (\{2^{2n} + m2^n + 1 : n, m \in \mathbb{N} \text{ and } m < n\} \cup \{\Sigma_{n\in F}\ 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\})$
IP^*	$2\mathbb{N} \setminus \{2^n - 2^m : n, m \in \mathbb{N} \text{ and } m < n\}$
AP^*	$\mathbb{N} \setminus \{\Sigma_{n\in F}\ 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$
Δ^*	$2\mathbb{N}$

In most cases it is at least routine, if not obvious, that the set has the specified property. We will explain now why this is true for J, central, SC, C^* and J^*. The set D is a member of an idempotent $p \in J(\mathbb{N})$, so $D+1 \in p+1$ and $p+1 \in J(\mathbb{N})$. The set $\{2^n + 2m : n, m \in \mathbb{N} \text{ and } m < n\}$ is the intersection of a thick set with a central* set, so is central. It is a consequence of [6, Theorem 3.2] that $\big\{x \in \mathbb{N} : (\sqrt{2}x - \lfloor\sqrt{2}x + 1/2\rfloor) \in \bigcup_{n=1}^{\infty} \big(1/(2n+1), 1/(2n)\big)\big\}$ and its complement are both strongly central so its intersection with $2\mathbb{N}$, which is IP^*, is strongly central. (We shall be giving a short proof of this consequence of [6, Theorem 3.2] in Theorem 3.3 below.) Since $\{\sum_{n\in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$ is not an AP-set, it is not a J-set so

$$2\mathbb{N} \setminus \{\textstyle\sum_{n\in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$$

is a C^*-set. And, by the Lemma 4.3 below, $\{2^{2n} + m2^n + 1 : n, m \in \mathbb{N} \text{ and } m < n\}$ is not a J-set, so $\mathbb{N} \setminus (\{2^{2n} + m2^n + 1 : n, m \in \mathbb{N} \text{ and } m < n\} \cup \{\sum_{n\in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\})$ is a J^*-set.

In all cases but two it is routine to verify that the listed set does not have any properties except those it is forced to by the implications in the diagram. The exceptions are AP and SC. One needs that the set listed for AP is not J and not Δ. It is trivially not a Δ-set. The fact that it is not a J-set is Lemma 4.3. One needs that the set listed for SC is not central* and not thick. It is trivially not thick. It is not central* because the complement of $\big\{x \in \mathbb{N} : (\sqrt{2}x - \lfloor\sqrt{2}x + 1/2\rfloor) \in \bigcup_{n=1}^{\infty} \big(1/(2n+1), 1/(2n)\big)\big\}$ is strongly central and thus central.

It was shown in [4, Theorem 6.1] that if $\alpha > 0$, $0 < \gamma < 1$, P is any of Δ, IP, central, central*, IP^*, or Δ^*, and $A \subseteq \mathbb{N}$ is a P-set, then so is $g_{\alpha,\gamma}[A]$. We shall show in the remainder of this paper that the same assertion holds if P is any of the properties in the left two columns of Figure 1. And we shall show that as long as $\alpha \leq 1$, the properties in the right column of Figure 1 are also preserved by $g_{\alpha,\gamma}$, but are not if $\alpha > 1$.

In the remainder of this section, we establish those results for which we have elementary proofs, not requiring the algebra of $\beta\mathbb{N}$.

Theorem 2.2. *Let $A \subseteq \mathbb{N}$, let $\alpha > 0$, and let $0 < \gamma < 1$.*
(1) If A is syndetic, then so is $g_{\alpha,\gamma}[A]$.
(2) If $\alpha \leq 1$ and A is thick, then so is $g_{\alpha,\gamma}[A]$.

Proof. (1). Pick $b \in \mathbb{N}$ such that for all $x \in \omega$, $\{x+1, x+2, \ldots, x+b\} \cap A \neq \emptyset$ and pick $k \in \mathbb{N}$ such that $\alpha < k$. We shall show that for all $y \in \mathbb{N}$, there exists $m \in g_{\alpha,\gamma}[A]$ such that $0 \leq m - y \leq bk$. So let $y \in \mathbb{N}$ be given and let x be the largest element of ω such that $g_{\alpha,\gamma}(x) < y$. Pick $t \in \{1, 2, \ldots, b\}$ such that $x + t \in A$ and let $m = g_{\alpha,\gamma}(x + t)$. Then $m \geq g_{\alpha,\gamma}(x + 1) \geq y$. Also $m \leq \alpha(x + t) + \gamma$ and $\alpha x + \gamma < y$ so $m - y < \alpha t < bk$.

(2). Let $b \in \mathbb{N}$. We need to show that there is some $m \in \mathbb{N}$ such that $\{m, m+1, \ldots, m+b\} \subseteq g_{\alpha,\gamma}[A]$. Pick $r \in \mathbb{N}$ such that $\alpha > \frac{1}{r}$ and pick $n \in \mathbb{N}$ such that $\{n, n+1, \ldots, n+lr\} \subseteq A$. Let $m = g_{\alpha,\gamma}(n)$. Since $\alpha \leq 1$ we have that for all $x \in \mathbb{N}$, $g_{\alpha,\gamma}(x+1) \leq g_{\alpha,\gamma}(x) + 1$ so it suffices to show that $g_{\alpha,\gamma}(n + lr) \geq m + l$. Since $m \leq \alpha n + \gamma$ and $l < \alpha lr$, this is true. □

Theorem 2.3. *Let $\alpha > 1$ and let $0 < \gamma < 1$. Then $g_{\alpha,\gamma}[\mathbb{N}]$ is not thick. In particular, none of the properties in the right column of Figure 1 are preserved by $g_{\alpha,\gamma}$.*

Proof. Pick $r \in \mathbb{N}$ such that $\frac{1}{r} < \alpha - 1$. Given $n \in \mathbb{N}$, let $k = g_{\alpha,\gamma}(n)$. Then $g_{\alpha,\gamma}(n + r) \geq k + r + 1$ so at least one of $\{k + 1, k + 2, \ldots, k + r\}$ is not in $g_{\alpha,\gamma}[\mathbb{N}]$. □

Theorem 2.4. *Let A be a piecewise syndetic subset of $\mathbb{N}$, let $\alpha > 0$, and let $0 < \gamma < 1$. Then $g_{\alpha,\gamma}[A]$ is piecewise syndetic.*

Proof. Pick $b \in \mathbb{N}$ such that for all $w \in \mathbb{N}$ there exists $x \in \mathbb{N}$ such that $\{x, x+1, \ldots, x+w\} \subseteq \bigcup_{t=0}^{b}(-t + A)$. Let $m = \lceil \alpha b + \alpha \rceil$. We claim that for all $y \in \mathbb{N}$ there exists $z \in \mathbb{N}$ such that $\{z, z+1, \ldots, z+y\} \subseteq \bigcup_{t=0}^{m}(-t + g_{\alpha,\gamma}[A])$. To this end, let $y \in \mathbb{N}$ and let $w = \lceil \frac{1}{\alpha} y \rceil$. Pick $x \in \mathbb{N}$ such that $\{x, x+1, \ldots, x+w\} \subseteq \bigcup_{t=0}^{b}(-t + A)$. Let $z = g_{\alpha,\gamma}(x)$. We claim that $\{z, z+1, \ldots, z+y\} \subseteq \bigcup_{t=0}^{m}(-t + g_{\alpha,\gamma}[A])$. To this end, let $k \in \{0, 1, \ldots, y\}$ be given. Now $y \leq \alpha w < y + \alpha$ so $z + y \leq \alpha \cdot (x+w) + \gamma < z + 1 + y + \alpha$. Thus $z + y \leq g_{\alpha,\gamma}(x+w)$. Pick the first $l \in \omega$ such that $z + k \leq g_{\alpha,\gamma}(x+l)$ and note that $l \in \{0, 1, \ldots, w\}$. Pick $t \in \{0, 1, \ldots, b\}$ such that $x + l + t \in A$. Let $v = g_{\alpha,\gamma}(x+l+t) - (z+k)$. Then $z + k + v \in g_{\alpha,\gamma}[A]$ so it suffices to show that $0 \leq v \leq m$. Since $g_{\alpha,\gamma}(x+l+t) \geq g_{\alpha,\gamma}(x) = z$, we have $v \geq 0$. By the choice of l, $z + k > g_{\alpha,\gamma}(x+l-1)$ so $\alpha x + \alpha l - \alpha < z + k$. Consequently $\alpha x + \alpha l + \alpha t + \gamma < z + k + \alpha t + \alpha \leq z + k + \alpha b + \alpha$ so $g_{\alpha,\gamma}(x+l+t) < z + k + \alpha b + \alpha \leq z + k + m$ as required. □

Theorem 2.5. *Let A be an AP-set, let $\alpha > 0$, and let $0 < \gamma < 1$. Then $g_{\alpha,\gamma}[A]$ is an AP-set.*

Proof. Let $r \in \mathbb{N} \setminus \{1, 2\}$. We need to show that $g_{\alpha,\gamma}[A]$ contains a length r arithmetic progression. By van der Waerden's Theorem, pick $m \in \mathbb{N}$ such that whenever a length m arithmetic progression is 2-colored, it contains a length r monochromatic arithmetic progression. For $i \in \{0, 1\}$, let $B_i = \{x \in \mathbb{N} : g_{\alpha,\gamma}(x) + \frac{i}{2} \leq x\alpha + \gamma < g_{\alpha,\gamma}(x) + \frac{i+1}{2}\}$. Pick a length m arithmetic progression P contained in A and pick $i \in \{0, 1\}$ and a length r arithmetic progression $\{a, a+d, \ldots, a+(r-1)d\}$ contained in $P \cap B_i$. We claim that

$$\{g_{\alpha,\gamma}(a), g_{\alpha,\gamma}(a+d), \ldots, g_{\alpha,\gamma}(a+(r-1)d)\}$$

is an arithmetic progression. (It is contained in $g_{\alpha,\gamma}[P]$ and is therefore contained in $g_{\alpha,\gamma}[A]$.)

To see this, let $k = g_{\alpha,\gamma}(a)$ and let $c = g_{\alpha,\gamma}(a+d) - k$. Now

$$k + \tfrac{i}{2} \leq a\alpha + \gamma < k + \tfrac{i+1}{2} \text{ and}$$
$$k + c + \tfrac{i}{2} \leq (a+d)\alpha + \gamma < k + c + \tfrac{i+1}{2} \text{ so}$$

$$(*) \qquad c - \tfrac{1}{2} < d\alpha < c + \tfrac{1}{2}\,.$$

We claim that for each $t \in \{0, 1, \ldots, r-1\}$, $g_{\alpha,\gamma}(a + td) = k + tc$. This is trivially true for $t \in \{0, 1\}$, so assume that $t \in \{1, 2, \ldots, r-2\}$ and that $g_{\alpha,\gamma}(a + td) = k + tc$. Let $m = g_{\alpha,\gamma}(a + (t+1)d)$. Then

$$\begin{aligned} k + tc + \tfrac{i}{2} \le (a + td)\alpha + \gamma < k + tc + \tfrac{i+1}{2} \text{ and} \\ m + \tfrac{i}{2} \le (a + (t+1)d)\alpha + \gamma < m + \tfrac{i+1}{2} \text{ so} \end{aligned}$$

$$(**) \qquad m - k - tc - \tfrac{1}{2} < d\alpha < m - k - tc + \tfrac{1}{2}\,.$$

Combining $(*)$ and $(**)$ we have

$$\begin{aligned} c - \tfrac{1}{2} < m - k - tc + \tfrac{1}{2} \text{ and} \\ m - k - tc - \tfrac{1}{2} < c + \tfrac{1}{2}\,. \end{aligned}$$

Thus we have that $k+(t+1)c-1 < m < k+(t+1)c+1$ and so $k+(t+1)c-1 = m$. □

Corollary 2.6. *Let $\alpha > 0$ and let $0 < \gamma < 1$. Then $\widetilde{g_{\alpha,\gamma}}[\mathcal{AP}] \subseteq \mathcal{AP}$.*

Proof. Let $p \in \mathcal{AP}$ and let $A \in \widetilde{g_{\alpha,\gamma}}(p)$. Pick $B \in p$ such that $g_{\alpha,\gamma}[B] \subseteq A$. Then B is an AP-set and by Theorem 2.5, $g_{\alpha,\gamma}[B]$ is an AP-set and thus A is an AP-set. □

3. The Functions f_α and h_α

We consider the circle group $\mathbb{T}$ to be $\mathbb{R}/\mathbb{Z}$ and represent the points of $\mathbb{T}$ by points in the interval $[-\frac{1}{2}, \frac{1}{2})$. (That is, if $x \in [-\frac{1}{2}, \frac{1}{2})$ we write x to represent the coset $x+\mathbb{Z}$.) The reason we do this is that we will be dealing extensively with intervals in $\mathbb{R}$, so it is convenient to have members of $\mathbb{T}$ represented by elements of $\mathbb{R}$.

Given $\alpha > 0$, we let $h_\alpha = g_{\alpha,1/2}$, so that for $x \in \mathbb{N}$, h_α is the nearest integer to αx (with ties broken by rounding up). We define $f_\alpha : \mathbb{N} \to \mathbb{T}$ by, for $x \in \mathbb{N}$, $f_\alpha(x) = \alpha x - h_\alpha(x)$ and we let $\widetilde{f_\alpha} : \beta\mathbb{N} \to \mathbb{T}$ be the continuous extension of f_α. Similarly $\widetilde{g_{\alpha,\gamma}} : \beta\mathbb{N} \to \beta\mathbb{N}$ and $\widetilde{h_\alpha} : \beta\mathbb{N} \to \beta\mathbb{N}$ are the continuous extensions of $g_{\alpha,\gamma}$ and h_α respectively.

Definition 3.1. Let $\alpha > 0$. Then $Z_\alpha = \{p \in \beta\mathbb{N} : \widetilde{f_\alpha}(p) = 0\}$.

We gather some basic facts about the objects we have just defined.

Lemma 3.2. *Let $\alpha > 0$ and let $0 < \gamma < 1$. The function $\widetilde{f_\alpha}$ is a homomorphism from $\beta\mathbb{N}$ into $\mathbb{T}$ and consequently, each idempotent is in Z_α. The* *restriction of* $\widetilde{h_\alpha}$ *to* Z_α *is an isomorphism onto* $Z_{1/\alpha}$ *whose inverse is the restriction of* $\widetilde{h_{1/\alpha}}$ *to* $Z_{1/\alpha}$*. For all* $p \in Z_\alpha$*,* $\widetilde{g_{\alpha,\gamma}}(p) = \widetilde{h_\alpha}(p)$*.*

Proof. The first conclusion follows from [13, Corollary 4.22]. The second and third conclusions are [4, Theorem 5.10] and [4, Theorem 5.8] respectively. □

As promised earlier, we provide a short proof of the consequence of [6, Theorem 3.2] that we have used in the previous section.

Theorem 3.3. *If* U *is an open subset of* $(0, \frac{1}{2})$ *with* $0 \in c\ell U$ *and* $\alpha > 0$ *is irrational, then* $f_\alpha^{-1}[U]$ *is strongly central.*

Proof. Let L be a minimal left ideal of $\beta\mathbb{N}$. Pick sequences $\langle a_n\rangle_{n=1}^\infty$ and $\langle b_n\rangle_{n=1}^\infty$ in $(0, \frac{1}{2})$ converging to 0 with $a_{n+1} < b_n < a_n$ for each n such that $\bigcup_{n=1}^\infty (b_n, a_n) \subseteq U$. For each $n \in \mathbb{N}$ pick c_n and d_n such that $b_n < d_n < c_n < a_n$ and pick by Kronecker's Theorem x_n such that $d_n < f_\alpha(x_n) < c_n$. Pick $p \in \mathbb{N}^* \cap \overline{\{x_n : n \in \mathbb{N}\}}$ and note that $p \in Z_\alpha$. Pick by [13, Theorem 1.61] an idempotent $q \in L \cap (p + \beta\mathbb{N})$ and pick $r \in \beta\mathbb{N}$ such that $q = p + r$. By Lemma 3.2 $\widetilde{f}_\alpha(q) = \widetilde{f}_\alpha(p) + \widetilde{f}_\alpha(r)$ and so $r \in Z_\alpha$.

We claim that $\{x_n : n \in \mathbb{N}\} \subseteq \{y \in \mathbb{N} : -y + f_\alpha^{-1}[U] \in r\}$ so that $f_\alpha^{-1}[U] \in p+r$ as required. Let $n \in \mathbb{N}$ and let $\epsilon = \min\{d_n - b_n, a_n - c_n\}$. Then $\{z \in \mathbb{N} : f_\alpha(z) \in (-\epsilon, \epsilon)\} \in r$ and $\{z \in \mathbb{N} : f_\alpha(z) \in (-\epsilon, \epsilon)\} \subseteq (-x_n + f_\alpha^{-1}[U]$. □

A major contrast between the preservation of largeness notions established in [4] and the results of our next section is that each of the notions considered in [4], namely IP-sets, Δ-sets, central sets, IP^*-sets, central* sets, and Δ^*-sets, is determined by members of Z_α so that $\widetilde{g_{\alpha,\gamma}}$ agrees with an isomorphism on Z_α at those points. The notion of strongly central had not been introduced at the time of the writing of [4]. We pause to observe that the methods of that paper are sufficient to show that it is preserved by the functions $g_{\alpha,\gamma}$.

Theorem 3.4. *Let* A *be a strongly central subset of* $\mathbb{N}$*, let* $\alpha > 0$*, and let* $0 < \gamma < 1$*. Then* $g_{\alpha,\gamma}[A]$ *is strongly central.*

Proof. Let L be a minimal left ideal of $\beta\mathbb{N}$. We need to show that there is an idempotent in $L \cap \overline{g_{\alpha,\gamma}[A]}$. Now $L \cap Z_{1/\alpha} \neq \emptyset$ because any idempotent in L is in this intersection, so $L \cap Z_{1/\alpha}$ is a minimal left ideal of $Z_{1/\alpha}$ by [13, Theorem 1.65(2)]. Therefore $\widetilde{h_{1/\alpha}}[L \cap Z_{1/\alpha}]$ is a minimal left ideal of Z_α by Lemma 3.2. Again by [13, Theorem 1.65(2)], we may pick a minimal left ideal M of $\beta\mathbb{N}$ such that $\widetilde{h_{1/\alpha}}[L \cap Z_{1/\alpha}] = M \cap Z_\alpha$. Pick an idempotent $p \in M \cap \overline{A}$. Then $p \in Z_\alpha$ so $\widetilde{h_\alpha}(p) \in \widetilde{h_\alpha} \circ \widetilde{h_{1/\alpha}}[L \cap Z_{1/\alpha}] = L \cap Z_{1/\alpha}$. Since $A \in p$, $g_{\alpha,\gamma}[A] \in \widetilde{g_{\alpha,\gamma}}(p) = \widetilde{h_\alpha}(p)$. Thus $\widetilde{h_\alpha}(p)$ is an idempotent in L and $g_{\alpha,\gamma}[A] \in \widetilde{h_\alpha}(p)$. □

In the rest of this paper, we need to consider points of $\beta\mathbb{N}$ which f_α can take anywhere in $\mathbb{T}$. The following lemma will simplify some of the computations. In this lemma we give specific values for τ but we only really care that τ exists.

Lemma 3.5. *Let $\alpha > 0$, let $p \in \beta\mathbb{N}$, and let $\mu = \widetilde{f_\alpha}(p)$. There exist $\tau \in \{\frac{1}{2}, \frac{3}{4}\}$ and $\epsilon > 0$ such that for all $\delta \in (0, \epsilon]$, $\{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu - \delta < \alpha x < g_{\alpha,\tau}(x) + \mu + \delta\} \in p$.*

Proof. Assume first that $\mu \neq -\frac{1}{2}$. Let $\tau = \frac{1}{2}$ and let $\epsilon = \min\{\frac{1}{2} - \mu, \frac{1}{2} + \mu\}$. Let $\delta \in (0, \epsilon]$ and let $A = \{x \in \mathbb{N} : f_\alpha(x) \in (\mu - \delta, \mu + \delta)\}$. Then by the continuity of $\widetilde{f_\alpha}$, $A \in p$. We note that $A = \{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu - \delta < \alpha x < g_{\alpha,\tau}(x) + \mu + \delta\}$.

Now assume that $\mu = -\frac{1}{2}$. Let $\tau = \frac{3}{4}$ and let $\epsilon = \frac{1}{4}$. Let $\delta \in (0, \epsilon]$ and let $A = \{x \in \mathbb{N} : f_\alpha(x) \in [-\frac{1}{2}, -\frac{1}{2} + \delta)\} \cup \{x \in \mathbb{N} : f_\alpha(x) \in (\frac{1}{2} - \delta, \frac{1}{2})\}$. Then by the continuity of $\widetilde{f_\alpha}$, $A \in p$. We claim that $A \subseteq \{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu - \delta < \alpha x < g_{\alpha,\tau}(x) + \mu + \delta\}$. (Actually, equality holds, but we don't care.) To this end, let $x \in A$ and let $k = h_\alpha(x)$. Assume first that $f_\alpha(x) \in [-\frac{1}{2}, -\frac{1}{2} + \delta)$. Then $-\frac{1}{2} \leq \alpha x - k < -\frac{1}{2} + \delta$ so $k < \alpha x + \frac{3}{4} < k + 1$. Thus $k = g_{\alpha,\tau}(x)$ and so $g_{\alpha,\tau}(x) + \mu - \delta < \alpha x < g_{\alpha,\tau}(x) + \mu + \delta$. Now assume that $f_\alpha(x) \in (\frac{1}{2} - \delta, \frac{1}{2})$. Then $\frac{1}{2} - \delta < \alpha x - k < \frac{1}{2}$. Thus $k + 1 = g_{\alpha,\tau}(x)$ and so $g_{\alpha,\tau}(x) + \mu - \delta < \alpha x < g_{\alpha,\tau}(x) + \mu + \delta$. □

Lemma 3.6. *Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$, let $\alpha > 0$, and let $\epsilon > 0$. There exists $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $k \in F$, $f_\alpha\big(\sum_{t \in H} k(t)\big) \in (-\epsilon, \epsilon)$.*

Proof. Let $A = \{x \in \mathbb{N} : f_\alpha(x) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})\}$. Since $\widetilde{f_\alpha}$ is a homomorphism, for any idempotent in $\beta\mathbb{N}$, $\widetilde{f_\alpha}(p) = 0$ and so by Theorem 4.4, A is a C-set, and therefore a J-set. Pick by Theorem 4.1 some $a \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $k \in F$, $a + \sum_{t \in H} k(t) \in A$. Then given $k \in F$, $f_\alpha(a) + f_\alpha\big(\sum_{t \in H} k(t)\big) = f_\alpha\big(a + \sum_{t \in H} k(t)\big) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and $f_\alpha(a) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$, so $f_\alpha\big(\sum_{t \in H} k(t)\big) \in (-\epsilon, \epsilon)$. □

When we write that $\langle H_n \rangle_{n=1}^\infty$ is an increasing sequence in $\mathcal{P}_f(\mathbb{N})$ we mean that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$.

Lemma 3.7. *Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$, let $\alpha > 0$, and let $\epsilon > 0$. There is an increasing sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$ and each $k \in F$, $f_\alpha\big(\sum_{t\in H_n} k(t)\big) \in (-\frac{\epsilon}{2^n}, \frac{\epsilon}{2^n})$.*

Proof. Pick H_1 as guaranteed by Lemma 3.6 for $\frac{\epsilon}{2}$. Now let $n \in \mathbb{N}$ and assume that $H_1, H_2, \ldots, H_n$ have been chosen. Let $l = \max H_n$. For $k \in F$ define $r_k \in {}^{\mathbb{N}}\mathbb{N}$ by, for $t \in \mathbb{N}$, $r_k(t) = k(l+t)$. Pick by Lemma 3.6, $L \in \mathcal{P}_f(\mathbb{N})$ such that for all $k \in F$, $f_\alpha\big(\sum_{t\in L} r_k(t)\big) \in (-\frac{\epsilon}{2^n}, \frac{\epsilon}{2^n})$. Let $H_{n+1} = l + L$. □

4. Preservation of *J*-Sets, *C*-Sets, and C^*-Sets

We begin by showing as promised that in a commutative semigroup, the translate in the definition of J-sets can be taken to be in the given J-set.

Theorem 4.1. *Let $(S, +)$ be a commutative semigroup and let A be a J-set in S. For each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.*

Proof. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ be given and pick $c \in S$. Denote by $\overline{c}$ the sequence constantly equal to c. For $f \in F$ define $g_f \in {}^{\mathbb{N}}S$ by, for each $n \in \mathbb{N}$, $g_f(n) = c + f(n)$. Let $K = \{\overline{c}\} \cup \{g_f : f \in F\}$. Then $K \in \mathcal{P}_f({}^{\mathbb{N}}S)$ so pick $b \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $b + \sum_{t\in H} c \in A$ and for each $f \in F$, $b + \sum_{t\in H} g_f(t) \in A$. Let $a = b + \sum_{t\in H} c$. Then for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$. □

Lemma 4.2. *Let $A = \{2^{2n} + m2^n + 1 : n, m \in \mathbb{N}$ and $m < n\}$. If $a, d \in \mathbb{N}$ and $\{a, a+d, a+2d\} \subseteq A$, then there is some $n \in \mathbb{N}$ such that $\{a, a+d, a+2d\} \subseteq \big\{2^{2n} + m2^n + 1 : m \in \{1, 2, \ldots, n-1\}\big\}$.*

Proof. Suppose first we have some $k, n \in \mathbb{N}$ with $k < n$, some $r \in \{1, 2, \ldots, k-1\}$, and some $s \in \{1, 2, \ldots, n-1\}$ such that $a = 2^{2k} + r2^k + 1$ and $a + d = 2^{2n} + s2^n + 1$. Then $a \leq 2^{2n-2} + (n-2)2^{n-1} + 1 \leq 2^{2n-2} + 2^{n-1}2^{n-1} = 2^{2n-1}$ and $a+d > 2^{2n}$ so $d > 2^{2n-1}$. Thus $a+2d > 2^{2n} + 2^{2n-1} > 2^{2n} + (n-1)2^n + 1$. Also, $d < a + d \leq 2^{2n+1}$ so $a + 2d < 2^{2n+2}$. Since $2^{2n} + (n-1)2^n + 1 < a + 2d < 2^{2n+2}$, $a + 2d \notin A$.

Thus we must have some $n \in \mathbb{N}$ and $r, s \in \{1, 2, \ldots, n-1\}$ such that $a = 2^{2n} + r2^n + 1$ and $a + d = 2^{2n} + s2^n + 1$. So $d = (s-r)2^n$ and $a + 2d = 2^{2n} + (2s-r)2^n + 1 < 2^{2n+2}$. Therefore $a + 2d \in \big\{2^{2n} + m2^n + 1 : m \in \{1, 2, \ldots, n-1\}\big\}$. □

Lemma 4.3. *Let $A = \{2^{2n} + m2^n + 1 : n, m \in \mathbb{N}$ and $m < n\}$. Then A is not a J-set.*

Proof. Suppose that A is a J-set. Let

$$F = \{\langle 2^t\rangle_{t=1}^{\infty}, \langle 2^{t+1}\rangle_{t=1}^{\infty}, \langle 2^{2t}\rangle_{t=1}^{\infty}, \langle 2^{2t+1}\rangle_{t=1}^{\infty}\}.$$

Pick by Theorem 4.1 some $a \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $a + \sum_{t\in H} f(t) \in A$. Pick $n \in \mathbb{N}$ and $r \in \{1, 2, \ldots, n-1\}$ such that $a = 2^{2n} + r2^n + 1$. Let $b = \sum_{t\in H} 2^t$ and let $d = \sum_{t\in H} 2^{2t}$. Then $\{a, a+b, a+2b\} \subseteq A$ and $\{a, a+d, a+2d\} \subseteq A$ so by Lemma 4.2 we have $\{a, a+b, a+2b, a+d, a+2d\} \subseteq \big\{2^{2n} + m2^n + 1 : m \in \{1, 2, \ldots, n-1\}\big\}$. Pick $m \in \{1, 2, \ldots, n-1\}$ such that $a + b = 2^{2n} + m2^n + 1$. Then $b = (m - r)2^n$ so 2^n divides $\sum_{t\in H} 2^t$ and thus $\min H \geq n$. But then $d \geq 2^{2n}$ and so $a + d > 2^{2n} + (n-1)2^n + 1$, a contradiction. □

Theorem 4.4. *Let S be a semigroup. Then $J(S)$ is a closed two sided ideal of βS and a subset A of S is a C-set if and only if there is an idempotent $p \in J(S) \cap \overline{A}$.*

Proof. [7, Theorems 3.5 and 3.8]. □

Thus the relationship between C-sets and J-sets is very similar to the relationship between central sets and piecewise syndetic sets: however there are a couple of contrasts. First, $J(S)$ is closed while $K(\beta S)$ is commonly not closed. (In particular $K(\beta\mathbb{N})$ is not closed.) By [13, Corollary 4.41], $\{p \in \beta S : (\forall A \in p)(p$ is piecewise syndetic$)\} = c\ell K(\beta S)$. Second, $K(T)$ makes sense in an arbitrary compact Hausdorff right topological semigroup. We know of no reasonable meaning for $J(T)$ in that generality.

Theorem 4.5. *Let S be a semigroup. If $B \cup C$ is a J-set, then either B or C is a J-set. Consequently, a set $A \subseteq S$ is a J-set if and only if $\overline{A} \cap J(S) \neq \emptyset$.*

Proof. The first conclusion is [15, Theorem 2.14]. The second then follows from [13, Theorem 3.11]. □

We show now that spectra preserve J-sets. As we remarked previously, this proof is complicated by the fact that $\widetilde{f_\alpha}$ can take members of $J(\mathbb{N})$ anywhere in $\mathbb{T}$.

Theorem 4.6. *Let A be a J-set in $\mathbb{N}$, let $\alpha > 0$, and let $0 < \gamma < 1$. Then $g_{\alpha,\gamma}[A]$ is a J-set.*

Proof. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ be given. Pick by Theorem 4.5 some $p \in J(\mathbb{N})$ such that $A \in p$. Let $\mu = \widetilde{f_\alpha}(p)$. Pick τ and ϵ as guaranteed by Lemma 3.5. Pick $\delta > 0$ such that $\delta \leq \epsilon$ and, if $\mu+\gamma \notin \{0,1\}$, then $(\mu+\gamma-\delta, \mu+\gamma+\delta)\cap\{0,1\} = \emptyset$.

Pick by Lemma 3.7 an increasing sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$ and each $k \in F$, $f_{1/\alpha}\big(\sum_{t\in H_n} k(t)\big) \in (-\frac{\delta}{3\alpha 2^n}, \frac{\delta}{3\alpha 2^n})$. For $k \in F$ and $n \in \mathbb{N}$, let $r_k(n) = \sum_{t\in H_n} k(t)$. We claim that

$(*)$

for each $k \in F$ and each $L \in \mathcal{P}_f(\mathbb{N})$, we have $h_\alpha\big(\sum_{n\in L} (h_{1/\alpha} \circ r_k)(n)\big) = \sum_{n\in L} r_k(n)$ and $f_\alpha\big(\sum_{n\in L} (h_{1/\alpha} \circ r_k)(n)\big) \in (-\frac{\delta}{3}, \frac{\delta}{3})$.

To see this, let $k \in F$ and $L \in \mathcal{P}_f(\mathbb{N})$ be given. For $n \in L$, let $x_n = (h_{1/\alpha} \circ r_k)(n)$. Then $f_{1/\alpha}\big(r_k(n)\big) = \frac{1}{\alpha} r_k(n) - x_n$ so $x_n - \frac{\delta}{3\alpha 2^n} < \frac{1}{\alpha} r_k(n) < x_n + \frac{\delta}{3\alpha 2^n}$ so $\alpha x_n - \frac{\delta}{3\cdot 2^n} < r_k(n) < \alpha x_n + \frac{\delta}{3\cdot 2^n}$. Therefore, $\alpha \cdot \sum_{n\in L} x_n - \frac{\delta}{3} < \sum_{n\in L} r_k(n) < \alpha \cdot \sum_{n\in L} x_n + \frac{\delta}{3}$ and consequently $\sum_{n\in L} r_k(n) - \frac{\delta}{3} < \alpha \cdot \sum_{n\in L} x_n < \sum_{n\in L} r_k(n) + \frac{\delta}{3}$ as required for $(*)$.

We have that $\{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu - \frac{\delta}{3} < \alpha x < g_{\alpha,\tau}(x) + \mu + \frac{\delta}{3}\} \in p$ and, since $J(\mathbb{N}) \subseteq \beta\mathbb{N} \setminus \mathbb{N}$, $\{x \in \mathbb{N} : \alpha x > 3\} \in p$. Let

$$C_1 = \{x \in \mathbb{N} : \alpha x > 3 \text{ and } g_{\alpha,\tau}(x) + \mu - \tfrac{\delta}{3} < \alpha x < g_{\alpha,\tau}(x) + \mu\} \text{ and let}$$
$$C_2 = \{x \in \mathbb{N} : \alpha x > 3 \text{ and } g_{\alpha,\tau}(x) + \mu \leq \alpha x < g_{\alpha,\tau}(x) + \mu + \tfrac{\delta}{3}\}\,.$$

Pick $j \in \{1,2\}$ such that $C_j \in p$. Now $A \cap C_j \in p$ so $A \cap C_j$ is a J-set. Pick by Theorem 4.1 some $a \in A \cap C_j$ and some $L \in \mathcal{P}_f(\mathbb{N})$ such that for each $k \in F$, $a + \sum_{n\in L} (h_{1/\alpha} \circ r_k)(n) \in A \cap C_j$.

Let $x = g_{\alpha,\tau}(a)$ and for $k \in F$, let

$$z_k = g_{\alpha,\gamma}\big(a + \textstyle\sum_{n\in L} (h_{1/\alpha} \circ r_k)(n)\big), \text{ let}$$
$$w_k = g_{\alpha,\tau}\big(a + \textstyle\sum_{n\in L} (h_{1/\alpha} \circ r_k)(n)\big), \text{ and let}$$
$$y_k = h_\alpha\big(\textstyle\sum_{n\in L} (h_{1/\alpha} \circ r_k)(n)\big)\,.$$

We claim that it suffices to show that there is some $b \in \mathbb{N}$ such that $z_k = y_k + b$ for each $k \in F$. Assume that we have done this and let $M = \bigcup_{n\in L} H_n$. We claim that for each $k \in F$, $b + \sum_{t\in M} k(t) = z_k$ which will suffice since $z_k \in g_{\alpha,\gamma}[A]$. So let $k \in F$ be given. By $(*)$ $z_k - b = y_k = h_\alpha\big(\sum_{n\in L} (h_{1/\alpha} \circ r_k)(n)\big) = \sum_{n\in L} r_k(n) = \sum_{n\in L} \sum_{t\in H_n} k(t) = \sum_{t\in M} k(t)$ as required.

We show first that there is some $i \in \{-1, 0, 1\}$ such that for all $k \in F$, $z_k = w_k + i$.

<u>Case 1</u>. $j=1$. Then

$$w_k+\mu-\tfrac{\delta}{3}<\alpha\cdot\big(a+\textstyle\sum_{n\in L}(h_{1/\alpha}\circ r_k)(n)\big)<w_k+\mu \text{ so}$$
$$w_k+\mu+\gamma-\tfrac{\delta}{3}<\alpha\cdot\big(a+\textstyle\sum_{n\in L}(h_{1/\alpha}\circ r_k)(n)\big)+\gamma<w_k+\mu+\gamma\,.$$

If $\mu+\gamma\leq 0$, then $z_k=w_k-1$. If $0<\mu+\gamma\leq 1$, then $0<\mu+\gamma-\frac{\delta}{3}$ so $z_k=w_k$. If $1<\mu+\gamma$, then $1<\mu+\gamma-\frac{\delta}{3}$ so $z_k=w_k+1$.

<u>Case 2</u>. $j=2$. Then

$$w_k+\mu\leq\alpha\cdot\big(a+\textstyle\sum_{n\in L}(h_{1/\alpha}\circ r_k)(n)\big)<w_k+\mu+\tfrac{\delta}{3} \text{ so}$$
$$w_k+\mu+\gamma\leq\alpha\cdot\big(a+\textstyle\sum_{n\in L}(h_{1/\alpha}\circ r_k)(n)\big)+\gamma<w_k+\mu+\gamma+\tfrac{\delta}{3}\,.$$

If $\mu+\gamma<0$, then $\mu+\gamma+\frac{\delta}{3}<0$ so $z_k=w_k-1$. If $0\leq\mu+\gamma<1$, then $\mu+\gamma+\frac{\delta}{3}<1$ so $z_k=w_k$. If $1\leq\mu+\gamma$, then $z_k=w_k+1$.

Now to complete the proof that there is some $b\in\mathbb{N}$ with $z_k=y_k+b$ for each $k\in F$, we show that $x>2$ and $w_k=y_k+x$ for each $k\in F$. We know that

$$w_k+\mu-\tfrac{\delta}{3}<\alpha\cdot\big(a+\textstyle\sum_{n\in L}(h_{1/\alpha}\circ r_k)(n)\big)<w_k+\mu+\tfrac{\delta}{3}$$

and by $(*)$, $-\frac{\delta}{3}<\alpha\cdot\big(\sum_{n\in L}(h_{1/\alpha}\circ r_k)(n)\big)-y_k<\frac{\delta}{3}$ so $w_k-y_k+\mu-\frac{2\delta}{3}<\alpha a<w_k-y_k+\mu+\frac{2\delta}{3}$. Also $x+\mu-\frac{\delta}{3}<\alpha a<x+\mu+\frac{\delta}{3}$. Thus $x+\mu-\frac{\delta}{3}<w_k-y_k+\mu+\frac{2\delta}{3}$ and $w_k-y_k+\mu-\frac{2\delta}{3}<x+\mu+\frac{\delta}{3}$ so $x-\delta<w_k-y_k<x+\delta$ so $w_k-y_k=x$. Also, $3<\alpha a<x+\mu+\frac{\delta}{3}$, so $x\geq 3$. □

The proof of Theorem 4.6 uses the fact from Theorem 4.5 that any J-set is a member of an ultrafilter in $J(\mathbb{N})$, and therefore uses the Axiom of Choice. We suspect that with a bit more work, the proof can be rewritten to only depend on the fact that J-sets are partition regular. The proof of that fact in [15, Theorem 2.14] is elementary, using the Hales-Jewett Theorem, but is rather complicated.

Corollary 4.7. *Let $\alpha>0$, and let $0<\gamma<1$. Then $\widetilde{g_{\alpha,\gamma}}[J(\mathbb{N})]\subseteq J(\mathbb{N})$.*

Proof. Let $p\in J(\mathbb{N})$ and let $A\in\widetilde{g_{\alpha,\gamma}}(p)$. Pick $B\in p$ such that $g_{\alpha,\gamma}[B]\subseteq A$. Then B is a J-set and by Theorem 4.6, $g_{\alpha,\gamma}[B]$ is a J-set and thus A is a J-set. □

Corollary 4.8. *Let A be a C-set in $\mathbb{N}$, let $\alpha>0$, and let $0<\gamma<1$. Then $g_{\alpha,\gamma}[A]$ is a C-set.*

Proof. Pick by Theorem 4.4 an idempotent $p\in\overline{A}\cap J(\mathbb{N})$. Then $g_{\alpha,\gamma}[A]\in\widetilde{g_{\alpha,\gamma}}(p)$. By Corollary 4.7, $\widetilde{g_{\alpha,\gamma}}(p)\in J(\mathbb{N})$ while by Lemma 3.2, since p

is an idempotent and is therefore in Z_α, $\widetilde{g_{\alpha,\gamma}}(p) = \widetilde{h_\alpha}(p)$ and $\widetilde{h_\alpha}(p)$ is an idempotent. $\square$

While we know that $\widetilde{g_{\alpha,\gamma}}$ is a homomorphism on Z_α, it is not a homomorphism on $\beta\mathbb{N}$ or even on $\beta\mathbb{N}\setminus\mathbb{N}$. The following lemma is a partial result in that direction.

Lemma 4.9. *Let $\alpha > 0$, let $0 < \gamma < 1$, let $q \in \beta\mathbb{N}\setminus\mathbb{N}$, and let $r \in Z_\alpha$. Then $\widetilde{g_{\alpha,\gamma}}(q+r) = \widetilde{g_{\alpha,\gamma}}(q) + \widetilde{g_{\alpha,\gamma}}(r)$.*

Proof. We shall show that there is some $X \in q$ such that for all $x \in X$, $\widetilde{g_{\alpha,\gamma}}(x+r) = g_{\alpha,\gamma}(x) + \widetilde{g_{\alpha,\gamma}}(r)$. This will suffice since then the continuous functions $\widetilde{g_{\alpha,\gamma}} \circ \rho_r$ and $\rho_{\widetilde{g_{\alpha,\gamma}(r)}} \circ \widetilde{g_{\alpha,\gamma}}$ agree on a member of q and therefore agree at q. Once we have defined X, we will let $x \in X$ and produce $Y \in r$ such that for all $y \in Y$, $g_{\alpha,\gamma}(x+y) = g_{\alpha,\gamma}(x)+g_{\alpha,\gamma}(y)$, so that the continuous functions $\widetilde{g_{\alpha,\gamma}} \circ \lambda_x$ and $\lambda_{g_{\alpha,\gamma}(x)} \circ \widetilde{g_{\alpha,\gamma}}$ agree on a member of r.

Let $\mu = \widetilde{f_\alpha}(q)$. Pick by Lemma 3.5, $\tau \in \{\frac{1}{2}, \frac{3}{4}\}$ and $\epsilon \in (0, \frac{1}{2})$ such that for all $\delta \in (0, \epsilon]$, $\{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu - \delta < \alpha x < g_{\alpha,\tau}(x) + \mu + \delta\} \in q$. We consider two cases (with three subcases each). Only in cases 1b and 1c does the choice of the set Y depend on the choice of x.

<u>`Case 1`</u>. $\mu + \gamma \in \{0, 1\}$. Let $i = \mu + \gamma$.

<u>`Case 1a`</u>. $\{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu = \alpha x\} \in q$. Let $X = \{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu = \alpha x\}$. Then α is rational, since otherwise $|X| \leq 1$, and thus $Y = \{y \in \mathbb{N} : f_\alpha(y) = 0\} \in r$. Let $x \in X$ and $y \in Y$ and let $k = g_{\alpha,\tau}(x)$ and $l = h_\alpha(y)$. Then $k + \mu = \alpha x$ and $l = \alpha y$. Then $k + i = \alpha x + \gamma$, $k + i + l = \alpha \cdot (x + y) + \gamma$, and $l < \alpha y + \gamma < l + 1$ so $g_{\alpha,\gamma}(x + y) = k + i + l = g_{\alpha,\gamma}(x) + g_{\alpha,\gamma}(y)$.

<u>`Case 1b`</u>. $\{x \in \mathbb{N} : g_{\alpha,\tau}(x)+\mu-\epsilon < \alpha x < g_{\alpha,\tau}(x)+\mu\} \in q$. Let $X = \{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu - \epsilon < \alpha x < g_{\alpha,\tau}(x) + \mu\}$ and let $x \in X$ be given. Let $k = g_{\alpha,\tau}(x)$ and let $\eta = \min\{\gamma, 1 - \gamma, k + \mu - \alpha x\}$. Let $Y = \{y \in \mathbb{N} : f_\alpha(y) \in (-\eta, \eta)\}$, let $y \in Y$, and let $l = h_\alpha(y)$.

Now $k + i - 1 < k + \mu + \gamma - \epsilon < \alpha x + \gamma < k + \mu + \gamma = k + i$ so $g_{\alpha,\gamma}(x) = k + i - 1$. Also $l \leq l - \eta + \gamma < \alpha y + \gamma < l + \eta + \gamma \leq l + 1$ so $g_{\alpha,\gamma}(y) = l$. Finally, note that $\alpha y < l + \eta \leq k + l + \mu - \alpha x$ so that $k+l+i-1 < k+l+\mu+\gamma-\epsilon-\eta < \alpha\cdot(x+y)+\gamma < k+l+\mu+\gamma = k+l+i$ and thus $g_{\alpha,\gamma}(x + y) = k + l + i - 1$.

<u>`Case 1c`</u>. $\{x \in \mathbb{N} : g_{\alpha,\tau}(x)+\mu < \alpha x < g_{\alpha,\tau}(x)+\mu+\epsilon\} \in q$. Let $X = \{x \in \mathbb{N} : g_{\alpha,\tau}(x) + \mu < \alpha x < g_{\alpha,\tau}(x) + \mu + \epsilon\}$ and let $x \in X$ be given. Let $k = g_{\alpha,\tau}(x)$ and let $\eta = \min\{\gamma, 1 - \gamma, \alpha x - k - \mu\}$. Let $Y = \{y \in \mathbb{N} : f_\alpha(y) \in (-\eta, \eta)\}$, let $y \in Y$, and let $l = h_\alpha(y)$.

Now $k+i = k+\mu+\gamma < \alpha x+\gamma < k+\mu+\gamma+\epsilon < k+i+1$ so $g_{\alpha,\gamma}(x) = k+i$. Exactly as in case 1b, $g_{\alpha,\gamma}(y) = l$. Observe that $\alpha y > l-\eta \geq l-\alpha x+k+\mu$ so that $k+l+i = k+l+\mu+\gamma < \alpha\cdot(x+y)+\gamma < k+l+i+\epsilon+\eta < k+l+i+1$ and thus $g_{\alpha,\gamma}(x+y) = k+l+i$.

Case 2. $\mu+\gamma \notin \{0,1\}$. Pick $\delta \in (0,\epsilon)$ such that $\delta \leq \min\{\gamma, 1-\gamma\}$ and $(\mu+\gamma-\delta, \mu+\gamma+\delta) \cap \{0,1\} = \emptyset$. Let $X = \{x \in \mathbb{N} : g_{\alpha,\tau}(x)+\mu-\frac{\delta}{2} < \alpha x < g_{\alpha,\tau}(x)+\mu+\frac{\delta}{2}\}$ and let $Y = \{y \in \mathbb{N} : f_\alpha(y) \in (-\frac{\delta}{2}, \frac{\delta}{2})\}$. Let $k = g_{\alpha,\tau}(x)$ and let $l = h_\alpha(y)$. Then $l < l-\frac{\delta}{2}+\gamma < \alpha y+\gamma < l+\frac{\delta}{2}+\gamma < l+1$ so $g_{\alpha,\gamma}(y) = l$.

Case 2a. $\mu+\gamma < 0$. Then $k-1 < k+\mu+\gamma-\frac{\delta}{2} < \alpha x+\gamma < k+\mu+\gamma+\frac{\delta}{2} < k$ so $g_{\alpha,\gamma}(x) = k-1$. Also $k+l-1 < k+l+\mu+\gamma-\delta < \alpha\cdot(x+y)+\gamma < k+l+\mu+\gamma+\delta < k+l$ so $g_{\alpha,\gamma}(x+y) = k+l-1$.

Case 2b. $0 < \mu+\gamma < 1$. Then $k < k+\mu+\gamma-\frac{\delta}{2} < \alpha x+\gamma < k+\mu+\gamma+\frac{\delta}{2} < k+1$ so $g_{\alpha,\gamma}(x) = k$. Also $k+l < k+l+\mu+\gamma-\delta < \alpha\cdot(x+y)+\gamma < k+l+\mu+\gamma+\delta < k+l+1$ so $g_{\alpha,\gamma}(x+y) = k+l$.

Case 2c. $1 < \mu+\gamma$. Then $k+1 < k+\mu+\gamma-\frac{\delta}{2} < \alpha x+\gamma < k+\mu+\gamma+\frac{\delta}{2} < k+2$ so $g_{\alpha,\gamma}(x) = k+1$. Also $k+l+1 < k+l+\mu+\gamma-\delta < \alpha\cdot(x+y)+\gamma < k+l+\mu+\gamma+\delta < k+l+2$ so $g_{\alpha,\gamma}(x+y) = k+l+1$. □

Theorem 4.10. *Let $\alpha > 0$ and let $0 < \gamma < 1$. Then $\widetilde{g_{\alpha,\gamma}}[K(\beta\mathbb{N})] \subseteq K(\beta\mathbb{N})$ and consequently $\widetilde{g_{\alpha,\gamma}}[c\ell K(\beta\mathbb{N})] \subseteq c\ell K(\beta\mathbb{N})$.*

Proof. Of course the second conclusion follows from the first by continuity. To see that $\widetilde{g_{\alpha,\gamma}}[K(\beta\mathbb{N})] \subseteq K(\beta\mathbb{N})$, let $p \in K(\beta\mathbb{N})$. Pick a minimal left ideal L of $\beta\mathbb{N}$ such that $p \in L$ and pick an idempotent $r \in L$. Then $r \in Z_\alpha$ (since $\widetilde{f_\alpha}$ is a homomorphism). Now $K(Z_\alpha) = Z_\alpha \cap K(\beta\mathbb{N})$ by [13, Theorem 1.65]. By Lemma 3.2, $\widetilde{g_{\alpha,\gamma}}(r) = \widetilde{h_\alpha}(r)$ is an idempotent in $K(Z_{1/\alpha})$ and so $\widetilde{g_{\alpha,\gamma}}(r) \in K(\beta\mathbb{N})$. Now $p \in L = L+r$ so there is some $q \in L$ such that $p = q+r$. Thus by Lemma 4.9, $\widetilde{g_{\alpha,\gamma}}(p) = \widetilde{g_{\alpha,\gamma}}(q) + \widetilde{g_{\alpha,\gamma}}(r) \in K(\beta\mathbb{N})$. □

We see now that spectra preserve C^*-sets. We saw in Theorem 2.3 that they need not preserve thick, PS^*, J^*, or AP^*-sets. The reason for the distinction seems to be that C^*-sets depend on idempotents, which are in particular members of Z_α. (All of the properties shown to be preserved in [4] also depended on points which are members of Z_α for each $\alpha > 0$.)

Theorem 4.11. *Let A be a C^*-set in $\mathbb{N}$, let $\alpha > 0$, and let $0 < \gamma < 1$. Then $g_{\alpha,\gamma}[A]$ is a C^*-set.*

Proof. We need to show that $g_{\alpha,\gamma}[A]$ is a member of every idempotent in

$J(\mathbb{N})$, so let an idempotent $p \in J(\mathbb{N})$ be given. By Lemma 3.2 and Corollary 4.7, $\widetilde{h_{1/\alpha}}(p)$ is an idempotent in $J(\mathbb{N})$ so $A \in \widetilde{h_{1/\alpha}}(p)$. Again by Lemma 3.2, $\widetilde{g_{\alpha,\gamma}}\big(\widetilde{h_{1/\alpha}}(p)\big) = \widetilde{h_\alpha}\big(\widetilde{h_{1/\alpha}}(p)\big) = p$ and so $g_{\alpha,\gamma}[A] \in p$. □

In [13, Corollary 16.43] it was shown that if $\alpha \geq 1$, $0 < \gamma < 1$, P is any of the properties central, central*, IP, or IP^*, and $A \subseteq \mathbb{N}$, then $g_{\alpha,\gamma}[A]$ has property P if and only if A has property P. The arguments given there depended on the specific property. We see now that in fact, all that is required is the preservation of the property by all spectra.

Theorem 4.12. *Let $\mathcal{A}$ be a set of subsets of $\mathbb{N}$ which is closed under passage to supersets and assume that whenever $\alpha > 0$, $0 < \gamma < 1$, and $A \in \mathcal{A}$, one has that $g_{\alpha,\gamma}[A] \in \mathcal{A}$. Then whenever $\alpha \geq 1$, $0 < \gamma < 1$, and $A \subseteq \mathbb{N}$, one has $g_{\alpha,\gamma}[A] \in \mathcal{A}$ if and only if $A \in \mathcal{A}$.*

Proof. Let $\alpha \geq 1$, $0 < \gamma < 1$, and $A \subseteq \mathbb{N}$. The sufficiency holds by assumption. If $\alpha = 1$, then $g_{\alpha,\gamma}$ is the identity function so we can assume $\alpha > 1$. Notice that for all $x \in \mathbb{N}$, $g_{1/\alpha,(1-\gamma)/\alpha}\big(g_{\alpha,\gamma}(x)\big) = x$. (If $y = g_{\alpha,\gamma}(x)$, then $\frac{1}{\alpha}y \leq x + \frac{\gamma}{\alpha} < \frac{1}{\alpha}y + \frac{1}{\alpha}$ so $x < \frac{1}{\alpha}y + \frac{1-\gamma}{\alpha} \leq x + \frac{1}{\alpha} < x+1$.) Consequently, for any $A \subseteq \mathbb{N}$, $g_{1/\alpha,(1-\gamma)/\alpha}\big[g_{\alpha,\gamma}[A]\big] \subseteq A$. Thus if $g_{\alpha,\gamma}[A] \in \mathcal{A}$, one has $A \in \mathcal{A}$. □

There are a few basic constructions which have been used to produce examples of C-sets which are not central. By Theorem 4.12, if A is such an example, $\alpha > 1$, and $0 < \gamma < 1$, then $g_{\alpha,\gamma}[A]$ is another such example.

5. Preservation of Ideals

We show in this section that if $\alpha \leq 1$ and $0 < \gamma \leq 1$, then the ideals $K(\beta\mathbb{N})$, $c\ell K(\beta\mathbb{N})$, and $J(\mathbb{N})$ are preserved by $\widetilde{g_{\alpha,\gamma}}$ but fail dramatically to be preserved if $\alpha > 1$.

Lemma 5.1. *Let $p \in \beta\mathbb{N}$, let $0 < \alpha < 1$, and let $0 < \gamma < 1$. There exist $l \in \mathbb{Z}$, $\tau \in \{\frac{1}{2}, \frac{3}{4}\}$, and $A \in p$ such that for all $x \in A$, $g_{\alpha,\gamma}\big(l + g_{1/\alpha,\tau}(x)\big) = x$.*

Proof. Let $\mu = \widetilde{f_{1/\alpha}}(p)$ and pick by Lemma 3.5, $\tau \in \{\frac{1}{2}, \frac{3}{4}\}$ and $\epsilon > 0$ such that for all $\delta \in (0, \epsilon]$, $\{x \in \mathbb{N} : g_{1/\alpha,\tau}(x) + \mu - \delta < \frac{1}{\alpha}x < g_{1/\alpha,\tau}(x) + \mu + \delta\} \in p$. Let $\delta = \min\{\frac{1-\alpha}{2\alpha}, \epsilon\}$, let $A = \{x \in \mathbb{N} : g_{1/\alpha,\tau}(x) + \mu - \delta < \frac{1}{\alpha}x < g_{1/\alpha,\tau}(x) + \mu + \delta\}$, and let $x \in A$.

Now $(\mu - \frac{\gamma}{\alpha} - \delta + \frac{1}{\alpha}) - (\mu + \delta - \frac{\gamma}{\alpha}) = \frac{1}{\alpha} - 2\delta \geq \frac{1}{\alpha} - 2 \cdot \frac{1-\alpha}{2\alpha} = 1$ so pick $l \in \mathbb{Z}$ such that $\mu + \delta - \frac{\gamma}{\alpha} \leq l < \mu - \frac{\gamma}{\alpha} - \delta + \frac{1}{\alpha}$. Then $\alpha\mu + \alpha\delta - \gamma \leq \alpha l < \alpha\mu - \gamma - \alpha\delta + 1$.

Let $k = g_{1/\alpha,\tau}(x)$. We show that $g_{\alpha,\gamma}(l+k) = x$.

Now $\alpha k + \alpha\mu - \alpha\delta < x < \alpha k + \alpha\mu + \alpha\delta$ so $x - \alpha\mu - \alpha\delta + \alpha l + \gamma < \alpha\cdot(k+l) + \gamma < x - \alpha\mu + \alpha\delta + \alpha l + \gamma$, $0 \leq \alpha l - \alpha\mu - \alpha\delta + \gamma$, and $\alpha l - \alpha\mu + \alpha\delta + \gamma \leq 1$ so $x < \alpha\cdot(l+k) + \gamma < x + 1$ as required. □

Theorem 5.2. *Let I be a subset of $\beta\mathbb{N}$ which is a left ideal of $(\beta\mathbb{Z}, +)$ and assume that whenever $0 < \alpha$ and $0 < \gamma < 1$, one has $\widetilde{g_{\alpha,\gamma}}[I] \subseteq I$. Let $0 < \alpha \leq 1$ and let $0 < \gamma < 1$. Then $\widetilde{g_{\alpha,\gamma}}[I] = I$. In particular, $\widetilde{g_{\alpha,\gamma}}[K(\beta\mathbb{N})] = K(\beta\mathbb{N})$, $\widetilde{g_{\alpha,\gamma}}[c\ell K(\beta\mathbb{N})] = c\ell K(\beta\mathbb{N})$, $\widetilde{g_{\alpha,\gamma}}[J(\mathbb{N})] = J(\mathbb{N})$, and $\widetilde{g_{\alpha,\gamma}}[\mathcal{AP}] = \mathcal{AP}$.*

Proof. If $\alpha = 1$, then $g_{\alpha,\gamma}$ is the identity function, so we assume that $\alpha < 1$. We need to show that $I \subseteq \widetilde{g_{\alpha,\gamma}}[I]$. So let $p \in I$. By Lemma 5.1 we may pick $l \in \mathbb{Z}$, $\tau \in \{\frac{1}{2}, \frac{3}{4}\}$, and $A \in p$ such that for all $x \in A$, $g_{\alpha,\gamma}\big(l + g_{1/\alpha,\tau}(x)\big) = x$. By assumption $\widetilde{g_{1/\alpha,\tau}}(p) \in I$ and since I is a left ideal of $\beta\mathbb{Z}$, $l + \widetilde{g_{1/\alpha,\tau}}(p) \in I$. Since $\widetilde{g_{\alpha,\gamma}} \circ \lambda_l \circ \widetilde{g_{1/\alpha,\tau}}$ agrees with the identity on A, we have $p = \widetilde{g_{\alpha,\gamma}}\big(l + \widetilde{g_{1/\alpha,\tau}}(p)\big)$ and so $p \in \widetilde{g_{\alpha,\gamma}}[I]$.

To verify the "in particular" conclusions, by Corollary 4.7 and Theorem 4.10, we only need verify that each of $K(\beta\mathbb{N})$, $c\ell K(\beta\mathbb{N})$, $J(\mathbb{N})$, and $\mathcal{AP}$ are left ideals of $(\beta\mathbb{Z}, +)$. By [13, Lemma 1.43(c)] each minimal left ideal of $\beta\mathbb{N}$ is a left ideal of $\beta\mathbb{Z}$ so $K(\beta\mathbb{N})$ is the union of left ideals of $\beta\mathbb{Z}$ and is thus a left ideal of $\beta\mathbb{Z}$. Then by [13, Theorem 2.17], $c\ell K(\beta\mathbb{N})$ is a left ideal of $\beta\mathbb{Z}$. To see that $J(\mathbb{N})$ is a left ideal of $\beta\mathbb{Z}$, let $p \in J(\mathbb{N})$. Then $\beta\mathbb{Z} + p = c\ell(\mathbb{Z} + p)$ so it suffices to show that $\mathbb{Z} + p \subseteq J(\mathbb{N})$. So let $m \in \mathbb{Z}$ and let $A \in m + p$. We need to show that $A \cap \mathbb{N}$ is a J-set in $\mathbb{N}$ so let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ be given. We have that $-m + A \in p$ and $\{x \in \mathbb{N} : x > -m\} \in p$ so pick by Theorem 4.1, some $a \in (-m + A) \cap \{x \in \mathbb{N} : x > -m\}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $k \in F$, $a + \sum_{t\in H} k(t) \in (-m + A)$. Then $m + a \in \mathbb{N}$ and for all $k \in F$, $m + a + \sum_{t\in H} k(t) \in A$. The proof that $\mathcal{AP}$ is a left ideal of $\beta\mathbb{Z}$ is nearly identical. Given $p \in \mathcal{AP}$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$, and $A \in m + p$, $(-m + A) \cap \{x \in \mathbb{N} : x > -m\} \in p$ so it contains a length k arithmetic progression $\{a, a+d, \ldots, a+(k-1)d\}$. Then $\{m+a, m+a+d, \ldots, m+a+(k-1)d\}$ is a length k arithmetic progression in $A \cap \mathbb{N}$. □

We saw in Theorem 2.2 that if $\alpha \leq 1$ and $0 < \gamma < 1$, then $g_{\alpha,\gamma}$ preserves thick sets.

Corollary 5.3. *Let $0 < \alpha \leq 1$, let $0 < \gamma < 1$, and let $A \subseteq \mathbb{N}$. If A is a PS^*-set, then $g_{\alpha,\gamma}[A]$ is a PS^*-set. If A is a J^*-set, then $g_{\alpha,\gamma}[A]$ is a J^*-set. If A is an AP^*-set, then $g_{\alpha,\gamma}[A]$ is an AP^*-set.*

Proof. The proofs are essentially the same. We will do the proof for J^*-sets. Let A be a J^*-set. Then $J(\mathbb{N}) \subseteq \overline{A}$ so $J(\mathbb{N}) = \widetilde{g_{\alpha,\gamma}}[J(\mathbb{N})] \subseteq \widetilde{g_{\alpha,\gamma}}[\overline{A}] = \overline{g_{\alpha,\gamma}[A]}$. □

We saw in Theorem 2.3 that if $\alpha > 1$, then the conclusion of Corollary 5.3 fails badly. We see now that if $\alpha > 1$, then the conclusion of Theorem 5.2 also fails badly.

Theorem 5.4. *Let $\alpha > 1$ and let $0 < \gamma < 1$. There exists $x \in \mathbb{N}$ such that for every $p \in Z_{1/\alpha}$, $g_{\alpha,\gamma}[\mathbb{N}] \notin x + p$. Consequently, if I is a left ideal of $\beta\mathbb{N}$, then $I \setminus \widetilde{g_{\alpha,\gamma}}[\beta\mathbb{N}] \neq \emptyset$.*

Proof. Assume first that α is rational and pick $m, n \in \mathbb{N}$ such that $m > n$, $\alpha = \frac{m}{n}$, and m and n are relatively prime. Pick $i \in \{0, 1, \ldots, n-1\}$ such that $\frac{i}{n} \leq \gamma < \frac{i+1}{n}$. Pick $t \in \mathbb{N}$ such that $mt \equiv -i - 1 \pmod{n}$. (See for example [11, Theorem 56].) Pick $x \in \mathbb{N}$ such that $mt = xn - i - 1$. Then $\frac{m}{n} \cdot t + \gamma < \frac{m}{n} \cdot t + \frac{i+1}{n} = x$. Also, $\frac{m}{n} \cdot (t+1) + \gamma \geq \frac{m}{n} \cdot t + \frac{m}{n} + \frac{i}{n} = x + \frac{m-1}{n} \geq x + 1$. Therefore

$$(*) \qquad \text{if } l \in \mathbb{N} \text{ and } \frac{m}{n} \cdot l + \gamma \geq x, \text{ then } \frac{m}{n} \cdot l + \gamma \geq x + 1\,.$$

Now let $p \in Z_{1/\alpha} = Z_{n/m} = \overline{m\mathbb{N}}$ and suppose that $g_{\alpha,\gamma}[\mathbb{N}] \in x + p$. Then $(-x + g_{\alpha,\gamma}[\mathbb{N}]) \cap m\mathbb{N} \in p$ so pick $k \in \mathbb{N}$ such that $x + mk \in g_{\alpha,\gamma}[\mathbb{N}]$ and pick $r \in \mathbb{N}$ such that $x + mk = g_{\alpha,\gamma}(r)$. Then $x + mk \leq \frac{m}{n} \cdot r + \gamma < x + mk + 1$ so $x \leq \frac{m}{n} \cdot (r - nk) + \gamma < x + 1$, contradicting $(*)$.

Now assume that α is irrational, so that $\{\alpha t : t \in \mathbb{N}\}$ is dense mod 1. (See for example [11, Theorem 36].) Pick $x, t \in \mathbb{N}$ such that $x - \gamma - (\alpha - 1) < \alpha t < x - \gamma$. Then $x - (\alpha - 1) < \alpha t + \gamma < x$ and $\alpha \cdot (t+1) + \gamma > x + 1$. Let $\delta = \min\{x - (\alpha t + \gamma), \alpha \cdot (t+1) + \gamma - (x+1), \frac{1}{2}\}$. We claim that for any integer l,

$$(**) \qquad \text{if } \alpha l + \gamma > x - \delta, \text{ then } \alpha l + \gamma \geq x + 1 + \delta\,.$$

To see this, let $l \in \mathbb{Z}$. If $l \leq t$, then $\alpha l + \gamma \leq \alpha t + \gamma \leq x - \delta$. So assume that $l \geq t + 1$. Then $\alpha l + \gamma \geq \alpha t + \alpha + \gamma \geq x + 1 + \delta$.

Now let $p \in Z_{1/\alpha}$ and suppose that $g_{\alpha,\gamma}[\mathbb{N}] \in x + p$. Then $(-x + g_{\alpha,\gamma}[\mathbb{N}]) \cap \{y \in \mathbb{N} : f_{1/\alpha}(y) \in (-\frac{\delta}{\alpha}, \frac{\delta}{\alpha})\} \in p$ so pick $y \in \mathbb{N}$ such that $f_{1/\alpha}(y) \in (-\frac{\delta}{\alpha}, \frac{\delta}{\alpha})$ and $x + y \in g_{\alpha,\gamma}[\mathbb{N}]$. Pick $r \in \mathbb{N}$ such that $x + y = g_{\alpha,\gamma}(r)$. Let $k = h_{1/\alpha}(y)$. Then $k - \frac{\delta}{\alpha} < \frac{1}{\alpha} \cdot y < k + \frac{\delta}{\alpha}$ so $\alpha k - \delta < y < \alpha k + \delta$. Therefore $x + \alpha k - \delta < x + y \leq \alpha r + \gamma < x + y + 1 < x + \alpha k + \delta + 1$ so $x - \delta < \alpha \cdot (r - k) + \gamma < x + \delta + 1$, which is a contradiction to $(**)$.

To verify the last conclusion, pick x as guaranteed and let I be a left ideal of $\beta\mathbb{N}$. Then I contains a minimal left ideal hence contains an idempotent p which is necessarily in $Z_{1/\alpha}$. Then $x + p \in I$. Now $\widetilde{g_{\alpha,\gamma}}[\beta\mathbb{N}] = c\ell g_{\alpha,\gamma}[\mathbb{N}]$ so $x + p \notin \widetilde{g_{\alpha,\gamma}}[\beta\mathbb{N}]$. □

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