ROOK DOMINATION ON HEXAGONAL HEXAGON BOARDS

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Abstract

Chess-like game boards B_n are considered, which are hexagonal parts of the Euclidean tessellation of the plane by regular hexagons. For chess-like rooks on B_n the domination number $\gamma(n)$ is determined.

1. Introduction

Corresponding to a classical chessboard a hexagonal hexagon board B_n is defined as the following hexagonal part of the Euclidean tessellation of the plane by regular

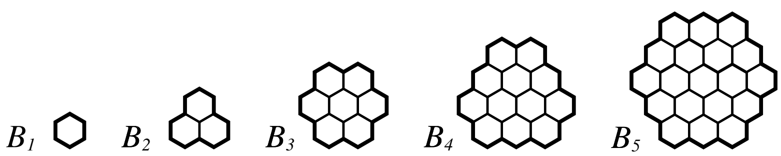

Figure 1: Hexagonal hexagon boards.

hexagons: If B_1 is one hexagon and if B_2 consists of three hexagons with a common vertex, then B_n for $n \geq 3$ consists of B_{n-2} together with all neighboring hexagons of B_{n-2} (see Figure 1). One may wonder whether Ronald Graham, who liked to look at combinatorial problems for chessboards, would have liked the corresponding problems for these hexagonal boards as well.

A rook can move on straight-line sequences of edge-adjacent hexagons, see [1], as on straight-line sequences of edge-adjacent squares on classical chessboards. Then the domination number $\gamma(n)$ denotes the smallest number of rooks, so that every hexagon of B_n is either occupied or threatened. Here we want to determine $\gamma(n)$.

Theorem 1. *For all $n \geq 1$ it holds that*

$$\gamma(n) = \begin{cases} \frac{n}{2} & \text{if} \quad n \equiv 0\,(mod\,4), \\ \frac{n+1}{2} & \text{if} \quad n \equiv 1\,(mod\,4), \\ \frac{n-1}{2} & \text{if} \quad n \equiv 3\,(mod\,4). \end{cases}$$

For the domination number of grids that threaten all neighboring hexagons on B_n, estimates can be found in [3].

The domination number for rooks on triangular hexagon boards seems to be more difficult to determine (see [5, 6]). The independence number $\beta(n)$ corresponding to $\gamma(n)$; that is, the maximum number of pairwise not threatening rooks on B_n is proven in [5] to be $\beta(n) = 2\lceil \frac{n}{2} \rceil - 1$.

In general, results for domination, independence, and other parameters for chessboards can be found in [4, 1].

2. Upper Bound

For $n \leq 3$, $\gamma(n)$ is easy to prove, as claimed in Theorem 1. Consider the four

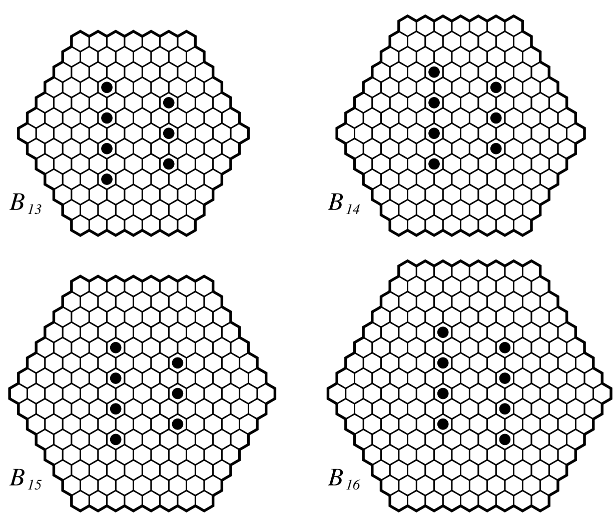

Figure 2: Rook domination for n = 13, 14, 15, and 16.

consecutive boards $B_n = B_{4t+i}$ for $i = 1, 2, 3, 4$ and $t \geq 1$. If the first row has

$\lfloor \frac{n+1}{2} \rfloor$ hexagons at the top, then we put a first rook in row $\lfloor \frac{n+5}{4} \rfloor$ at position $\lfloor \frac{n+5}{4} \rfloor$ from the left and a second rook in the next row at position $\lfloor \frac{n+5}{4} \rfloor$ from the right. Vertically below the first rook, $\lfloor \frac{n-1}{4} \rfloor$ further rooks are placed in the hexagons of the rows $\lfloor \frac{n+5}{4} \rfloor + 2j$ for $1 \leq j \leq \lfloor \frac{n-1}{4} \rfloor$ and vertically below the second rook, $\lfloor \frac{n}{4} \rfloor - 1$ further rooks are placed in the hexagons of the rows $\lfloor \frac{n+5}{4} \rfloor + 2j + 1$ for $1 \leq j \leq \lfloor \frac{n}{4} \rfloor - 1$ (see Figure 2 for $t = 3$). Note that this construction is also possible for $n = 4$.

Together, there are $1 + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n-1}{4} \rfloor$ rooks, that is, $2t + 1 = \frac{n+1}{2}$, $2t + 1 = \frac{n}{2}$, $2t + 1 = \frac{n-1}{2}$, and $2t + 2 = \frac{n}{2}$ rooks for $i = 1, 2, 3$, and 4, respectively, as claimed in Theorem 1. Now it is easy to check that the rooks on the three families of adjacent parallel straight lines of hexagons dominate each hexagon of B_n.

3. Lower Bound

The straight-line sequences of edge-adjacent hexagons of B_n are called x-lines if they have x hexagons. There are three n-lines (diagonals) which, for odd n, have the central hexagon in common and which, for even n, pairwise have one of the three central hexagons in common. The six x-lines at the border of B_n have $x = \frac{n+1}{2}$ for odd n and alternatingly $x = \frac{n+2}{2}$ and $x = \frac{n}{2}$ for even n, say $x = \frac{n}{2}$ at the top.

The strategy for the proofs of the lower bounds is to check all x-lines with $x = n, n-1, n-2, ...$

(i) whether it must contain a rook,

(ii) whether it must not contain a rook, or

(iii) whether both cases containing a rook or not having a rook are to be distinguished.

There are two possibilities for (i):

$(\text{i})_1$: Since each of the assumed γ rooks from outside an x-line can threaten at most two hexagons of the x-line and since each of p pairs of the γ rooks that threaten one and the same hexagon threatens at most three hexagons of the x-line, it follows that at most $2\gamma - p$ hexagons are threatened and $2\gamma - p < x$ forces a rook on the x-line if $p < x$.

$(\text{i})_2$: If for a hexagon of an x-line the other two lines through this hexagon both contain no rook, then a rook on this x-line is forced.

If for (ii) an x-line has a hexagon, so that one of the two other lines through the hexagon contains a rook and so that the other line, say an y-line, has been chosen as without a rook because of $2\gamma - p = y$, $p < y$, and this hexagon is threatened only once, then no rook on this x-line is forced.

For (iii) it must hold for the x-line $2\gamma - p \geq x$, $p < x$.

3.1. $\mathbf{n \equiv 3 \pmod 4}$

Assume that $\gamma(n) \le \frac{n-3}{2}$. Because $2\frac{n-3}{2} < n-2$, all $(n-i)$-lines for $i \le 2$ each have a rook. If a rook is assumed on every $(n-j)$-line for $j \le i-1$, then on every $(n-i)$-line there are $p = i-1$ hexagons which are threatened twice (see Figure 3). Hence, with $(\mathrm{i})_1$ and because of $2\gamma - p \le 2\frac{n-3}{2} - (i-1) < n-i$ for $i < \frac{n+1}{2}$ it follows

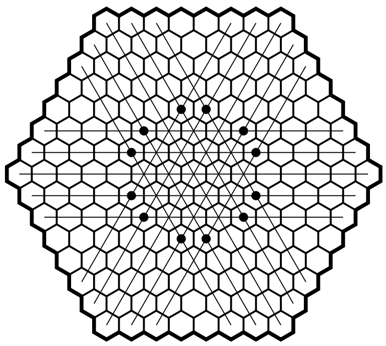

Figure 3: Case $i = 3$ for B_{15}.

that all $(n-i)$-lines must each contain a rook. Since all $(n-i)$-lines for $i \ge \frac{n-3}{4}$ are required to cover all hexagons of B_n, there are at least $2i+1 \ge 2\frac{n-3}{4}+1 = \frac{n-1}{2}$ parallel x-lines, each containing a rook, contradicting $\gamma \le \frac{n-3}{2}$.

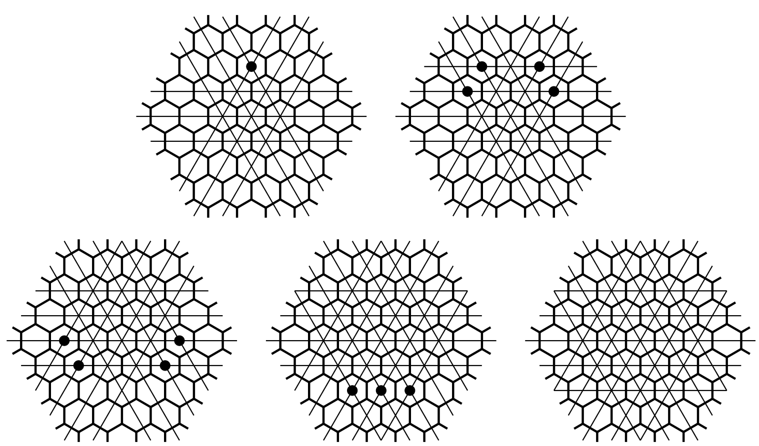

Figure 4: All $(n-t)$-lines with a rook for $t = 2$.

3.2. $\mathbf{n \equiv 1 \pmod 4}$

For $n = 4s + 1$ it is assumed $\gamma(n) \leq \frac{n-1}{2}$. Because $2\frac{n-1}{2} < n$, every n-line must have a rook. If all x-lines for $n \geq x \geq n - t + 1$ each contain a rook, then on each $(n - t)$-line there are $p = t - 1$ hexagons that have already been threatened twice. Hence, with (iii) and because of $2\frac{n-1}{2} - (t - 1) = n - t$ for $t < \frac{n+1}{2}$, every $(n - t)$-line could contain a rook or not contain a rook. Note that in the later case, only exactly one threat is allowed for all hexagons of an $(n - t)$-line, except for those $p = t - 1$ hexagons that are twice threatened.

If there is a rook on one of the six $(n - t)$-lines, then each of the two neighboring $(n - t)$-lines now has t doubly threatened hexagons, so that with $(\text{i})_1$ and because of $2\frac{n-1}{2} - t < n - t$ for $t < \frac{n}{2}$ these $(n - t)$-lines also contain a rook. Now the next two neighboring $(n - t)$-lines also have t doubly threatened hexagons, so they also must have a rook as before. Finally, the sixth $(n - t)$-line then has $t + 1$ doubly threatened hexagons, and it must have a rook as before (see Figure 4). So if one of the $(n - t)$-lines contains a rook, then all six must contain a rook. Thus there are only two possibilities: that all $(n - t)$-lines have a rook and that all are without a rook.

Now the six $(n - t)$-lines are supposed to be the largest that are without a rook.

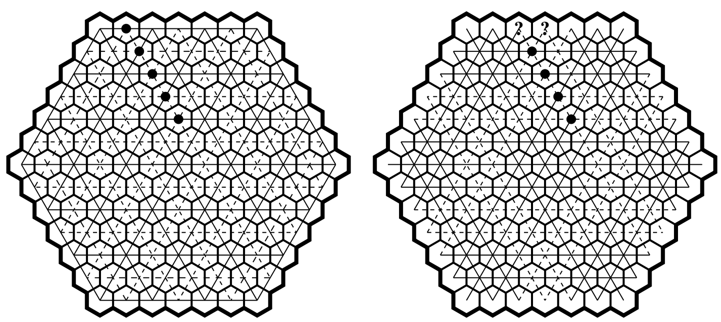

Figure 5: Cases $t = 1$ and $t = 2$ for B_{13}.

$t = 1$: In this case all $(n - i)$-lines for $i = 0$, 1, 2, ... are alternatingly with or without a rook if i is even or odd, respectively. This is true for $i = 0$ and $i = 1$ because $t = 1$. If all $(n - 2j - 1)$-lines are without a rook ($j \geq 0$) then every $(n - 2j - 2)$-line is forced to have a rook because of a common hexagon together with an $(n-1)$-line (without rook) and an $(n-2j-1)$-line (without rook) and with $(\text{i})_2$. If then all $(n - 2j - 2)$-lines have a rook then every $(n - 2j - 3)$-line is forced to be without a rook because of a common hexagon together with an $(n - 1)$-line (without rook) and an $(n - 2j - 2)$-line (with rook) and with (ii) (see Figure 5).

Since then on every x-line with a rook hexagons occur that are threatened only once, all x-lines with a rook are required for a domination of all hexagons of B_n. So there are $\lceil\frac{4s+1}{2}\rceil = 2s+1 = \frac{n+1}{2}$ parallel x-lines that each have a rook in contradiction to $\gamma(n) \le \frac{n-1}{2}$. It can be noted that for $n = 4s-1$, two x-lines, each with a rook, are omitted, so that $\gamma(n) = \gamma(4s-1) = \lceil\frac{4s+1}{2}\rceil - 2 = 2s-1 = \frac{n-1}{2}$ could apply.

$t \ge 2$: Any $(n-x-t)$-line for $1 \le x \le t-1$ is forced to be without a rook because of a common hexagon together with an $(n-t)$-line (without rook) and an $(n-x)$-line (with rook) and with (ii). Next, any $(n-x-t)$-line for $t \le x \le 2t-1$ is forced to have a rook because of a common hexagon together with an $(n-t)$-line (without rook) and an $(n-x)$-line (without rook) and with $(\text{i})_2$ (see Figure 5). Then on the one hand any $(n-3t)$-line is forced to be without a rook because of a common hexagon together with an $(n-t)$-line (without rook) and an $(n-2t)$-line (with rook) and with (ii). On the other hand any $(n-3t)$-line is forced to have a rook because of a common hexagon together with an $(n-t-1)$-line (without rook) and an $(n-2t+1)$-line (without rook) and with $(\text{i})_2$ (see Figure 5). This is a contradiction and thus a domination cannot exist for $n-3t \ge \frac{n+1}{2}$, that is, for $n \ge 6t+1$. For $n < 6t+1$ there are rooks on each $(n-x)$-line for $0 \le x \le t-1$ and for $2t \le x \le 3t-1$. Then B_n has $2(t-1)+1 \ge \frac{n+1}{2}$ rooks on parallel lines if $n \le 4t-3$ and $2(t-1)+1+2j \ge \frac{n+1}{2}$ rooks on parallel lines if $n \ge 4t+2j-1$, $j \ge 1$. Both cases are contradicting $\gamma(n) \le \frac{n-1}{2}$ which finishes the proof for $n \equiv 1\,(mod\,4)$.

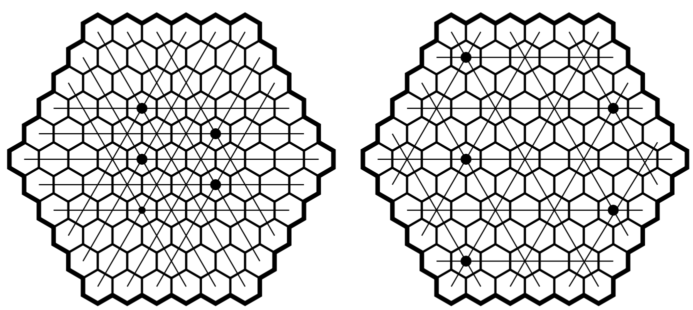

Figure 6: Two dominations for $n = 11$.

It can be noted that for $n = 4t-1$, that is, for $j = 0$, there are only $2(t-1)+1 = \frac{n-1}{2}$ rooks on parallel lines so that $\gamma(n) = \frac{n-1}{2}$ could apply. Together with the note in case $t = 1$ there are only two possibilities for $\gamma(n) = \frac{n-1}{2}$ if $n \equiv 3\,(mod\,4)$ both of which are similar by a factor of 2. Then all maximum independences of rooks for the triple threatened hexagons determine all minimum dominations of rooks for B_n (see Figure 6). This gives the following.

Corollary 1. *For $n \equiv 3 \,(mod\, 4)$, all minimum dominations with rooks on B_n are determined by all maximum independences with rooks on $B_{(n-1)/2}$.*

3.3. n ≡ 0 (mod 2)

Assume that $\gamma(n) < \frac{n}{2}$. Then $2(\frac{n}{2}-1) < n-1$ implies that all $(n-i)$-lines for $i \leq 1$ must have a rook. If a rook is assumed on every $(n-j)$-line for $j \leq i-1$, then on the six $(n-i)$-lines there are alternatingly $p = i$ and $p = i-2$ hexagons which are threatened twice (see Figure 7). With $(\mathrm{i})_1$ and because of $2\frac{n-2}{2} - i < n-i$ for

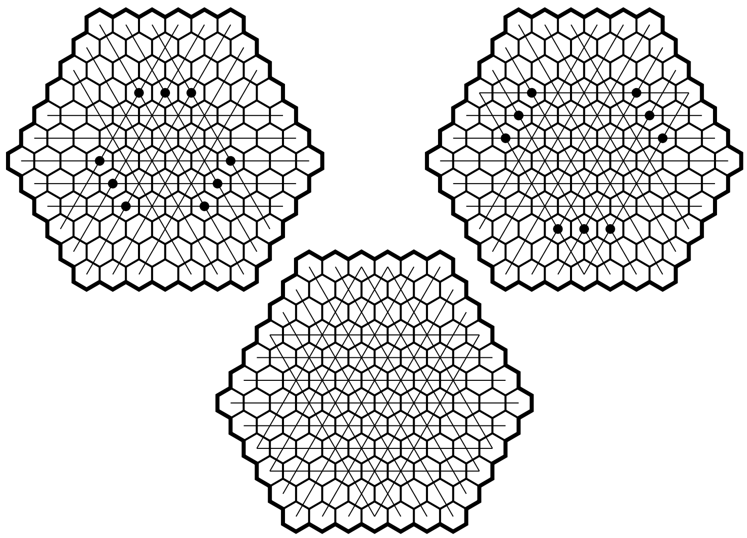

Figure 7: Case $i = 3$ for B_{12}.

$i < \frac{n}{2}$ those three $(n-i)$-lines with i doubly threatened hexagons also must contain a rook. Then the three remaining $(n-i)$-lines also each have $p = i$ hexagons which are threatened twice (see Figure 7) and as before these $(n-i)$-lines also must have a rook. Now all $(n-i)$-lines for $i \geq \lfloor \frac{n}{4} \rfloor$ are needed to cover all hexagons of B_n so that there are at least $2i+1 \geq 2\lfloor \frac{n}{4} \rfloor + 1 \geq \frac{n}{2}$ parallel lines, each containing a rook, contradicting $\gamma(n) < \frac{n}{2}$. This completes the proof of Theorem 1.

It has to be remarked that after the submission it was noticed that in [2] the domination number has already been determined, but only for odd n and with a completely different proof. The rooks are referred to as queens in [2], although queens should be able to move in six directions.

References

[1] J.-P. Bode and H. Harborth, Independent chess pieces on euclidean boards, *J. Combin. Math. Combin. Comput.* **33** (2000), 209–223.

[2] A.P. Burger and C.M. Mynhardt, Queens on hexagonal boards, *J. Combin. Math. Combin. Comput.* **31** (1999), 97–111.

[3] H. Harborth and H. Nienborg, Grid domination on hexagonal boards, *Congr. Numer.* **233** (2019), 11–20.

[4] S.M. Hedetniemi, S.T. Hedetniemi, and R. Reynolds, Combinatorial problems on chessboards, II, in *Domination in Graphs*, M. Dekker, New York, 1998, 133–162.

[5] J.D.E. Konhauser, D. Velleman, and S. Wagon, *Which Way did the Bicycle Go?* The Mathematical Association of America, Washington DC, MAA, 1996.

[6] S. Wagon, Graph theory problems from hexagonal and traditional chess, *College Math. J.* **45** (2014), 278–287.