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ON THE SUM OF RECIPROCAL GENERALIZED FIBONACCI NUMBERS

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Abstract

The Fibonacci Zeta functions are defined by $\zeta_F(s) = \sum_{k=1}^{\infty} F_k^{-s}$. Several aspects of the function have been studied. In this article we generalize the results by Ohtsuka and Nakamura, who treated the partial infinite sum $\sum_{k=n}^{\infty} F_k^{-s}$ for all positive integers n.

1. Introduction

The so-called *Fibonacci and Lucas Zeta functions*, defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s}\,,$$

respectively, have been considered in several different ways. In [8] the analytic continuation of these series is discussed. In [2] it is shown that the numbers $\zeta_F(2), \zeta_F(4), \zeta_F(6)$ (respectively, $\zeta_L(2), \zeta_L(4), \zeta_L(6)$) are algebraically independent, and that each of $\zeta_F(2s)$ (respectively, $\zeta_L(2s)$) $(s = 4, 5, 6, \dots)$ can be written as a rational (respectively, algebraic) function of these three numbers over $\mathbb{Q}$. Similar results are obtained in [2] for the alternating sums

$$\zeta_F^*(2s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^{2s}} \quad \left(\text{respectively,} \zeta_L^*(2s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^{2s}}\right) \quad (s = 1, 2, 3, \dots)\,.$$

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From the main theorem in [4] it follows that for any positive distinct integers s_1, s_2, s_3 the numbers $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, and $\zeta_F(2s_3)$ are algebraically independent if and only if at least one of s_1, s_2, s_3 is even. Other types of algebraic independence, including the functions

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s},$$

are discussed in [6]. In [5] Fibonacci zeta functions and Lucas zeta functions including

$$\zeta_F(1), \zeta_F(2), \zeta_F(3), \zeta_F^*(1), \zeta_L(1), \zeta_L(2), \zeta_L^*(1)$$

are expanded as non-regular continued fractions whose components are Fibonacci or Lucas numbers.

In [9] the partial infinite sums of reciprocal Fibonacci numbers were studied. In this paper we shall generalize their results, given in Propositions 1 and 2 below. Here, $\lfloor \cdot \rfloor$ denotes the floor function.

Proposition 1. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

Proposition 2. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

2. Main Results

Let a be a positive integer. Let $\{G_n\}$ be a general Fibonacci sequence defined by $G_{k+2} = aG_{k+1} + G_k$ $(k \geq 0)$ with $G_0 = 0$ and $G_1 = 1$.

Theorem 3. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_n - G_{n-1} & \text{if } n \text{ is even and } n \geq 2; \\ G_n - G_{n-1} - 1 & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

Theorem 4. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} aG_{n-1}G_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ aG_{n-1}G_n & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

We need some identities in order to prove Theorems 1 and 2.

Lemma 5. *For $n \geq 1$, we have*

(1) $G_n^2 - G_{n-1}G_{n+1} = (-1)^{n-1}$

(2) $G_{n-1}G_{n+3} - G_nG_{n+2} = (-1)^n(a^2+1)$

(3) $G_nG_{n+2} + G_{n-1}G_{n+1} = G_{2n+1}$

(4) $G_{n+1}G_{n+2} - G_{n-1}G_n = aG_{2n+1}$.

Proof. Every proof is done by induction and omitted. □

Proof of Theorem 3. Using Lemma 5 (1), for $n \geq 1$ we have

$$\begin{aligned}
&\frac{1}{G_n - G_{n-1}} - \frac{1}{G_n} - \frac{1}{G_{n+1}} - \frac{1}{G_{n+2} - G_{n+1}} \\
&\qquad = \frac{G_{n+2} - G_{n+1} - G_n + G_{n-1}}{(G_n - G_{n-1})(G_{n+2} - G_{n+1})} - \frac{G_{n+1} + G_n}{G_nG_{n+1}} \\
&\qquad = \frac{G_{n+2}(G_{n-1}G_{n+1} - G_n^2) + G_{n-1}(G_nG_{n+2} - G_{n+1}^2)}{G_nG_{n+1}(G_n - G_{n-1})(G_{n+2} - G_{n+1})} \\
&\qquad = \frac{(-1)^n(G_{n+2} - G_{n-1})}{G_nG_{n+1}(G_n - G_{n-1})(G_{n+2} - G_{n+1})}. \qquad (1)
\end{aligned}$$

If n is even with $n \geq 2$, since the right-hand side of the identity (1) is positive, we get

$$\frac{1}{G_n - G_{n-1}} > \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2} - G_{n+1}}. \qquad (2)$$

By applying inequality (2) repeatedly we have

$$\begin{aligned}
\frac{1}{G_n - G_{n-1}} &> \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2} - G_{n+1}} \\
&> \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2}} + \frac{1}{G_{n+3}} + \frac{1}{G_{n+4} - G_{n+3}} \\
&> \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2}} + \frac{1}{G_{n+3}} + \frac{1}{G_{n+4}} + \frac{1}{G_{n+5}} + \cdots.
\end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{G_k} < \frac{1}{G_n - G_{n-1}}. \qquad (3)$$

In a similar way, if n is odd with $n \geq 1$, then

$$\sum_{k=n}^{\infty} \frac{1}{G_k} > \frac{1}{G_n - G_{n-1}}\,. \tag{4}$$

On the other hand, if n is even with $n \geq 2$, then by Lemma 5, parts (1) and (4)

$$\begin{aligned}
&\frac{1}{G_n - G_{n-1} + 1} - \frac{1}{G_n} - \frac{1}{G_{n+1}} - \frac{1}{G_{n+2} - G_{n+1} + 1} \\
&\quad = -\frac{2(-1)^{n-1} + (-1)^{n-1}G_{n+2} + (-1)^n G_{n-1} + aG_{2n+1} + G_n + G_{n+1}}{G_n G_{n+1}(G_n - G_{n-1} + 1)(G_{n+2} - G_{n+1} + 1)} \\
&\quad = -\frac{(aG_{2n+1} - G_{n+2}) + (G_{n-1} + G_n + G_{n+1} - 2)}{G_n G_{n+1}(G_n - G_{n-1} + 1)(G_{n+2} - G_{n+1} + 1)} < 0\,.
\end{aligned}$$

Hence, by applying the inequality

$$\frac{1}{G_n - G_{n-1} + 1} < \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2} - G_{n+1} + 1}$$

repeatedly, we obtain

$$\begin{aligned}
\frac{1}{G_n - G_{n-1} + 1} &< \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2} - G_{n+1} + 1} \\
&< \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2}} + \frac{1}{G_{n+3}} + \frac{1}{G_{n+4} - G_{n+3} + 1} \\
&< \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2}} + \frac{1}{G_{n+3}} + \frac{1}{G_{n+4}} + \frac{1}{G_{n+5}} + \cdots\,.
\end{aligned}$$

Thus,

$$\frac{1}{G_n - G_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k}\,.$$

Together with (3) we have

$$\frac{1}{G_n - G_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k} < \frac{1}{G_n - G_{n-1}}\,,$$

so

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = G_n - G_{n-1}\,.$$

In a similar manner, if n is odd with $n \geq 1$, then

$$\frac{1}{G_n - G_{n-1} - 1} - \frac{1}{G_n} - \frac{1}{G_{n+1}} - \frac{1}{G_{n+2} - G_{n+1} - 1}$$
$$= \frac{2(-1)^{n-1} + (-1)^n G_{n+2} + (-1)^{n-1} G_{n-1} + aG_{2n+1} - G_n - G_{n+1}}{G_n G_{n+1}(G_n - G_{n-1} - 1)(G_{n+2} - G_{n+1} - 1)}$$
$$= \frac{aG_{n+1}(G_{n+1} - 1) + G_n(aG_n - a - 2) + 2}{G_n G_{n+1}(G_n - G_{n-1} - 1)(G_{n+2} - G_{n+1} - 1)} \geq 0\,,$$

where the equality holds only for $n = a = 1$. Hence,

$$\frac{1}{G_n - G_{n-1} - 1} > \sum_{k=n}^{\infty} \frac{1}{G_k}\,.$$

Together with (4) we have

$$\frac{1}{G_n - G_{n-1}} < \sum_{k=n}^{\infty} \frac{1}{G_k} < \frac{1}{G_n - G_{n-1} - 1}\,,$$

so

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = G_n - G_{n-1} - 1\,.$$

□

Proof of Theorem 4. By Lemma 5(1)

$$\frac{1}{aG_{n-1}G_n - 1} - \frac{1}{G_n^2} - \frac{1}{aG_nG_{n+1} - 1} = \frac{a(G_nG_{n+1} - G_{n-1}G_n)}{(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)} - \frac{1}{G_n^2}$$
$$= \frac{a^2G_n^4 - (aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}{G_n^2(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}$$
$$= \frac{a^2G_n^2(-1)^{n-1} + aG_n(G_{n-1} + G_{n+1}) - 1}{G_n^2(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}$$
$$\geq \frac{2aG_{n-1}G_n - 1}{G_n^2(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}$$
$$> 0\,.$$

Therefore,

$$\begin{aligned}\frac{1}{aG_{n-1}G_n - 1} &> \frac{1}{G_n^2} + \frac{1}{aG_nG_{n+1} - 1}\\ &> \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{aG_{n+1}G_{n+2} - 1}\\ &> \qquad\vdots\\ &> \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{G_{n+2}^2} + \frac{1}{G_{n+3}^2} + \cdots .\end{aligned}$$

Thus, we have

$$\frac{1}{aG_{n-1}G_n - 1} > \sum_{k=n}^{\infty} \frac{1}{G_k^2} . \tag{5}$$

In a similar way,

$$\frac{1}{aG_{n-1}G_n + 1} - \frac{1}{G_n^2} - \frac{1}{aG_nG_{n+1} + 1} \le -\frac{2aG_{n-1}G_n + 1}{G_n^2(aG_{n-1}G_n + 1)(aG_nG_{n+1} + 1)} < 0 .$$

Thus, we have

$$\frac{1}{aG_{n-1}G_n + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k^2} . \tag{6}$$

On the other hand, by Lemma 5(1) and (3),

$$\begin{aligned}&\frac{1}{aG_{n-1}G_n} - \frac{1}{G_n^2} - \frac{1}{G_{n+1}^2} - \frac{1}{aG_{n+1}G_{n+2}}\\ &\qquad = \frac{G_{n-2}}{aG_{n-1}G_n^2} - \frac{G_{n+3}}{aG_{n+1}^2G_{n+2}}\\ &\qquad = \frac{G_{n-2}G_{n+1}^2G_{n+2} - G_{n-1}G_n^2G_{n+3}}{aG_{n-1}G_n^2G_{n+1}^2G_{n+2}}\\ &\qquad = \frac{a^2(G_n^2 - G_{n-1}G_{n+1})(G_nG_{n+2} + G_{n-1}G_{n+1})}{aG_{n-1}G_n^2G_{n+1}^2G_{n+2}}\\ &\qquad = \frac{a(-1)^{n-1}G_{2n+1}}{G_{n-1}G_n^2G_{n+1}^2G_{n+2}} .\end{aligned}$$

If n is even with $n \geq 2$, then

$$\begin{aligned} \frac{1}{aG_{n-1}G_n} &< \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{aG_{n+1}G_{n+2}} \\ &< \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{G_{n+2}^2} + \frac{1}{G_{n+3}^2} + \frac{1}{aG_{n+3}G_{n+4}} \\ &< \qquad \vdots \\ &< \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{G_{n+2}^2} + \frac{1}{G_{n+3}^2} + \frac{1}{G_{n+4}^2} + \frac{1}{G_{n+5}^2} + \cdots . \end{aligned}$$

Hence, we have

$$\sum_{k=n}^{\infty} \frac{1}{G_k^2} > \frac{1}{aG_{n-1}G_n} . \tag{7}$$

Similarly, if n is odd with $n \geq 1$, then

$$\sum_{k=n}^{\infty} \frac{1}{G_k^2} < \frac{1}{aG_{n-1}G_n} . \tag{8}$$

If n is even with $n \geq 2$, then by equations (5) and (7) we obtain

$$aG_{n-1}G_n - 1 < \left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} < aG_{n-1}G_n .$$

Thus,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right\rfloor = aG_{n-1}G_n - 1 .$$

If n is odd with $n \geq 1$, then by equations (6) and (8) we obtain

$$aG_{n-1}G_n < \left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} < aG_{n-1}G_n + 1 .$$

Thus,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right\rfloor = aG_{n-1}G_n .$$

□

The following results are proved in similar manners. Such reciprocal sums of Fibonacci-type numbers have been studied by several authors (e.g. [1], [3], [6], [11]).

Theorem 6. *We have*

$$\text{(1)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}} \right)^{-1} \right\rfloor = G_{2n} - G_{2n-2} - 1 \qquad (n \geq 1)$$

$$\text{(2)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}} \right)^{-1} \right\rfloor = G_{2n-1} - G_{2n-3} \qquad (n \geq 2)$$

$$\text{(3)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}G_{2k+1}} \right)^{-1} \right\rfloor = G_{4n-1} - G_{4n-3} \qquad (n \geq 1)$$

$$\text{(4)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}G_{2k+2}} \right)^{-1} \right\rfloor = G_{4n+1} - G_{4n-1} - 1 \qquad (n \geq 1)$$

$$\text{(5)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}^2} \right)^{-1} \right\rfloor = G_{4n-1} - G_{4n-3} - 1 \qquad (n \geq 1)$$

$$\text{(6)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}^2} \right)^{-1} \right\rfloor = G_{4n-3} - G_{4n-5} \qquad (n \geq 2)$$

$$\text{(7)}\qquad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}G_{2k}} \right)^{-1} \right\rfloor = G_{4n-2} - G_{4n-4} \qquad (n \geq 1).$$

3. Generalized Fibonacci Numbers

Let c be a non-negative integer. Let $\{H_n\}$ be a generalized Fibonacci sequence defined by $H_{k+2} = H_{k+1} + H_k$ $(k \geq 0)$ with $H_0 = c$ and $H_1 = 1$.

Note that $H_n = F_{n+1}$ if $c = 1$, and $H_n = L_n$ (Lucas numbers) if $c = 2$ ([7, Corollary 5.5 (5.14)]).

The sequence H_n can be defined also as the total number of matchings in the connected planar graph on n vertices with $n-2+c$ total edges, of which $c-1$ edges are between one pair of vertices. The $c = 1$ and $c = 2$ cases are stated in [10, A45 and A204], and the proof for $c > 2$ is a inductive counting argument. An similar result for the Fibonacci type sequence $G_{k+2} = aG_{k+1} + G_k$, $G_0 = 0$, $G_1 = 1$ can be generated by counting the total matchings in a path (as defined in [12] on $k-1$ vertices with a loops at each vertex.

Theorem 7. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{H_k} \right)^{-1} \right\rfloor = \begin{cases} H_{n-2} - 1 & \text{if } n \text{ is even and } n \ge n_0; \\ H_{n-2} & \text{if } n \text{ is odd and } n \ge n_1\,. \end{cases}$$

Remark. n_0 and n_1 are determined depending only on the value of c. For example, if $H_k = L_k$ (Lucas number) or $c = 2$, then $n_0 = 2$ and $n_1 = 3$.
Precisely speaking, $n_0 = 2$ if $c = 1, 2$; $n_0 = 4$ if $c \le 4$; $n_0 = 6$ if $c \le 10$; $n_0 = 8$ if $c \le 26$; $n_0 = 10$ if $c \le 68$; $n_0 = 12$ if $c \le 178$; $n_0 = 14$ if $c \le 466$; $n_0 = 16$ if $c \le 1220$; $n_0 = 18$ if $c \le 3194$; $n_0 = 20$ if $c \le 8362$.
Similarly, $n_1 = 1$ if $c = 1$; $n_1 = 3$ if $c = 2$; $n_1 = 5$ if $c \le 6$; $n_1 = 7$ if $c \le 16$; $n_1 = 9$ if $c \le 42$; $n_1 = 11$ if $c \le 110$; $n_1 = 13$ if $c \le 288$; $n_1 = 15$ if $c \le 754$; $n_1 = 17$ if $c \le 1974$; $n_1 = 19$ if $c \le 5168$.

Theorem 8. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{H_k^2} \right)^{-1} \right\rfloor = \begin{cases} H_{n-1}H_n + g(c) - 1 & \text{if } n \text{ is even and } n \ge n_2; \\ H_{n-1}H_n - g(c) & \text{if } n \text{ is odd and } n \ge n_3\,, \end{cases}$$

where

$$g(c) = \begin{cases} \dfrac{c(c+1)}{3} & \text{if } c \equiv 0, 2 \pmod 3; \\ \dfrac{c(c+1)+1}{3} & \text{if } c \equiv 1 \pmod 3)\,. \end{cases}$$

Remark. Note that $g(c)$ is an integer. If $H_k = L_k$, then we take $n_2 = 2$ and $n_3 = 1$. Precisely speaking, we can determine n_2 and n_3 as follows:

c	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n_2	2	2	4	4	4	6	4	6	6	6	6	6	6	6	8	6	8	8	6	8
n_3	1	1	3	5	3	5	5	5	5	7	5	5	7	7	7	7	7	7	7	7

We need some lemmata in order to prove Theorems 7 and 8. Every proof of the lemmata is done by induction and omitted.

Lemma 9. *For $n \ge 1$, $H_n = cF_{n-1} + F_n$.*

Lemma 10. *We have*

(1) $H_n^2 - H_{n-1}H_{n+1} = H_nH_{n+1} - H_{n-1}H_{n+2} = (-1)^n(c^2 + c - 1)$

(2) $H_{n-1}H_{n+1} - H_{n-2}H_{n+2} = (-1)^{n-1}2(c^2 + c - 1)$

(3) $H_{n+4}H_n - H_{n+2}H_{n-2} = H_{n+1}(H_{n+3} - H_{n-1})$

(4) $H_{n+1}H_{n+2} - H_{n-1}H_n = H_n^2 + H_{n+1}^2 = cH_{2n} + H_{2n+1}$.

Proof of Theorem 7. By Lemma 10 (2)

$$\begin{aligned}\frac{1}{H_{n-2}} - \frac{2}{H_n} - \frac{1}{H_{n+1}} &= \frac{(H_n - H_{n-2})H_{n+1} - H_{n-2}(H_n + H_{n+1})}{H_{n-2}H_nH_{n+1}}\\ &= \frac{H_{n-1}H_{n+1} - H_{n-2}H_{n+2}}{H_{n-2}H_nH_{n+1}}\\ &= \frac{(-1)^{n-1}2(c^2+c-1)}{H_{n-2}H_nH_{n+1}}\,.\end{aligned}$$

Hence, if $c \ge 1$ and n is even, then by

$$\begin{aligned}\frac{1}{H_{n-2}} &< \frac{1}{H_n} + \frac{1}{H_{n+1}} + \frac{1}{H_n}\\ &< \frac{1}{H_n} + \frac{1}{H_{n+1}} + \frac{1}{H_{n+2}} + \frac{1}{H_{n+3}} + \frac{1}{H_{n+2}}\\ &< \frac{1}{H_n} + \frac{1}{H_{n+1}} + \frac{1}{H_{n+2}} + \frac{1}{H_{n+3}} + \frac{1}{H_{n+4}} + \frac{1}{H_{n+5}} + \cdots\,,\end{aligned}$$

we have

$$\frac{1}{H_{n-2}} < \sum_{k=n}^{\infty} \frac{1}{H_k}\,. \tag{9}$$

In a similar manner, if $c \ge 1$ and n is odd, then

$$\frac{1}{H_{n-2}} > \sum_{k=n}^{\infty} \frac{1}{H_k}\,. \tag{10}$$

On the other hand, if n is even, then by Lemma 10 (2)

$$\frac{1}{H_{n-2}-1} - \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_n - 1} \tag{11}$$

$$= \frac{(-1)^{n-1}2(c^2+c-1)H_n + H_{n+2}(H_{n-2} + H_n - 1)}{H_nH_{n+1}(H_{n-2}-1)(H_n-1)}$$

$$= \frac{-2(c^2+c-1)H_n + H_{n+2}(H_{n-2} + H_n - 1)}{H_nH_{n+1}(H_{n-2}-1)(H_n-1)}\,. \tag{12}$$

The numerator is positive if n is large enough for a fixed c. For example, one can take n so that $H_{n+2} > 2(c^2+c-1)$ since H_n is monotone increasing for n. Exactly speaking, if $c=1$, then the right-hand side of (12) is positive for $n\ge 2$. If $2\le c\le 4$, then $n\ge 4$. If $5\le c\le 9$, then $n\ge 6$. If $10\le c\le 24$, then $n\ge 8$. If $25\le c\le 62$, then $n\ge 10$. If $63\le c\le 161$, then $n\ge 12$. If $162\le c\le 422$, then $n\ge 14$. If $423\le c\le 1104$, then $n\ge 16$.

If n is odd, then

$$\frac{1}{H_{n-2}+1}-\frac{1}{H_n}-\frac{1}{H_{n+1}}-\frac{1}{H_n+1} \tag{13}$$

$$=\frac{(-1)^{n-1}2(c^2+c-1)H_n-H_{n+2}(H_{n-2}+H_n+1)}{H_nH_{n+1}(H_{n-2}+1)(H_n+1)}$$

$$=\frac{2(c^2+c-1)H_n-1H_{n+2}(H_{n-2}+H_n+1)}{H_nH_{n+1}(H_{n-2}+1)(H_n+1)}. \tag{14}$$

The numerator is negative if n is large enough for a fixed c. For example, if $c=1$, then the right-hand side of (14) is negative for $n\ge 1$. If $c=2$, then $n\ge 3$. If $3\le c\le 6$, then $n\ge 5$. If $7\le c\le 15$, then $n\ge 7$. If $16\le c\le 38$, then $n\ge 9$. If $39\le c\le 100$, then $n\ge 11$. If $101\le c\le 261$, then $n\ge 13$. If $262\le c\le 682$, then $n\ge 15$.

When n is even, repeating the inequality

$$\frac{1}{H_{n-2}-1}-\frac{1}{H_n}-\frac{1}{H_{n+1}}-\frac{1}{H_n-1}>0\,,$$

we have

$$\frac{1}{H_{n-2}-1}>\sum_{k=n}^{\infty}\frac{1}{H_k}\,. \tag{15}$$

Together with (9), we obtain

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{H_k}\right)^{-1}\right\rfloor=H_{n-2}-1\,.$$

When n is odd, repeating the inequality

$$\frac{1}{H_{n-2}+1}-\frac{1}{H_n}-\frac{1}{H_{n+1}}-\frac{1}{H_n+1}<0\,,$$

we have

$$\frac{1}{H_{n-2}+1}<\sum_{k=n}^{\infty}\frac{1}{H_k}\,. \tag{16}$$

Together with (10), we obtain

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{H_k}\right)^{-1}\right\rfloor=H_{n-2}\,.$$

Proof of Theorem 8. By Lemma 10 (1)

$$\frac{1}{H_{n-1}H_n+(-1)^n g(c)-1}-\frac{1}{H_n^2}-\frac{1}{H_nH_{n+1}+(-1)^{n+1}g(c)-1}$$

$$=\frac{H_n^2+(-1)^{n+1}2g(c)}{(H_{n-1}H_n+(-1)^n g(c)-1)(H_nH_{n+1}+(-1)^{n+1}g(c)-1)}-\frac{1}{H_n^2}$$

$$=\frac{(-1)^n(c^2+c-1-3g(c))H_n^2+\big(g(c)\big)^2+H_n(H_{n+1}+H_{n-1})-1}{H_n^2(H_{n-1}H_n+(-1)^n g(c)-1)(H_nH_{n+1}+(-1)^{n+1}g(c)-1)}\,.$$

Suppose that n is even with $n\ge 2$. Then the numerator is

$$\big(c^2+c-3g(c)-1\big)H_n^2+\big(g(c)\big)^2+H_n(H_{n+1}+H_{n-1}-1)$$
$$\ge H_n(H_{n-1}-H_{n-2})+\big(g(c)\big)^2-1\ge 0$$

(the equalities hold only for $n=2$ and $c=1$). Suppose that n is odd with $n\ge 1$. Then the numerator is

$$\big(3g(c)-c^2-c+1\big)H_n^2+\big(g(c)\big)^2+H_n(H_{n+1}+H_{n-1}-1)$$
$$\ge H_n^2+H_n(H_{n+1}+H_{n-1})+\big(g(c)\big)^2-1>0\,.$$

Therefore, for all $n\ge 1$

$$\frac{1}{H_{n-1}H_n+(-1)^n g(c)-1}>\sum_{k=n}^{\infty}\frac{1}{H_k^2}\,. \tag{17}$$

Similarly,

$$\frac{1}{H_{n-1}H_n+(-1)^n g(c)+1}-\frac{1}{H_n^2}-\frac{1}{H_nH_{n+1}+(-1)^{n+1}g(c)+1}$$

$$=\frac{H_n^2+(-1)^{n+1}2g(c)}{(H_{n-1}H_n+(-1)^n g(c)-1)(H_nH_{n+1}+(-1)^{n+1}g(c)-1)}-\frac{1}{H_n^2}$$

$$=\frac{(-1)^n(c^2+c-1-3g(c))H_n^2+\big(g(c)\big)^2-H_n(H_{n+1}+H_{n-1})-1}{H_n^2(H_{n-1}H_n+(-1)^n g(c)+1)(H_nH_{n+1}+(-1)^{n+1}g(c)+1)}\,.$$

If n is even, then the numerator is less than or equal to

$$-H_n(H_{n+1}+H_n+H_{n-1})+\big(g(c)\big)^2-1\,.$$

If n is odd, then the numerator is less than or equal to

$$-H_n(H_{n-1}-H_{n-2})+\big(g(c)\big)^2-1\,.$$

Thus, in any case, for $n \geq n_5$ (n_5 is large) both values are negative. Therefore,

$$\frac{1}{H_{n-1}H_n + (-1)^n g(c) + 1} < \sum_{k=n}^{\infty} \frac{1}{H_k^2}\,. \tag{18}$$

By Lemma 10, parts (1) and (4)

$$\frac{1}{H_{n-1}H_n + (-1)^n g(c)} - \frac{1}{H_n^2} - \frac{1}{H_{n+1}^2} - \frac{1}{H_{n+1}H_{n+2} + (-1)^n g(c)}$$

$$= \frac{H_{n+1}H_{n+2} - H_{n-1}H_n}{\big(H_{n-1}H_n + (-1)^n g(c)\big)\big(H_{n+1}H_{n+2} + (-1)^n g(c)\big)} - \frac{H_n^2 + H_{n+1}^2}{H_n^2 H_{n+1}^2}$$

$$= \frac{(cH_{2n} + H_{2n+1})\big((-1)^n(c^2+c-1)H_nH_{n+1}}{\big(H_{n-1}H_n + (-1)^n g(c)\big)\big(H_{n+1}H_{n+2} + (-1)^n g(c)\big)H_n^2H_{n+1}^2}\,.$$

$$+ \frac{(-1)^{n+1}g(c)(H_{n+1}H_{n+2} + H_{n-1}H_n) - (g(c))^2\big)}{\big(H_{n-1}H_n + (-1)^n g(c)\big)\big(H_{n+1}H_{n+2} + (-1)^n g(c)\big)H_n^2H_{n+1}^2}\,.$$

Hence, if n is even with $n \geq n_6$ (large), then by

$$(c^2+c-1)H_nH_{n+1} - g(c)(H_{n+1}H_{n+2} + H_{n-1}H_n) - (g(c))^2 < 0$$

we have

$$\frac{1}{H_{n-1}H_n + g(c)} < \sum_{k=n}^{\infty} \frac{1}{H_k^2}\,. \tag{19}$$

If n is odd with $n \geq n_7$ (large), then by

$$-(c^2+c-1)H_nH_{n+1} + g(c)(H_{n+1}H_{n+2} + H_{n-1}H_n) - (g(c))^2 > 0$$

we have

$$\frac{1}{H_{n-1}H_n - g(c)} > \sum_{k=n}^{\infty} \frac{1}{H_k^2}\,. \tag{20}$$

In conclusion, if n is even, by (17) and (19) we obtain

$$\frac{1}{H_{n-1}H_n + g(c)} < \sum_{k=n}^{\infty} \frac{1}{H_k^2} < \frac{1}{H_{n-1}H_n + g(c) - 1}\,.$$

If n is odd, by (18) and (20) we obtain

$$\frac{1}{H_{n-1}H_n - g(c) + 1} < \sum_{k=n}^{\infty} \frac{1}{H_k^2} < \frac{1}{H_{n-1}H_n - g(c)}\,.$$

4. The Sum of Reciprocal Jacobsthal Numbers

It would be interesting to find similar results for the sum $\sum_{k=n}^{\infty} U_k^{-1}$, where the sequence $\{U_n\}_n$ is defined by $U_n = aU_{n-1} + bU_{n-2}$ $(n \geq 2)$ with $U_0 = c$ and $U_1 = d$ for arbitrary fixed integers a, b, c and d.

Here, we mention the result for the sum of reciprocal Jacobsthal numbers, defined by $J_n = J_{n-1} + 2J_{n-2}$ $(n \geq 2)$ with $J_0 = 0$ and $J_1 = 1$ (*Cf.* [7, Ch.39]).

Theorem 11. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{J_k} \right)^{-1} \right\rfloor = \begin{cases} J_{n-1} - 1 & \text{if } n \text{ is even and } n \geq 2; \\ J_{n-1} & \text{if } n \text{ is odd and } n \geq 1\,. \end{cases}$$

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