DISTANCE GRAPHS AND ARITHMETIC PROGRESSIONS

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Abstract

A set of positive integers D is called *lonely* if there exist real numbers $\alpha, \delta \in (0,1)$ such that each point of the dilation αD is at distance at least δ from the nearest integer. We prove that for every lonely set there is a 2-coloring of the integers without arbitrarily long monochromatic arithmetic progressions with steps $d \in D$. This result is a step towards a more general conjecture by Brown, Graham, and Landman, stating that a similar 2-coloring exists whenever the set of allowable steps D violates the restricted version of van der Waerden's theorem.

– Dedicated to the memory of Ron Graham

1. Introduction

Let $D \subseteq \mathbb{N}$ be a fixed subset of the set of positive integers. Consider a graph G_D on the set of vertices $\mathbb{N}$ in which two vertices $a, b \in \mathbb{N}$, with $a < b$, are joined by an edge if and only if $b - a \in D$. Investigations of such graphs were initiated by Eggleton, Erdős, and Skilton [3] in connection with the famous Hadwiger-Nelson problem concerning the chromatic number of the plane (see [13]).

Let $\chi(D)$ denote the chromatic number of G_D. A challenging problem is to characterize sets D with finite chromatic number. For example, if D consists of all even integers, then $\chi(D) = \infty$, since there is an infinite clique in G_D. On the other hand, if D consists of all odd numbers, then $\chi(D) = 2$, since the sets of even and odd integers are independent in G_D. This shows that the chromatic number $\chi(D)$ may radically differ for sets that are just translates one of another.

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The main conjecture in this matter was posed independently by Katznelson [9] and Ruzsa (personal communication), using different terminology of topological dynamics and additive number theory, respectively. To make a precise statement, let us denote by $\|x\|$ the distance from x to the nearest integer. A set D is called *lonely*, if there exist real numbers $\alpha, \delta > 0$ such that the inequality $\|\alpha d\| \geqslant \delta$ is satisfied for all $d \in D$. For example, the set of all odd integers is lonely, as can be seen by taking $\alpha = \delta = 1/2$.

Conjecture 1 (Katznelson-Ruzsa)**.** The chromatic number $\chi(D)$ is finite if and only if the set D is a finite union of lonely sets.

That the loneliness condition on D is sufficient for the finiteness of $\chi(D)$ was proved by Katznelson [9] and independently (implicitly) by Ruzsa, Tuza and Voigt [12]. Both papers solve a problem posed by Erdős whether $\chi(D)$ is finite for sets with exponential growth (so called *lacunary* sets).

In this note we prove a result concerning arithmetic progressions whose steps are restricted to lonely sets. The celebrated theorem of van der Waerden [14] asserts that any finite coloring of $\mathbb{N}$ admits arbitrarily long monochromatic arithmetic progressions. Clearly, this is not necessarily true if we restrict the set of allowable steps of arithmetic progressions to some fixed set D. In particular, in a proper coloring of G_D with $\chi(D)$ colors, there are no nontrivial monochromatic arithmetic progressions (with steps from D) at all.

The problem of characterizing sets D for which the restricted van der Waerden's theorem holds was undertaken by Brown, Graham, and Landman [2]. They posed the following intriguing conjecture.

Conjecture 2 (Brown, Graham, Landman, [2])**.** Let D be a set of positive integers. Suppose that there is a finite coloring of $\mathbb{N}$ such that all monochromatic arithmetic progressions with steps in D have bounded length. Then there is a 2-coloring of $\mathbb{N}$ with the same property.

In this note we prove that lonely sets satisfy this conjecture. In particular, this solves an open problem, posed in [1] (see also [10]), of determining the least number of colors needed to avoid long monochromatic arithmetic progressions with steps belonging to the set of Fibonacci numbers.

2. The Result

Let us consider the *torus* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is geometrically just a circle of unit circumference with a distinguished point 0. For $x \in \mathbb{R}$, there is a unique point on $\mathbb{T}$ corresponding to x whose circular coordinate is the factional part $\{x\}$ of x. We will denote real numbers and their corresponding points on the torus by the same symbols. By $\|x\|$ we denote the circular distance from $x \in \mathbb{T}$ to the point 0. More precisely, $\|x\|$ equals $\{x\}$ or $1 - \{x\}$, where $\{x\}$ is the fractional part of x. Notice also that for any two numbers $x, y \in \mathbb{R}$, the number $\|x - y\|$ is equal to the circular distance between the corresponding points on the torus $\mathbb{T}$.

Suppose now that D is a lonely set, that is, the inequality $\|\alpha d\| \geqslant \delta$ holds for some $\alpha, \delta \in (0, 1)$ and all $d \in D$. For a fixed $\alpha \in (0, 1)$, let $\delta_\alpha(D) = \inf\{\|\alpha d\| : d \in D\}$, and let $\lambda(D) = \sup\{\delta_\alpha(D) : \alpha \in (0, 1)\}$. We shall call it the *loneliness constant* of the set D.

Theorem 1 (Katznelson [9])**.** *Let D be a lonely set with the loneliness constant $\lambda(D)$. Then $\chi(D) \leqslant \lceil \frac{1}{\lambda(D)} \rceil$.*

Proof. Put $k = \lceil \frac{1}{\lambda(D)} \rceil$ and partition the torus $\mathbb{T}$ into half-open arcs $A_j = \left[\frac{j}{k}, \frac{j+1}{k}\right)$, for $j = 0, 1, \ldots, k-1$. Define a k-coloring $c : \mathbb{N} \to \{0, 1, \ldots, k-1\}$ by $c(n) = j$ if and only if $\alpha n \in A_j$. We claim that this is a proper coloring of the distance graph G_D. Indeed, suppose that $c(a) = c(b) = j$ for some $a, b \in \mathbb{N}$. Then $\alpha a, \alpha b \in A_j$, which implies that $\|\alpha a - \alpha b\| < \frac{1}{k}$. Hence, $\|\alpha(a-b)\| < \frac{1}{k}$, and in consequence $a - b$ cannot be an element of D (since every $d \in D$ satisfies $\|\alpha d\| \geqslant \lambda(D) \geqslant \frac{1}{k}$. □

Now we use similar methods to prove Theorem 2, which is an effective version of both Lemma 7.4 in [7] and Corollary 8.11 in [4].

Theorem 2. *Let D be a lonely set with the loneliness constant λ. Then there exists a 2-coloring of $\mathbb{N}$ such that no arithmetic progression of length $\ell = \lceil \frac{1}{2\lambda} \rceil + 1$ and step $d \in D$ is monochromatic.*

Proof. Let D be a lonely set with loneliness $\lambda > 0$. Consider a red-blue coloring of the torus $\mathbb{T} = R \cup B$, where $R = [0, \frac{1}{2})$ and $B = [\frac{1}{2}, 1)$. Define a red-blue coloring of $\mathbb{N}$ so that the color of a number $n \in \mathbb{N}$ coincides with the color of the corresponding point $\alpha n \in \mathbb{T}$ on the torus. Here α is a constant satisfying the loneliness condition $\|\alpha d\| \geqslant \lambda$ for all $d \in D$.

Now, consider any arithmetic progression $a, a+d, a+2d, \ldots, a+kd$ with $d \in D$. We claim that it is not monochromatic, provided that $k \geqslant \frac{1}{2\lambda}$. Indeed, consider the corresponding points $\alpha a, \alpha(a+d), \alpha(a+2d), \ldots, \alpha(a+kd)$. These points also form an arithmetic progression with step $s = \|\alpha d\| \geqslant \lambda$ on the torus $\mathbb{T}$ (going clockwise or counterclockwise). So, the length of the whole arc spanned between the first and the last point of this progression equals at least $ks \geqslant \frac{1}{2\lambda}\lambda = \frac{1}{2}$. On the other hand, $s \leqslant \frac{1}{2}$, so, there must exist two points in the progression dropping into different parts of the partition of the torus. This proves the theorem. □

Recall that a set $D = \{d_1, d_2, \ldots\}$ is *lacunary* if there exists a real number $\theta > 0$ such that $\frac{d_{i+1}}{d_i} \geqslant 1 + \theta$, for all $i = 1, 2, \ldots$. It was independently proved by Katznelson [9] and Ruzsa, Tuza, and Voigt [12] that every lacunary set is lonely.

For instance, the set of Fibonacci numbers $F = \{1, 2, 3, 5, 8, 13, \dots\}$ is well-known to be lacunary. Hence, F is lonely and satisfies the assertion of Theorem 2. As mentioned at the end of the Introduction, this answers a question posed in [1] (see also [10]).

Moreover, as proved by Peres and Schlag in [11], every finite union of lacunary sets is a lonely set. By their results one may derive the dependence between loneliness and lacunary constants and obtain thereby an upper bound on the length of monochromatic arithmetic progressions in this case. More specifically, let $t \geqslant 1$ be a fixed integer, and suppose that D_i is a lacunary set with the lacunary constant $1+\theta_i$, for $i = 1, 2 \dots, t$. Then, as proved in [11], the set $D = \bigcup_{i=1}^{t} D_i$ is lonely with the loneliness constant satisfying

$$\lambda(D) \geqslant \frac{1}{240M \log_2 M},$$

where $M = \sum_{i=1}^{t} \theta_i^{-1}$ (provided that $M \geqslant 4$).

Finally, notice that Theorem 2 has the following consequence for finite unions of lonely sets.

Corollary 1. *Let $t \geqslant 1$ be an integer and let $D = D_1 \cup D_2 \cup \dots \cup D_t$, where each D_i is a lonely set. Then there exists a 2^t-coloring of $\mathbb{N}$ and a constant $\ell = \ell(D)$ such that no arithmetic progression of length ℓ and step $d \in D$ is monochromatic.*

Proof. It is enough to take the product coloring whose components are the 2-colorings guaranteed by Theorem 2 for each of the sets D_i. Clearly, there can be no monochromatic progression with step in D of length greater than $\lceil \frac{1}{2\lambda} \rceil$, where $\lambda = \min \lambda(D_i)$. □

3. Further Remarks

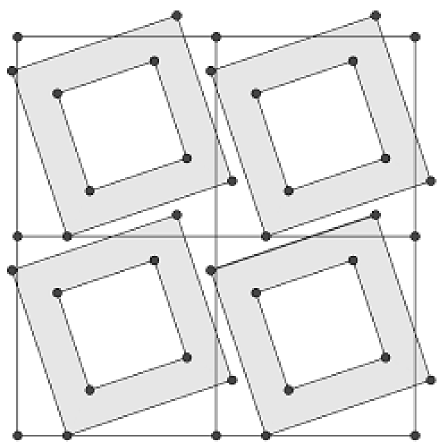
We conclude this short note with some reflections on further possible applications of the torus coloring method to other problems for distance graphs.

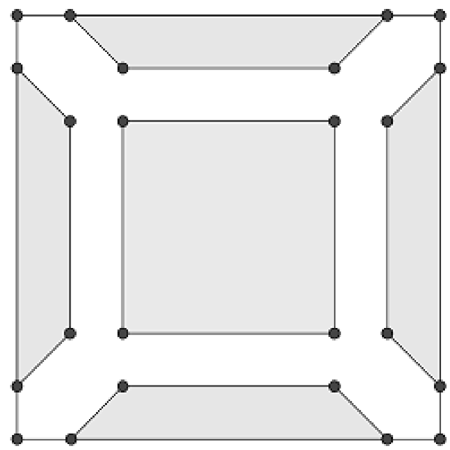
The first and natural attempt would be to extend Theorem 2 to all sets D with finite chromatic number $\chi(D)$. A natural attempt here is to consider finite unions of lonely sets (cf. Conjecture 8.13 in [4]). By Corollary 1, we know that a desired coloring (avoiding long arithmetic progressions with steps in D) can be obtained by using at most 2^t colors, where t is the number of lonely sets in the union.

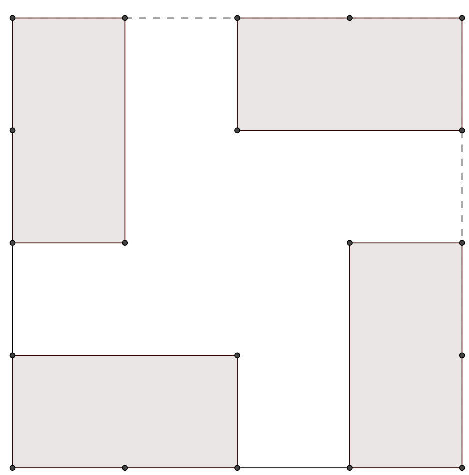
We will now propose a series of 2-colorings of the integers which we believe to show (but are unable to prove) that a union of two lonely sets is not 2-large. Given a partition of the 2-Torus $\mathbb{T}^2 = A \cup B$, and some $(\alpha, \beta) \in \mathbb{T}^2$, we may define the coloring $f_{A,B,\alpha,\beta} : \mathbb{N} \to \{1, 2\}$ by

$$f_{A,B,\alpha,\beta}(n) = \begin{cases} 1 & \text{if } (n\alpha, n\beta) \in A \\ 2 & \text{if } (n\alpha, n\beta) \in B \end{cases}. \tag{1}$$

Informally, Theorem 2 was proven by analyzing the coloring $f_{R,B,\alpha}$. Now let $S = L_1 \cup L_2$ be a union of two lonely sets. Suppose that $\alpha, \beta \in \mathbb{R}$ and $\delta > 0$ is such that $||n\alpha|| > \delta$ for every $n \in L_1$ and $||n\beta|| > \delta$ for every $n \in L_2$. We conjecture that for at least one of the partitions of $\mathbb{T}^2$ that are shown below, the coloring $f_{A,B,\alpha,\beta}$ will not contain long monochromatic arithmetic progressions with steps in S.







We hope that at least one of the above partitions will also give insight into partitions of higher dimensional Tori that will allow us to generalize Theorem 2 to finite unions of lonely sets.

Another direction could be to look at arbitrary forward paths in distance graphs, not only following arithmetic progressions. It is known, for instance, that for the set F of Fibonacci numbers, there is a 6-coloring of G_F avoiding arbitrarily long monochromatic forward paths. On the other hand, two colors are not sufficient for this property (see [1]). Other related problems and results can be found in [6].

It is also known that there is a set H with infinite chromatic number $\chi(H)$ and a finite coloring of $\mathbb{N}$ avoiding 3-term monochromatic arithmetic progressions with steps in H (see [8] or Section 9.1 of [5]). On the other hand, there is no finite coloring of $\mathbb{N}$ avoiding arbitrarily long monochromatic forward paths in the graph G_H.

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