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SPOTTED TILINGS AND n-COLOR COMPOSITIONS

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Abstract

Spotted tilings are presented as a new combinatorial interpretation of n-color compositions. Previous and new results are proven using this tool, an interpretation of the Fibonacci numbers, and a case of Terquem's problem. The spotted tilings allow for MacMahon's zig-zag graphs to be applied to n-color compositions, addressing a question of Agarwal in the article where these compositions were introduced in 2000.

1. Background

For a given positive integer n, the compositions of n are ordered t-tuples $(c_1, \ldots, c_t)$ of positive integers with $c_1 + \cdots + c_t = n$. For instance, there are four compositions of 3, namely $(3), (2, 1), (1, 2)$, and $(1, 1, 1)$. The individual c_i are called parts of the composition. Compositions are sometimes referred to as ordered partitions.

Compositions of n may be represented graphically as tilings of a $1 \times n$ board, where a part k corresponds to a $1 \times k$ rectangle. This is essentially MacMahon's construction using nodes on a line [7]. Figure 1 shows the tilings for the compositions of 3. Notice that each tiling has a vertical bar at the leftmost edge, possibly vertical bars separating the rectangles corresponding to parts, and a vertical bar at the rightmost edge.

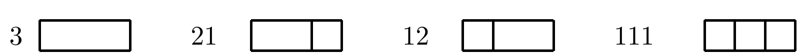

Figure 1: The 4 tilings representing the compositions of 3.

Agarwal [1] introduced the concept of n-colored compositions, where a part k has one of k possible colors, denoted by a subscript $1, \ldots, k$. There are eight n-colored compositions of 3, namely

$$(3_1),(3_2),(3_3),(2_1,1_1),(2_2,1_1),(1_1,2_1),(1_1,2_2),(1_1,1_1,1_1).$$

Let $CC(n)$ denote the number of n-color compositions of n, so $CC(3) = 8$. These compositions, inspired by the analogous concept of n-color partitions, have been further considered in several articles and monographs.

We introduce a combinatorial tool, spotted tilings, to provide a graphic representation of n-colored compositions. A part k_i corresponds to a $1 \times k$ rectangle with a spot in position i. Spotted tilings for the n-color compositions of 3 are shown in Figure 2. Notice that the bars and spots alternate, since there is one spot per part.

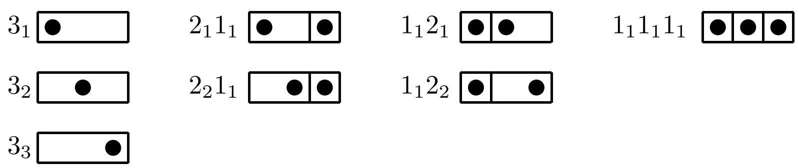

Figure 2: The 8 spotted tilings for $CC(3)$.

We will use this new tool to prove previous and new results about colored compositions (§2) and to develop an analog of MacMahon's zig-zag graphs (§3), addressing a question of Agarwal [1].

There are two combinatorial results we will use in the sequel. The Fibonacci numbers are given by $F_0 = 0$, $F_1 = 1$ and the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The following theorem is proven using generating functions in [6].

Theorem 1. *Let $Co(n)$ denote the number of compositions of n whose parts are all odd. Then $Co(n) = F_n$.*

Proof. Proceed by induction. Certainly $Co(0) = 0$, as 0 cannot be written as a sum of positive odd integers. Also, $Co(1) = 1$, since the single composition of 1 consists of an odd part. Assume the claim is true for $n - 1$ and $n - 2$. The compositions counted by $Co(n)$ include the odd part compositions of $n - 1$ with an additional part 1 included at the end, and the odd-part compositions of $n-2$ with the last part increased by 2 (which results in another odd part). Since any odd part composition of n ends in either 1 or a larger odd number, $Co(n-1) + Co(n-2)$ counts all odd part compositions of n by the induction hypothesis. $\square$

Terquem [9] considered the problem of counting subsequences of $\{1, \ldots, n\}$ where the terms alternate parity. For instance, such 3-term subsequences of $\{1, \ldots, 5\}$ with first term odd are $\{1, 2, 3\}$, $\{1, 2, 5\}$, $\{1, 4, 5\}$, and $\{3, 4, 5\}$. Church and Gould [4] give a combinatorial proof of Terquem's general problem. We give a different proof of the special case needed below.

Theorem 2. *The number of subsequences* $\{x_1, \ldots, x_{2j-1}\}$ *of* $\{1, ..., 2k-1\}$ *where the parity of* x_i *matches the parity of* i *is given by* $\binom{j+k-1}{2j-1}$.

Proof. Let $T(j,k)$ denote the number of desired subsequences. Since there are k odd numbers in $\{1, ..., 2k-1\}$, we have $T(1,k) = k$. To determine $T(j,k)$ inductively, there are two cases to consider. If a subsequence counted by $T(j,k)$ does not include the number $2k-1$, then it is among the subsequences counted by $T(j,k-1) = \binom{j+k-2}{2j-1}$. If a subsequence counted by $T(j,k)$ does end with the number $2k-1$, then the second to last entry is one of the even numbers $2j-2, \ldots, 2k-2$ and the number of possible $\{x_1, \ldots, x_{2j-3}\}$ are counted by $T(j-1, j-1), \ldots, T(j-1, k-1)$, respectively. Together, this gives

$$\begin{aligned} T(j,k) &= T(j,k-1) + T(j-1,k-1) + \cdots + T(j-1,j-1) \\ &= \binom{j+k-2}{2j-1} + \binom{j+k-3}{2j-3} + \cdots + \binom{2j-3}{2j-3} \\ &= \binom{j+k-2}{2j-1} + \binom{j+k-2}{2j-2} \\ &= \binom{j+k-1}{2j-1} \end{aligned}$$

using the binomial coefficient identities known as the "hockey-stick theorem" and Pascal's lemma (identities 135 and 127 of [3]). □

2. n-color Composition Enumerations

In this section, we use spotted tilings to give new proofs of enumeration results for $CC(n)$ and two special classes, $CC(n,k)$, the number of n-color compositions with k parts, and $CCe(n)$, the number of n-color compositions with each part even. We also prove a new enumeration result for $CCo(n)$, the number of n-color compositions with each part odd.

Theorem 3. ([1], Theorem 1b, d) *The number of n-color compositions of n satisfies the formula* $CC(n) = F_{2n}$ *and the recurrence relation* $CC(n) = 3 \cdot CC(n-1) - CC(n-2)$. *The corresponding generating function is* $\frac{q}{1-3q+q^2}$.

Proof. We establish $CC(n) = Co(2n)$ by a bijection between n-color compositions of n and (normal) compositions of $2n$ with all parts odd.

Consider a spotted tiling for an n-color composition of n. Recall that the bars and spots alternate. Consider the spot signifying part k_i to be positioned at $i - \frac{1}{2}$ within the $1 \times k$ rectangle. Then the distance between a spot and an adjacent bar has the form $m + \frac{1}{2}$ for some integer $m \geq 0$. Create a (normal) tiling of $2n$ by replacing spots with bars and doubling all distances, which are now of the form $2m+1$, odd numbers. Figure 3 shows the transition from spotted tilings counted by $CC(3)$ to tilings counted by $Co(6)$.

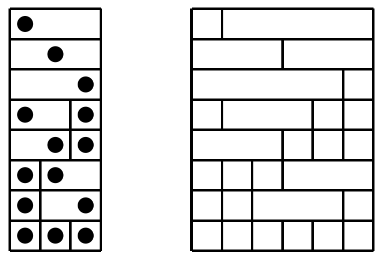

Figure 3: Row by row correspondence between $CC(3)$ and $Co(6)$.

The process is invertible because the bars in an all-odd composition of $2n$ alternate even / odd, considering the leftmost bar to be at 0. Because $2n$ is even, there are at least bars at 0, some odd position, and $2n$. To create the corresponding spotted tiling, halve all distances and replace the bars at any half-integer positions with spots.

By Theorem 1, $CC(n) = F_{2n}$. The recurrence relation for F_{2n} is known, but we demonstrate the recurrence in terms of spotted tilings to introduce ideas used in subsequent proofs. We establish a bijection between the n-color compositions counted by $3 \cdot CC(n-1)$ and by $CC(n) + CC(n-2)$.

Given three copies of the spotted tilings counted by $CC(n-1)$, perform the following operations.

(a) To the first set, add 1_1 at the end of each composition.

(b) To the second set, for each composition, replace its last part k_j with $(k+1)_j$.

(c) To the third set, for each composition,

 (c1) if the last part is k_1, replace it with $(k+1)_{k+1}$.

 (c2) if the last part is k_j with $j > 1$, replace it with $(k-1)_{j-1}$.

In terms of spotted tilings, operation (a) adds a box at the right hand side. Operations (b) and (c1) both extend the final rectangle by length one; (b) keeps the spots fixed, while (c1) moves the last spot from the first position of a $1 \times k$ rectangle to the last position of the new $1 \times (k+1)$ rectangle. The results of (a), (b), and (c1) are spotted tilings counted by $CC(n)$ with no repetition—notice that (b) produces tilings where the spot is not in the rightmost position, while tilings produced by (a) and (c1) all have a spot in the rightmost position. Operation (c2) moves the spot of the final rectangle one position to the left and decreases the rectangle by length one, producing a tiling counted by $CC(n-2)$. Both (c1) and (c2) can be considered to move the last spot one position to the left, allowing "wrap around" from the leftmost to rightmost position.

We demonstrate this with $3 \cdot CC(3) = CC(4) + CC(2)$. Since operations (a) and (b) are less complicated, Figure 4 shows them only applied to 3_2, while all 8 spotted tilings counted by $CC(3)$ are shown under (c1) and (c2).

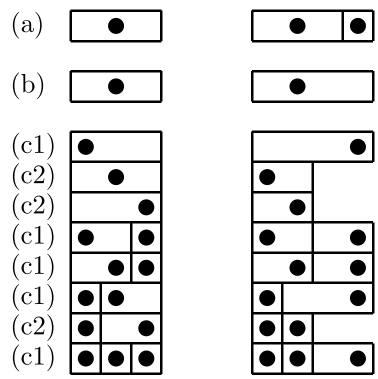

Figure 4: For three copies of $CC(3)$, examples of operations (a) and (b), and row by row details of (c1) and (c2).

For the reverse direction, there are three cases for n-color compositions counted by $CC(n)$. If the last part is 1_1, remove it; this is the inverse of operation (a). If the last part is k_j for $k \geq 2$ and $j < k$, replace it with $(k-1)_j$; this is the inverse of operation (b). If the last part is k_k for $k \geq 2$, replace it with $(k-1)_1$; this is the inverse of operation (c1). For an n-color composition counted by $CC(n-2)$ with last part k_j, replace it with $(k+1)_{j+1}$; this is the inverse operation of (c2).

We have established $CC(n) = 3 \cdot CC(n-1) - CC(n-2)$. The generating function follows from the recurrence (denominator) and initial values of the sequence (numerator); see [10] for more details. $\square$

Theorem 4. ([1], Theorem 1a, c) *The number of n-color compositions of n with m parts satisfies the formula $CC(n,m) = \binom{n+m-1}{2m-1}$. The corresponding generating function is $\frac{q^m}{(1-q)^{2m}}$.*

Proof. Consider the spotted tiling associated to an n-color composition of n with m parts. The spotted tiling will include m spots and $m+1$ bars, more specifically bars at 0, $2n$, and $m-1$ other positions. As in the previous proof, consider the spot signifying part k_i to be positioned at $i - \frac{1}{2}$ within the $1 \times k$ rectangle. Doubling all distances places spots at odd positions and bars at even positions. Since spots and bars alternate, the positions of the m spots and $m-1$ internal bars create a subsequence $\{x_1, \ldots, x_{2m-1}\}$ of $\{1, \ldots, 2n-1\}$ where the parity of x_i matches the parity of i. From Theorem 2, we know there are $\binom{n+m-1}{2m-1}$ such subsequences.

For generating functions associated with binomial coefficients, see [10]. □

Since $CC(n) = \sum_m CC(n, m)$, combining Theorems 3 and 4 gives

$$\sum_{m=1}^{n} \binom{n+m-1}{2m-1} = F_{2n}$$

which is equivalent to the even index case of an identity connecting Fibonacci numbers to diagonals in Pascal's triangle (identity 4 in [3]).

Theorem 5. ([5], Theorems 1.2, 5.1) *The number of n-color compositions of n having only even parts satisfies the formula $CCe(n) = 4 \cdot CCe(n-2) - CCe(n-4)$ with initial values $CCe(0) = 0$ and $CCe(2) = 2$. The corresponding generating function is $\frac{2q^2}{1-4q^2+q^4}$.*

Proof. Note that $CCe(n) = 0$ for all odd n. We establish a bijection between the n-color compositions counted by $CCe(n) + CCe(n-4)$ and by $4 \cdot CCe(n-2)$.

Similar to the recurrence relation proof of Theorem 3, we perform the following operations on four copies of the spotted tilings counted by $CC(n-2)$.

(a) To the first set, add 2_1 at the end.

(b) To the second set, add 2_2 at the end.

(c) For the third set, replace the final part k_j with $(k+2)_j$.

(d) For the fourth set,

- (d1) if the last part is k_1, replace it with $(k+2)_{k+1}$,
- (d2) if the last part is k_2, replace it with $(k+2)_{k+2}$,
- (d3) if the last part is k_j with $j > 2$, replace it with $(k-2)_{j-2}$.

Notice that the three (d) operations each move the spot in the last rectangle two to the left, with the convention that the rightmost position is to the left of the leftmost position. Adding parts 2_j and changing between part lengths k, $k+2$, and $k-2$ maintains the parity restriction. Figure 5 shows the results on four copies of the spotted tilings counted by $CCe(4)$.

As in the proof of Theorem 3, operations (a), (b), (c), (d1), and (d2) produce n-color compositions counted by $CCe(n)$ with no repetition, by inspection of the last part. Operation (d3) produces n-color compositions counted by $CCe(n-4)$.

For the inverse map, there are again several cases for an n-color composition counted by $CCe(n)$. If the last part is 2_1 or 2_2, remove it. If the last part is k_j

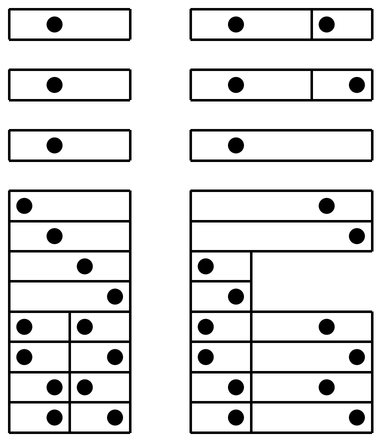

Figure 5: For three copies of $CCe(4)$, examples of operations (a), (b), and (c), and row by row details of (d1), (d2), and (d3).

with $j \leq k-2$, replace it with $(k-2)_j$. If the last part is k_{k-1} or k_k, replace it with $(k-2)_1$ or $(k-2)_2$, respectively. These are inverse operations for (a), (b), (c), (d1), and (d2), respectively. Given an n-color composition counted by $CCe(n-4)$, replace the last part k_j with $(k+2)_{j+2}$, the inverse of operation (d3).

We have established $CCe(n) = 4 \cdot CCe(n-2) - CCe(n-4)$. The initial conditions are easily checked. The sequence begins $0, 0, 2, 0, 8, 0, 30, 0, 112, 0, 418$. Removing the zeros, this is double the sequence A001353 in [8]. The generating function follows from the recurrence relation and the initial values. □

Theorem 6. *The number of n-color compositions of n having only odd parts satisfies the recurrence $CCo(n) = CCo(n-1) + 2 \cdot CCo(n-2) + CCo(n-3) - CCo(n-4)$ with initial values $0, 1, 1, 4$. The corresponding generating function is $\frac{q+q^3}{1-q-2q^2-q^3+q^4}$.*

Proof. We establish a bijection between the n-color compositions counted by $CCo(n) + CCo(n-4)$ and by $CCo(n-1) + 2 \cdot CCo(n-2) + CCo(n-3)$.

Similar to the previous proofs, we perform the following operations on spotted tilings counted by $CCo(n-1) + 2 \cdot CCo(n-2) + CCo(n-3)$.

(a) Given an n-color composition counted by $CCo(n-1)$, add 1_1 at the end.

(b) For the first set of n-color composition counted by $CCo(n-2)$, replace the final part k_j with $(k+2)_j$.

(c) For the second set of n-color composition counted by $CCo(n-2)$,

(c1) if the last part is k_1, replace it with $(k+2)_{k+1}$, (c2) if the last part is k_2, replace it with $(k+2)_{k+2}$,

(c3) if the last part is k_j with $j \geq 3$, replace it with $(k-2)_{j-2}$.

(d) Given an n-color composition counted by $CCo(n-3)$, add 3_3 at the end.

As in the proof of Theorem 5, the three (c) operations move the spot in the last rectangle two to the left, with the wrap around convention. Adding parts 1_1 and 3_3 and changing between part lengths k, $k+2$, and $k-2$ maintains the parity requirement. Figure 6 shows the results on $CCo(5) + 2 \cdot CCo(4) + CCo(3)$.

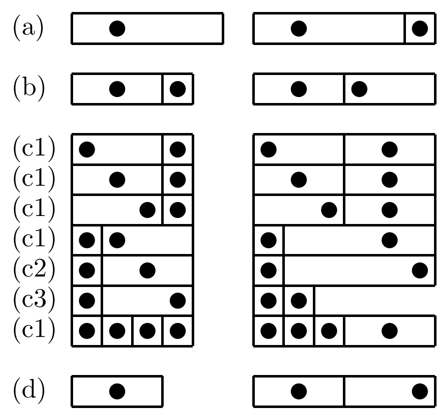

Figure 6: For $n = 6$, examples of operations (a), (b), and (d), and row by row details of (c1), (c2), and (c3).

Notice that neither operations (c1) nor (c2) produces an n-color composition of $n+2$ with last part 3_3. As in the previous proofs, operations (a), (b), (c1), (c2), and (d) produce n-color compositions counted by $CCo(n)$ with no repetition, by inspection of the last part. Operation (c3) produces n-color compositions counted by $CCo(n-4)$.

For the inverse map, consider an n-color composition counted by $CCo(n)$. If the last part is 1_1 or 3_3, remove it. If the past part is k_j with $j \leq k-2$, replace it with $(k-2)_j$. If the last part is k_{k-1} or k_k, replace it with $(k-2)_1$ or $(k-2)_2$, respectively. There are inverse operations for (a), (d), (b), (c1), and (c2), respectively. Given an n-color composition counted by $CCo(n-4)$, replace the last part k_j with $(k+2)_{j+2}$, the inverse operation of (c3).

We have established $CCo(n) = CCo(n-1) + 2 \cdot CCo(n-2) + CCo(n-3) - CCo(n-4)$. The initial conditions are easily checked. The sequence begins 0, 1, 1, 4, 7, 15, 32, 65, 137, 284, 591, which is A119749 in [8]. The generating function follows from the recurrence relation and the initial values. $\square$

3. Conjugable n-color Compositions

In his foundational work *Combinatory Analysis*, MacMahon introduced the zig-zag graph of a composition as a way of defining conjugacy. A composition is broken up with one rectangular part per row, such that the first segment of a part is directly beneath the last segment of its predecessor; see Figure 7 for an example. MacMahon explains, "Whereas the composition is read from left to right in successive rows from top to bottom, the conjugate is read from top to bottom in successive columns from left to right." [7, Section IV, chapter 1, §129]

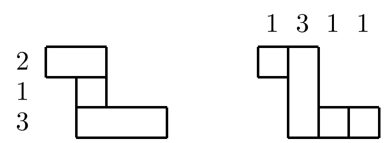

Figure 7: Mahonian zig-zag graphs showing that the conjugate of (2,1,3) is (1,3,1,1).

In the paper where Agarwal introduced n-color compositions [1], he concluded with two questions, including "What will be the shape of MacMahon's zig-zag graph in the case of n-colour compositions?" The notion of spotted tilings introduced here allows a possible answer to this question—simply carry the spots into the zig-zag graph. But for which n-color compositions does conjugation of the spotted zig-zag graph lead to a valid spotted tiling?

Consider the composition in Figure 7. Can $(2, 1, 3)$ be colored so that it is conjugable? In order for the first column to include a spot, the 2 must become 2_1. The second column automatically has a spot since 1 can only become 1_1. To avoid another spot in the second column, the 3 must become 3_2 or 3_3, but either possibility leaves some column without a spot. So no coloring of $(2, 1, 3)$ gives a conjugable n-color composition. In contrast, Figure 8 shows that $(2_1, 1_1, 3_2, 1_1)$ is conjugable.

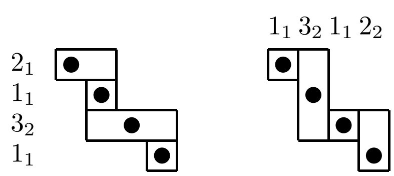

Figure 8: Spotted zig-zag graphs showing that the conjugate of $(2_1, 1_1, 3_2, 1_1)$ is $(1_1, 3_2, 1_1, 2_2)$.

Definition. An n-color composition is *conjugable* if its zig-zag graph has exactly one spot per column.

Our final theorem characterizes and counts conjugable n-color compositions.

Theorem 7. *(a) Conjugable n-color compositions have the form*

$$(2_1^a, 1_1, 2_2^b, (3_2, 2_1^a, 1_1, 2_2^b)^c)$$

where exponents denote repetition, $a, b, c \geq 0$, and a, b may vary in each occurrence.
(b) The number of conjugable n-color compositions of n is 0 if n is even, $2^{(n-1)/2}$ if n is odd.

Proof. The analysis of conjugable n-color compositions is most easily done in terms of the zig-zag graphs. By construction, there is one spot per row. By definition, there is one spot per column. Since a subsequent part is positioned with its first segment under the last segment of its predecessor, its spot must appear to the right of the spot above it. In order to have one spot per column, it is exactly one position to the right. The spots, therefore, lie on the diagonal.

The parts 3_1, 3_3, and k_j for any $k \geq 4$ cannot occur in a conjugable n-color compositions, since each would leave at least one column without a spot. As spots must appear on the diagonal, the allowed parts $1_1, 2_1, 2_2, 3_2$ may occur within the constraints shown in Figure 9. The description in part (a) follows.

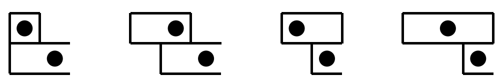

Figure 9: Parts 1_1 and 2_2 can only be followed by some allowed k_2 while parts 2_1 and 3_2 can only be followed by some allowed k_1.

The spotted zig-zag graph of a conjugable n-color composition of n with m parts fits in an $m \times m$ square with m spots on the main diagonal and exactly $m-1$ more segments in the two adjacent diagonals to "connect the spots." Thus $n = 2m-1$ and there are no conjugable n-color compositions of even n.

To count the conjugable n-color compositions of odd $n = 2m-1$, consider the original representation as a tiling of a $1 \times n$ board. Spots on the main diagonal of the zig-zag graph here means that the first, third, $\ldots, (2m-1)$-st positions contain spots, leaving $m-1$ gaps where there are two choices for the position of a vertical bar; see Figure 10. Thus there are $2^{m-1} = 2^{(n-1)/2}$ such compositions. $\square$

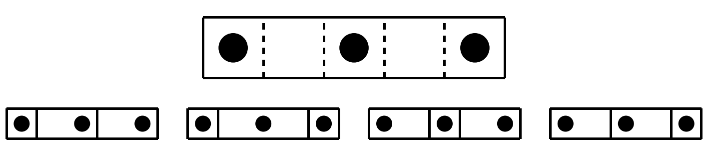

Figure 10: Between adjacent spots, there are two choices for the position of a vertical bar, so there are 4 conjugable n-color compositions of 5.

In conclusion, spotted tilings are a combinatorial interpretation of n-color compositions in the spirit of MacMahon that allow for bijective proofs of various enumeration results and an analog of the zig-zag graph and conjugacy. The interested reader will want to compare the spotted tilings introduced here to the combinatorial interpretation of n-color compositions using lattice paths given in [2].

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