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STRONGLY IMAGE PARTITION REGULAR MATRICES

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Abstract

A $u \times v$ matrix A with rational entries is *image partition regular over* $\mathbb{N}$ provided that whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are monochromatic. We say that A is *strongly image partition regular over* $\mathbb{N}$ provided that for every IP-set C in $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are in C. Many characterizations of image partition regular matrices are known. We provide here two sufficient conditions and one necessary condition for a matrix with rank u to be strongly image partition regular and show that such matrices can be expanded horizontally at will. We provide several examples showing that our results are sharp.

1. Introduction

We let $\mathbb{N}$ be the set of positive integers and $\omega = \mathbb{N} \cup \{0\}$.

Definition 1.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with rational entries.

(1) The matrix A is *kernel partition regular over* $\mathbb{N}$ provided that whenever $\mathbb{N}$ is finitely colored, there exists monochromatic $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = \vec{0}$.

(2) The matrix A is *image partition regular over* $\mathbb{N}$ (IPR/$\mathbb{N}$) provided that whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are monochromatic.

In 1933 Richard Rado [11] characterized kernel partition regular matrices in terms of the "columns condition".

Definition 1.2. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. For $i \in \{1, 2, \ldots, v\}$, let $\vec{c}_i$ be column i of A. Then A satisfies the *columns condition* if and only if there exist $m \in \mathbb{N}$ and a partitiion $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, v\}$ such that

(1) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$ and

(2) for each $j \in \{2, 3, \ldots, m\}$, if any, $\sum_{i \in I_j} \vec{c}_i$ is a linear combination over $\mathbb{Q}$ of $\{\vec{c}_i : i \in \bigcup_{t=1}^{j-1} I_t\}$.

Theorem 1.3. (Rado [11]). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is kernel partition regular over $\mathbb{N}$ if and only if A satisfies the columns condition.*

Say that a subset C of $\mathbb{N}$ is *large* if for every kernel partition regular matrix A, there exists $\vec{x}$ in the kernel of A with all entries of $\vec{x}$ in C. Rado conjectured that if a large subset of $\mathbb{N}$ is finitely colored, then there will be a monochromatic large subset. This conjecture was proved by Walter Deuber in 1973 [2] using what he called (m, p, c)-*sets*. These (m, p, c)-sets are images of certain "first entries" matrices. Part of Deuber's results included the fact that first entries matrices are image partition regular over $\mathbb{N}$.

We follow the custom of denoting the entries of a matrix by the lower case letter corresponding to the name of the matrix.

Definition 1.4. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with rational entries. Then A is a *first entries matrix* if and only if no row of A is $\vec{0}$ and whenever $i, j \in \{1, 2, \ldots, u\}$ and $k = \min\{t \in \{1, 2, \ldots, v\} : a_{i,t} \neq 0\} = \min\{t \in \{1, 2, \ldots, v\} : a_{j,t} \neq 0\}$, then $a_{i,k} = a_{j,k} > 0$. An element b of $\mathbb{Q}$ is a *first entry* of A if and only if there is some row i of A such that $b = a_{i,k}$ where $k = \min\{t \in \{1, 2, \ldots, v\} : a_{i,t} \neq 0\}$.

Image partition regular matrices were first characterized in 1993 [5]. One of these characterizations involves first entries matrices.

Theorem 1.5. [5]. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with rational entries. Then A is image partition regular over $\mathbb{N}$ if and only if there exist $m \in \mathbb{N}$ and a $u \times m$ first entries matrix B such that for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.*

Since the publication of [5] several other characterizations of IPR/$\mathbb{N}$ matrices have been obtained. Theorem 15.24 in [8] lists twelve statements that are equivalent to IPR/$\mathbb{N}$. Some of these, first obtained in [6], are included in the following theorem. Two that are of interest to us involve "central" sets. Central sets were introduced by Hillel Furstenberg in [4] and defined in terms of topological dynamics.

Theorem 1.6. [6]. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *A is image partition regular over* $\mathbb{N}$.

(b) *For each central set* C *in* $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\} \neq \emptyset$.

(c) *For each central set* C *in* $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ *is central in* $\mathbb{N}^v$.

(d) *For each column* $\vec{c} \in \mathbb{Q}^u$, $\begin{pmatrix} A & \vec{c} \end{pmatrix}$ *is image partition regular over* $\mathbb{N}$.

(e) *For each row* $\vec{r} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ *there exists* $b \in \mathbb{Q} \setminus \{0\}$ *such that* $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$ *is image partition regular over* $\mathbb{N}$.

It was an idea of Vitaly Bergelson [1] to characterize central sets in terms of the algebra of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. See [8, Definition 4.42] for the algebraic definition of central set and [8, Chapter 19] for a proof of the equivalence of the algebraic and dynamical definitions of central. We will not go into the precise definitions in this paper since we will not be using the algebra of the Stone-Čech compactification of a discrete semigroup here. What is important for us here is that central sets are IP-sets.

Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Definition 1.7. Let $(S,+)$ be a commutative semigroup and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S. Then $FS(\langle x_n \rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. If $k, m \in \mathbb{N}$ and $k \leq m$, then $FS(\langle x_n \rangle_{n=k}^m) = \big\{ \sum_{n\in F} x_n : \emptyset \neq F \subseteq \{k, k+1, \ldots, m\}\big\}$.

Definition 1.8. Let $(S,+)$ be a commutative semigroup and let $C \subseteq S$. Then C is an *IP-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq C$.

For readers familar with the algebra of the Stone-Čech compactification βS of a discete semigroup S, we remark that a subset C of S is an IP-set if and only if C is a member of an idempotent in βS. See [8, Theorem 5.12].

Lemma 1.9. *Let* C *be an IP-set in* $\mathbb{N}$ *and let* $m \in \mathbb{N}$. *There is an increasing sequence* $\langle x_n \rangle_{n=1}^\infty$ *in* $\mathbb{N}$ *such that* $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq C \cap m\mathbb{N}$.

Proof. By [8, Lemma 6.6] $C \cap m\mathbb{N}$ is an IP-set so one can pick $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq C \cap m\mathbb{N}$. By combining successive terms, we may presume that $\langle x_n \rangle_{n=1}^\infty$ is increasing. $\square$

Definition 1.10. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *strongly image partition regular over* $\mathbb{N}$ (SIPR/$\mathbb{N}$) provided whenever C is an IP-set in $\mathbb{N}$, there exists $\vec{x}$ in $\mathbb{N}^v$ such that $A\vec{x} \in C^u$.

We shall see in Section 2 that strongly image partition matrices are indeed image partition regular. It is easy to see that the converse fails. The simplest nontrivial instance of van der Waerden's theorem [12] tells us that the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

is image partition regular. On the other hand, if $a \in \mathbb{N} \setminus \{1, 2\}$ a simple consideration of the base a expansions shows that $FS(\langle a^t \rangle_{t=1}^{\infty})$ does not contain any length 3 arithmetic progressions, so that matrix is not strongly image partition regular over $\mathbb{N}$.

We shall see in Section 3 that, if one adds the assumption that the rank of A is u, where u is the number of rows, one gets a substantial collection of SIPR/$\mathbb{N}$ matrices. Further, in this section we develop sufficient conditions for such a matrix to be SIPR/$\mathbb{N}$ as well as one necessary condition. These conditions are in terms of the inverse of a matrix consisting of u linearly independent columns of A.

Section 4 is primarily devoted to examples.

In Section 5 we will extend the notion of strong image partition regularity to infinite matrices.

2. Strongly Image Partition Regular Over S

In this section we present some results that apply to arbitrary commutative semigroups. Unfortunately there have been different definitions in the literature for the notion of image partition regularity over a commutative semigroup. We use here the definition that we used in [9]. (See the discussion in [9] for reasons for the choice.)

If a commutative semigroup has an identity, we denote that identity by 0. If not, then of course $S \setminus \{0\} = S$. If S is cancellative and $x \in S$, then by $-x$ we mean the inverse of x in the group of differences of S.

Definition 2.1. Let S be a commutative semigroup, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix. If S is cancellative, and therefore embeddable in a group, then A is *appropriate for* S provided no row of A is zero and the entries of A come from $\mathbb{Z}$. If S is not cancellative, then A is *appropriate for* S provided no row of A is zero and the entries of A come from ω.

Definition 2.2. Let S be a commutative semigroup, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix which is appropriate for S. Then A is *image partition regular over* S (IPR/S) if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that the entries of $A\vec{x}$ are monochromatic.

In [8, Definition 5.9] what we are calling *image partition regular* here was called *strongly image partition regular*.

Definition 2.3. Let S be an infinite commutative semigroup, let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix which is appropriate for S. Then A is *strongly image partition regular over S* (SIPR/S) if and only if whenever C is an IP-set contained in $S \setminus \{0\}$, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that $A\vec{x} \in C^u$.

Since we have defined *strongly* image partition regular, we pause to show that very weak hypotheses guarantee that a SIPR/S matrix is in fact IPR/S.

Theorem 2.4. *Let S be an infinite commutative semigroup, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix which is appropriate for S. Assume that $S \setminus \{0\}$ is an IP-set in S and that A is SIPR/S. Then A is IPR/S.*

Proof. Let $r \in \mathbb{N}$ and assume that $S \setminus \{0\} = \bigcup_{i=1}^r D_i$. Pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq S \setminus \{0\}$. Then $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq \bigcup_{i=1}^r D_i$ so by [8, Corollary 5.15] pick a sequence $\langle y_n \rangle_{n=1}^\infty$ and $i \in \{1, 2, \ldots, r\}$ such that $C = FS(\langle y_n \rangle_{n=1}^\infty) \subseteq D_i$. Pick $\vec{x} \in (S \setminus \{0\})^v$ such that $A\vec{x} \in C^u$. Then the entries of $A\vec{x}$ are all in D_i. $\square$

It is easy to see that if S is weakly cancellative, that is if for each $x, y \in S$, $\{z \in S : x + z = y\}$ is finite, then $S \setminus \{0\}$ is an IP-set in S. In fact, if $\{x \in S : \{y \in S : x + y = 0\}$ is infinite$\}$ is finite, then it is routine to construct a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq S \setminus \{0\}$.

We see now that if S satisfies this weak hypothesis, then SIPR/S matrices satisfy a conclusion similar to Theorem 1.6(c).

Theorem 2.5. *Let S be an infinite commutative semigroup, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix which is appropriate for S. Assume that $S \setminus \{0\}$ is an IP-set in S and that A is SIPR/S. Then for each IP-set C in $S \setminus \{0\}$, $\{x \in S^v : A\vec{x} \in C^u\}$ is an IP-set in S^v.*

Proof. Let C be an IP-set in $S \setminus \{0\}$ and pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq C$. Pick $\vec{y}(1) \in (S \setminus \{0\})^v$ and $F_{1,1}, F_{1,2}, \ldots, F_{1,u}$ in $\mathcal{P}_f(\mathbb{N})$ such that

$$A\vec{y}(1) = \begin{pmatrix} \sum_{t \in F_{1,1}} x_t \\ \vdots \\ \sum_{t \in F_{1,u}} x_t \end{pmatrix}.$$

Let $n \in \mathbb{N}$ and assume we have chosen $\vec{y}(n)$ and $F_{n,1}, F_{n,2}, \ldots, F_{n,u}$. Let $m = \max \bigcup_{i=1}^u F_{n,i}$. Then $FS(\langle x_t \rangle_{t=m+1}^\infty)$ is an IP-set in $S \setminus \{0\}$ so pick $\vec{y}(n+1) \in (S \setminus \{0\})^v$ and $F_{n+1,1}, F_{n+1,2}, \ldots, F_{n+1,u}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\min \bigcup_{i=1}^u F_{n+1,i} > m$ and

$$A\vec{y}(n+1) = \begin{pmatrix} \sum_{t \in F_{n+1,1}} x_t \\ \vdots \\ \sum_{t \in F_{n+1,u}} x_t \end{pmatrix}.$$

Given $H \in \mathcal{P}_f(\mathbb{N})$ and $i \in \{1,2,\ldots,u\}$, let $K_i = \bigcup_{n\in H} F_{n,i}$. Then

$$A\big(\sum_{n\in H} \vec{y}\,(n)\big) = \begin{pmatrix} \sum_{t\in K_1} x_t \\ \vdots \\ \sum_{t\in K_u} x_t \end{pmatrix}.$$

□

In the generality of Theorem 2.5 we do not see that we can guarantee that $FS(\langle \vec{y}\,(n)\rangle_{n=1}^{\infty}) \subseteq (S \setminus \{0\})^v$; that is, that $0 \notin FS(\langle \vec{y}\,(n)\rangle_{n=1}^{\infty})$.

Definition 2.3 applies to the semigroup $(\mathbb{N},+)$ and differs from Definition 1.10 because in the latter the entries of A were allowed to be fractions. We see now that this makes no essential difference.

Theorem 2.6. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, let $d \in \mathbb{N}$ such that all entries of dA are in $\mathbb{Z}$. If for every IP-set C in $\mathbb{N}$, $\{\vec{a} \in \mathbb{N}^v : (dA)\vec{a} \in C^u\} \neq \emptyset$, then for every IP-set C in $\mathbb{N}$, $\{\vec{b} \in \mathbb{N}^v : A\vec{b} \in C^u\} \neq \emptyset$.*

Proof. Let C be an IP-set in $\mathbb{N}$. Pick $\vec{a} \in \mathbb{N}^v$ such that $(dA)\vec{a} \in C^u$. let $\vec{b} = d\vec{a}$. Then $A\vec{b} \in C^u$. □

3. Strongly Image Partition Regular Over $\mathbb{N}$

We begin by showing that if the rank of the $u \times v$ matrix is u, then the property of being SIPR/$\mathbb{N}$ shares one of the strong conclusions applying to the property of being IPR/$\mathbb{N}$, namely the condition of Theorem 1.6(d).

Definition 3.1. Let S be a semigroup. A subset D of S is an *IP*-set* provided it has nonempty intersection with every IP-set in S.

Lemma 3.2. *Let $k, v \in \mathbb{N}$. Then $\{\vec{x} \in \mathbb{N}^v : \text{for all } i \in \{1,2,\ldots,v\}\,,\, x_i > k\}$ is an IP*-set in $\mathbb{N}^v$.*

Proof. Let $D = \{\vec{x} \in \mathbb{N}^v : \text{for all } i \in \{1,2,\ldots,v\}\,,\, x_i > k\}$ and let C be an IP-set in $\mathbb{N}^v$. Pick a sequence $\langle \vec{x}_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}^v$ such that $FS(\langle \vec{x}_n\rangle_{n=1}^{\infty}) \subseteq C$. Then $\sum_{n=1}^{k+1} \vec{x}_n \in C \cap D$. □

Theorem 3.3. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with rational entries such that $\operatorname{rank}(A) = u$ and A is SIPR/$\mathbb{N}$. Let $\vec{y} \in \mathbb{Q}^u$. Then $\begin{pmatrix} A & \vec{y} \end{pmatrix}$ is SIPR/$\mathbb{N}$.*

Proof. Since the columns of A span $\mathbb{Q}^u$, pick $\vec{z} \in \mathbb{Q}^v$ such that $A\vec{z} = \vec{y}$. Pick $m \in \mathbb{N}$ such that $m\vec{z} \in \mathbb{Z}^v$ and let $k = \max\big(\{1\} \cup \{mz_i : i \in \{1,2,\ldots,v\}\}\big)$. Let C be an IP-set in $\mathbb{N}$. Now $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is an IP-set in $\mathbb{N}^v$ by Theorem 2.5 and

$\{\vec{x} \in \mathbb{N}^v : \text{for all } i \in \{1,2,\ldots,v\}\,,\, x_i > k\}$ is an IP*-set in $\mathbb{N}^v$ so pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$ and $x_i > k$ for each $i \in \{1,2,\ldots,u\}$.

Define $\vec{w} \in \mathbb{Q}^{v+1}$ by $w_j = x_j - mz_j$ if $j \leq v$ and $w_{v+1} = m$. Note that $\vec{w} \in \mathbb{N}^{v+1}$. Also $\begin{pmatrix} A & \vec{y} \end{pmatrix} \vec{w} = A\vec{x} - A(m\vec{z}) + m\vec{y} = A\vec{x} \in C^u$. □

The rank$(A) = u$ hypothesis cannot be simply omitted as seen by considering the matrix $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. We saw in the introduction that the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ is not SIPR/$\mathbb{N}$. On the other hand, the rank$(A) = u$ assumption is not necessary since any column can be added to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the result will be SIPR/$\mathbb{N}$. (We will show in the next section that any $2 \times v$ matrix which is IPR/$\mathbb{N}$ is SIPR/$\mathbb{N}$.)

We have two sufficient conditions for a $u \times v$ matrix with rank u to be SIPR/$\mathbb{N}$ and one necessary condition.

Theorem 3.4. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with rational entries and rank u, and assume that B consists of u linearly independent columns of A. Let $D = B^{-1}$ and for $i \in \{1,2,\ldots,u\}$, let $\vec{c}_i$ be column i of D. Assume there is nonempty $I \subseteq \{1,2,\ldots,u\}$ such that all entries of $\sum_{i\in I} \vec{c}_i$ are positive. Then A is SIPR/$\mathbb{N}$.*

Proof. By Theorem 3.3 we may presume that $A = B$. Pick $m \in \mathbb{N}$ such that for each $(i,j) \in \{1,2,\ldots,u\} \times \{1,2,\ldots,u\}$, $md_{i,j} \in \mathbb{Z}$. Let C be an IP-set in $\mathbb{N}$. By Lemma 1.9 we may pick an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq C \cap m\mathbb{N}$.

Pick $n \in \mathbb{N}$ such that for each $i \in \{1,2,\ldots,u\}$, $x_n \sum_{j\in I} d_{i,j} + \sum_{j\notin I} d_{i,j} x_1 > 0$.

For $j \in \{1,2,\ldots,u\}$, let $\alpha_j = x_n$ if $j \in I$ and let $\alpha_j = x_1$ if $j \notin I$. Let

$$\vec{y} = B^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_u \end{pmatrix}.$$

Then $A\vec{y} \in C^u$ so it suffices to show that $\vec{y} \in \mathbb{N}^u$. Let $i \in \{1,2,\ldots,u\}$. Then

$$\begin{array}{rcl} y_i & = & \sum_{j=1}^u d_{i,j}\alpha_j \\ & = & \sum_{j\in I} d_{i,j} x_n + \sum_{j\notin I} d_{i,j} x_1\,. \end{array}$$

Since x_n and x_1 are in $m\mathbb{N}$, $y_i \in \mathbb{Z}$. By the choice of x_n, $y_i \in \mathbb{N}$. □

Theorem 3.5. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with rational entries and rank u, and assume that B consist of u linearly independent columns of A. Let $D = B^{-1}$ and assume that the first nonzero entry of each row of D is positive. Then A is SIPR/$\mathbb{N}$.*

Proof. By Theorem 3.3 we may presume that $A = B$. Pick $m \in \mathbb{N}$ such that for each $(i,j) \in \{1,2,\ldots,u\} \times \{1,2,\ldots,u\}$, $md_{i,j} \in \mathbb{Z}$. Let C be an IP-set in $\mathbb{N}$. By Lemma 1.9 we may pick an increasing sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq C \cap m\mathbb{N}$.

For $i \in \{1,2,\ldots,u\}$, let $\mu(i) = \min\{j \in \{1,2,\ldots,u\} : d_{i,j} \neq 0\}$. Let $I = \big\{\mu(i) : i \in \{1,2,\ldots,u\}\big\}$, let $k = |I|$, and let $m_1, m_2, \ldots, m_k$ enumerate I in order. Note that $m_1 = 1$.

If $k = 1$, so for all $i \in \{1,2,\ldots,u\}$, $\mu(i) = 1$, let $\alpha_j = x_1$ if $j > 1$ and pick $n_1 > 1$ such that for all $i \in \{1,2,\ldots,u\}$, $d_{i,1}x_{n_1} + \sum_{j=2}^u d_{i,j}\alpha_j > 0$. Let $\alpha_1 = x_{n_1}$.

Now assume that $k > 1$. For $j \in \{1,2,\ldots,u\} \setminus I$, if any, let $\alpha_j = x_1$. Pick $n_k > 1$ such that for each i with $\mu(i) = m_k$, $d_{i,m_k}x_{n_k} + \sum_{j=m_k+1}^u d_{i,j}\alpha_j > 0$ and let $\alpha_{m_k} = x_{n_k}$.

Given $l \in \{1,2,\ldots,k-1\}$, having chosen n_{l+1} and $\alpha_{m_{l+1}}$, pick $n_l > 1$ such that for each i with $\mu(i) = m_l$, $d_{i,m_l}x_{n_l} + \sum_{j=m_l+1}^u d_{i,j}\alpha_j > 0$ and let $\alpha_{m_l} = x_{n_l}$.

Let $\vec{y} = B^{-1}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_u \end{pmatrix}$.

It suffices to show that $\vec{y} \in \mathbb{N}^u$ so let $i \in \{1,2,\ldots,u\}$ and pick l such that $\mu(i) = m_l$. Then

$$\begin{array}{rcl} y_i & = & \sum_{j=m_l}^u d_{i,j}\alpha_j \\ & = & d_{i,m_l}x_{n_l} + \sum_{j=m_l+1}^u d_{i,j}\alpha_j\,. \end{array}$$

Since each α_j is in $m\mathbb{Z}$, $y_i \in \mathbb{Z}$. By the choice of x_{n_l}, $y_i \in \mathbb{N}$. □

Theorem 3.6. *Let $u \in \mathbb{N}$. Let A be a $u \times u$ matrix with rational entries and rank u, let $D = A^{-1}$, and for $i \in \{1,2,\ldots,u\}$, let $\vec{c}_i$ be column i of D. If A is SIPR/$\mathbb{N}$, then there exists a nonempty subset I of $\{1,2,\ldots,u\}$ such that all entries of $\sum_{i\in I}\vec{c}_i$ are nonnegative.*

Proof. Suppose not. For each nonempty $I \subseteq \{1,2,\ldots,u\}$ let $\vec{f}(I) = \sum_{i\in I}\vec{c}_i$ and pick $s(I) \in \{1,2,\ldots,u\}$ such that $\vec{f}(I)_{s(I)} < 0$. For $\vec{x}, \vec{y} \in \mathbb{Q}^u$, let $||\vec{x} - \vec{y}|| = \max\big\{|x_i - y_i| : i \in \{1,2,\ldots,u\}\big\}$.

For this paragraph fix nonempty $I \subseteq \{1,2,\ldots,u\}$ and let χ_I be the characteristic function of I. Note that $D\chi_I = \sum_{i\in I}\vec{c}_i = \vec{f}(I)$. Pick $\epsilon(I) > 0$ such that if $\vec{x} \in \mathbb{Q}^u$ and $||\vec{x} - \chi_I|| < \epsilon(I)$, then $||D\vec{x} - \vec{f}(I)|| < |\vec{f}(I)_{s(I)}|$.

Let $\epsilon = \min\big\{\epsilon(I) : \emptyset \neq I \subseteq \{1,2,\ldots,u\}\big\}$. Inductively choose a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that for each n, $\epsilon x_{n+1} > \sum_{i=1}^n x_i$. Pick $F_1, F_2, \ldots, F_u \in \mathcal{P}_f(\mathbb{N})$ and $\vec{y} \in \mathbb{N}^u$ such that $A\vec{y} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_u \end{pmatrix}$ where for each $i \in \{1,2,\ldots,u\}$, $\alpha_i = \sum_{t\in F_i} x_t$. Pick k such that $\alpha_k = \max\big\{\alpha_i : i \in \{1,2,\ldots,u\}\big\}$ and let $m_k = \max F_k$. We can presume that $m_k > 1$. Now

$$\frac{1}{\alpha_k} D \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_u \end{pmatrix} = D \begin{pmatrix} \alpha_1/\alpha_k \\ \vdots \\ \alpha_u/\alpha_k \end{pmatrix}.$$

Note that, if $i \in \{1, 2, \ldots, u\}$ and $\max F_i < m_k$, then by the choice of the sequence, $0 < \alpha_i/\alpha_k < \epsilon$ while if $\max F_i = m_k$, then $|\alpha_i/\alpha_k - 1| < \epsilon$. To verify the latter statement note that $\alpha_i/\alpha_k \leq 1$ and $\alpha_i/\alpha_k \geq x_{m_k}/(\sum_{t=1}^{m_k} x_t) =$ $x_{m_k}/(x_{m_k} + \sum_{t=1}^{m_k-1} x_t) > x_{m_k}/(x_{m_k} + \epsilon x_{m_k}) = 1/(1+\epsilon) > 1 - \epsilon$.

Let $I = \{i \in \{1, 2, \ldots, u\} : \max F_i = m_k\}$ and let $\vec{x} = \begin{pmatrix} \alpha_1/\alpha_k \\ \vdots \\ \alpha_u/\alpha_k \end{pmatrix}$. Then $||\vec{x} - \chi_I|| < \epsilon \leq \epsilon(I)$ so $||D\vec{x} - \vec{f}(I)|| < |\vec{f}(I)_{s(I)}|$. Now $D\vec{x} = (1/\alpha_k) D \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_u \end{pmatrix} =$ $(1/\alpha_k)\vec{y}$ so $||(1/\alpha_k)\vec{y} - \vec{f}(I)|| < |\vec{f}(I)_{s(I)}|$. Then $|(1/\alpha_k) y_{s(I)} - \vec{f}(I)_{s(I)}| < |\vec{f}(I)_{s(I)}|$ so $(1/\alpha_k) y_{s(I)} < 0$ and thus $y_{s(i)} < 0$, a contradiction. □

Finally we have a special situation where one column of A^{-1} has one zero entry and the rest of its entries are positive.

Theorem 3.7. *Let $u \in \mathbb{N} \setminus \{1\}$, let A be a $u \times u$ matrix with rational entries and rank u, and let $D = A^{-1}$. Assume we have $i, j \in \{1, 2, \ldots, u\}$ such that*

(1) $d_{i,j} = 0$ and

(2) if $k \in \{1, 2, \ldots, u\} \setminus \{i\}$, then $d_{k,j} > 0$.

The following statements are equivalent.

(a) A is SIPR/$\mathbb{N}$.

(b) A is IPR/$\mathbb{N}$.

(c) There exists $\vec{y} \in \mathbb{N}^u$ such that $A\vec{y} \in \mathbb{N}^u$.

(d) There exists $l \in \{1, 2, \ldots, u\}$ such that $d_{i,l} > 0$.

Proof. That $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are trivial.

To see that $(c) \Rightarrow (d)$, pick $\vec{y} \in \mathbb{N}^u$ such that $\vec{z} = A\vec{y} \in \mathbb{N}^u$. Suppose that for each $l \in \{1, 2, \ldots, u\}$, $d_{i,l} \leq 0$.

Then $\vec{y} = D\vec{z}$ so $y_i = \sum_{l=1}^{u} d_{i,l} z_i \leq 0$, a contradiction.

To see that $(d) \Rightarrow (a)$, let C be an IP-set in $\mathbb{N}$. Pick $m \in \mathbb{N}$ such that all entries of mD are in $\mathbb{Z}$. By Lemma 1.9, pick an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C \cap m\mathbb{N}$.

Pick $l \in \{1,2,\ldots,u\} \setminus \{j\}$ such that $d_{i,l} > 0$. For $t \in \{1,2,\ldots,u\} \setminus \{j,l\}$ let $\alpha_t = x_1$. Pick n_1 such that $d_{i,l}x_{n_1} + \sum_{t\in\{1,2,\ldots,u\}\setminus\{j,l\}} d_{i,t}\alpha_t > 0$ and let $\alpha_l = x_{n_1}$. Pick n_2 such that for each $k \in \{1,2,\ldots,u\}\setminus\{i\}$, $d_{k,j}x_{n_2} + \sum_{t\in\{1,2,\ldots,u\}\setminus\{j\}} d_{k,t}\alpha_t > 0$ and let $\alpha_j = x_{n_2}$. If $\vec{y} = D\vec{\alpha}$, then $A\vec{y} = \vec{\alpha} \in C^u$. □

4. Examples

The following theorem will be used in some of the examples of this section.

Theorem 4.1. *Let $v \in \mathbb{N} \setminus \{1\}$ and let A be a $1 \times v$ or $2 \times v$ matrix with rational entries such that A is IPR/$\mathbb{N}$. Then A is SIPR/$\mathbb{N}$.*

Proof. If A has only one row, our claim is immediate from Theorem 3.3 and the trivial fact that the matrix (c) is IPR/$\mathbb{N}$ if and only if $c > 0$, in which case it is also SIPR/$\mathbb{N}$. So we may suppose that A has two rows. By [8, Theorem 15.24(g)] we may pick $m \in \{1,2\}$, a $v \times m$ matrix G with entries from ω and no row equal to $\vec{0}$, $c \in \mathbb{N}$, and a $2 \times m$ first entries matrix B with entries from ω whose only first entry is c such that $AG = B$. (The fact that $m \leq 2$ is not part of the statement of Theorem 15.24(g), but in the proof that (a) implies (g), $\{I_1, I_2, \ldots, I_m\}$ is a partition of $\{1,2,\ldots,u\}$.)

Let C be an IP-set in $\mathbb{N}$. We will show that there is some $\vec{y} \in \mathbb{N}^m$ such that $B\vec{y} \in C^2$. Then letting $\vec{x} = G\vec{y}$, we have that $\vec{x} \in \mathbb{N}^v$ and $A\vec{x} \in C^2$.

Assume first that $\text{rank}(B) = 1$ so that $B = \begin{pmatrix} c \\ c \end{pmatrix}$ or for some $b \in \omega$, $B = \begin{pmatrix} c & b \\ c & b \end{pmatrix}$. In this case, our claim follows because it holds for matrices with only one row.

So assume that $\text{rank}(B) = 2$. By switching rows if need be we either have that $B = \begin{pmatrix} c & a \\ 0 & c \end{pmatrix}$ for some $a \in \omega$ or $B = \begin{pmatrix} c & a \\ c & b \end{pmatrix}$ for some $a, b \in \omega$ with $a < b$. In the first case, our claim follows from Theorem 3.7. So assume that $B = \begin{pmatrix} c & a \\ c & b \end{pmatrix}$ for some $a, b \in \omega$ with $a < b$. Pick by Lemma 1.9 a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq c(b-a)\mathbb{N}$. Pick $n \in \mathbb{N} \setminus \{1\}$ such that $x_n > \frac{ax_1}{b-a}$. Let $y_1 = \frac{x_n}{c} - \frac{ax_1}{c(b-a)}$ and let $y_2 = \frac{x_1}{b-a}$. Then $\vec{y} \in \mathbb{N}^2$ and $B\vec{y} = \begin{pmatrix} x_n \\ x_n + x_1 \end{pmatrix} \in C^2$. □

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. A is a first entries matrix so is IPR/$\mathbb{N}$ and so by Theorem 4.1 A is SIPR/$\mathbb{N}$. Now $A^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ so A does not satisfy the hypotheses of either Theorem 3.4 or Theorem 3.5 so neither of these sufficient conditions is necessary.

Now let $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Then $B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$ so B satisfies the hypothese of Theorem 3.4 but not of Theorem 3.5. And $C^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ so C satisfies the hypothese of Theorem 3.5 but not of Theorem 3.4. Therefore the two sufficient conditions are independent.

Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then A is a first entries matrix and

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 2 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}$$

so A satisfies the hypothesis of Theorem 3.6. It is a consequence of the next theorem that A is not SIPR/$\mathbb{N}$, so the necessary condition of Theorem 3.6 is not sufficient.

Theorem 4.2. *Let* $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ *and let* $C = FS(\langle 2^{4t}\rangle_{t=1}^{\infty})$. *Then* $\{\vec{y} \in \mathbb{N}^3 : A\vec{y} \in C^3\} = \emptyset$.

Proof. Suppose we have $\vec{y} \in \mathbb{N}^3$ such that $A\vec{y} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \in C^3$. Pick $F, G, H \in \mathcal{P}_f(\mathbb{N})$ such that $\alpha_1 = \sum_{t\in F} 2^{4t}$, $\alpha_2 = \sum_{t\in G} 2^{4t}$, and $\alpha_3 = \sum_{t\in H} 2^{4t}$ where $F, G, H \in \mathcal{P}_f(\mathbb{N})$.

Then multiplying by A^{-1} we see that $\alpha_2 > 0$, $4\alpha_3 > \alpha_1 - \alpha_2$, and $\alpha_1 - \alpha_2 > 2\alpha_3$.

Let $m = \max H$. Then $2^{4m} \leq \alpha_3 < 2^{4m+1}$ so $2^{4m+1} \leq 2\alpha_3 < 2^{4m+2}$ and $2^{4m+2} \leq 4\alpha_3 < 2^{4m+3}$. Therefore $2^{4m+1} < \alpha_1 - \alpha_2 < 2^{4m+3}$.

Now $\alpha_1 - \alpha_2 = \sum_{t\in F\setminus G} 2^{4t} - \sum_{t\in G\setminus F} 2^{4t}$. Since $\alpha_1 > \alpha_2$, $F \setminus G \neq \emptyset$. Let $k = \max(F \setminus G)$.

Case 1. $G \setminus F = \emptyset$. Then $2^{4k} \leq \alpha_1 - \alpha_2 < 2^{4k+1}$.

Case 2. $G \setminus F \neq \emptyset$. Let $r = \max(G \setminus F)$ and note that $r < k$. Then $2^{4k} \leq \sum_{t\in F\setminus G} 2^{4t} < 2^{4k+1}$ and $-2^{4r+1} < -\sum_{t\in G\setminus F} 2^{4t} \leq -2^{4r}$ so $2^{4k-1} < 2^{4k} - 2^{4r+1} < \alpha_1 - \alpha_2 < 2^{4k+1} - 2^{4r} < 2^{4k+1}$.

Thus, in either case, $2^{4k-1} < \alpha_1 - \alpha_2 < 2^{4k+1}$. Thus $2^{4k-1} < \alpha_1 - \alpha_2 < 2^{4m+3}$ and $2^{4m+1} < \alpha_1 - \alpha_2 < 2^{4k+1}$. Since $2^{4m+1} < 2^{4k+1}$, $m \leq k - 1$. So $2^{4k-1} < 2^{4m+3} \leq 2^{4(k-1)+3} = 2^{4k-1}$, a contradiction. □

We saw in Theorem 3.3 that a strong analogue of Theorem 1.6(d) is valid for SIPR/$\mathbb{N}$ matrices. We shall show now that the natural analogues of Theorem 1.6(e) are not valid for SIPR/$\mathbb{N}$ matrices using two examples. One of these starts with a square matrix and the other ends up with a square matrix. The matrices $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ are first entries matrices so are SIPR/$\mathbb{N}$ by Theorem 4.1. We will see that they cannot be extended by adding a multiple of the row $\begin{pmatrix} 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ nor by adding a multiple of the row $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

Let $b \in \mathbb{Q} \setminus \{0\}$, let $A = \begin{pmatrix} b & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ and let $B = \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$. If $b = 1$, we saw in the introduction that A is not SIPR/$\mathbb{N}$ and it is a consequence of Theorem 4.2 that B is not SIPR/$\mathbb{N}$. Further, since $B^{-1} = \begin{pmatrix} \frac{1}{b} & 0 & 0 \\ \frac{1}{2b} & 2 & -\frac{1}{2} \\ -\frac{1}{2b} & -1 & \frac{1}{2} \end{pmatrix}$ it is an immediate consequence of Theorem 3.6 that B is not SIPR/$\mathbb{N}$ if $b \neq 1$. We establish now a stronger result.

Theorem 4.3. *Let $b \in \mathbb{Q} \setminus \{0, 1\}$. Then neither A nor B is IPR/$\mathbb{N}$.*

Proof. First suppose that A is IPR/$\mathbb{N}$. Then by [8, Theorem 15.24(b)] there exist positive rationals s and t such that

$$D = \begin{pmatrix} bs & 0 & -1 & 0 & 0 \\ s & t & 0 & -1 & 0 \\ s & 2t & 0 & 0 & -1 \end{pmatrix}$$

satisfies the columns condition. For $i \in \{1, 2, 3, 4, 5\}$, let $\vec{c}_i$ be column i of D. In particular, there exists nonempty $I_1 \subseteq \{1, 2, 3, 4, 5\}$ such that $\sum_{i \in I_1} \vec{c}_i = \vec{0}$. One cannot have $I_1 \subseteq \{3, 4, 5\}$. If $2 \in I_1$, then $t = 2t$ contradicting the fact that $t > 0$. So $2 \notin I_1$ and $1 \in I_1$. But then from row 2 one sees that $s = 1$ while from row 1 one sees that $s = \frac{1}{b}$.

Similarly, if one assumes that B is IPR/$\mathbb{N}$ one easily derives a contradiction from the assumption that there exist positive rationals r, s, and t such that

$$\begin{pmatrix} br & 0 & 0 & -1 & 0 & 0 \\ 0 & s & t & 0 & -1 & 0 \\ r & 2s & 4t & 0 & 0 & -1 \end{pmatrix}$$

satisfies the columns condition. □

Our original motive for the current study was [10, Question 4.9].

Definition 4.4. A Q-set in $\mathbb{N}$ is a set which contains a set of the form $\{x_n - x_m : m < n \text{ in } \mathbb{N}\}$ for some increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$.

We remark that every IP-set in $\mathbb{N}$ contains a Q-set in $\mathbb{N}$. Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. If $y_n = \sum_{i=1}^n x_i$, then $\{y_n - y_m : m, n \in \mathbb{N}, n > m\} \subseteq FS(\langle x_n\rangle_{n=1}^\infty)$.

Question 4.5. [10] *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from ω which is IPR/$\mathbb{N}$ such that* $\text{rank}(A) = u$.

(1) If C is an IP-set in $\mathbb{N}$, must $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ be an IP-set in $\mathbb{N}^v$?

(2) If C is a Q-set in $\mathbb{N}$, must $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ be a Q-set in $\mathbb{N}^v$?

By Theorems 2.5 and 4.1, the answer to (1) is "yes" if $u = 2$, even without the rank assumption. By Theorem 4.2, the alnswer to (1) is "no" if $u = 3$.

The proof of the following theorem is very similar to the proof of Theorem 4.1, but the conclusion is weaker; we cannot assert that $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^2\}$ is a Q-set. That is, we cannot assert the existence of a sequence $\langle \vec{z}(n)\rangle_{n=1}^\infty$ in $\mathbb{N}^v$ such that $\vec{z}(n) - \vec{z}(m) \in \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^2\}$ whenever $m < n$ in $\mathbb{N}$.

Theorem 4.6. *Let $v \in \mathbb{N} \setminus \{1\}$ and let A be a $2 \times v$ matrix with rational entries such that A is IPR/$\mathbb{N}$. If C is a Q-set in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^2\} \neq \emptyset$.*

Proof. By [8, Theorem 15.24(g)] we may pick $m \in \{1, 2\}$, a $v \times m$ matrix G with entries from ω and no row equal to $\vec{0}$, $c \in \mathbb{N}$, and a $2 \times m$ first entries matrix B with entries from ω whose only first entry is c such that $AG = B$.

Let C be a Q-set in $\mathbb{N}$. We will show that there is some $\vec{y} \in \mathbb{N}^m$ such that $B\vec{y} \in C^2$. Then letting $\vec{x} = G\vec{y}$, we have that $\vec{x} \in \mathbb{N}^v$ and $A\vec{x} \in C^2$.

Assume first that $\text{rank}(B) = 1$ so that $B = \begin{pmatrix} c \\ c \end{pmatrix}$ or for some $b \in \omega$, $B = \begin{pmatrix} c & b \\ c & b \end{pmatrix}$. In the former case, let $b = 0$. Pick an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $\{x_n - x_m : m < n \text{ in } \mathbb{N}\} \subseteq C$. By thinning the sequence we may assume that for all m and n, $x_n \equiv x_m \pmod{c}$. Pick $n \in \mathbb{N}$ such that $x_n - x_1 > b$. If $B = \begin{pmatrix} c \\ c \end{pmatrix}$, let $y = \dfrac{x_n - x_1}{c}$ so that $By = \begin{pmatrix} x_n - x_1 \\ x_n - x_1 \end{pmatrix} \in C^2$. If $B = \begin{pmatrix} c & b \\ c & b \end{pmatrix}$, let $\vec{y} = \begin{pmatrix} \frac{x_n - x_1}{c} - b \\ c \end{pmatrix}$. Then $B\vec{y} = \begin{pmatrix} x_n - x_1 \\ x_n - x_1 \end{pmatrix} \in C^2$.

Now assume that $\text{rank}(B) = 2$. By switching rows if need be we either have that $B = \begin{pmatrix} c & a \\ 0 & c \end{pmatrix}$ for some $a \in \omega$ or $B = \begin{pmatrix} c & a \\ c & b \end{pmatrix}$ for some $a, b \in \omega$ with $a < b$.

Assume first that $B = \begin{pmatrix} c & a \\ 0 & c \end{pmatrix}$ for some $a \in \omega$. Pick an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $\{x_n - x_m : m < n \text{ in } \mathbb{N}\} \subseteq C$. By thinning the sequence we may assume that for all m and n, $x_n \equiv x_m \pmod{c^2}$. Pick $n \in \mathbb{N}$ such that $c(x_n - x_1) > a(x_2 - x_1)$. Let $y_1 = \dfrac{x_n - x_1}{c} - \dfrac{a(x_2 - x_1)}{c^2}$ and let $y_2 = \dfrac{x_2 - x_1}{c}$. Then $\vec{y} \in \mathbb{N}^2$ and $B\vec{y} = \begin{pmatrix} x_n - x_1 \\ x_2 - x_1 \end{pmatrix} \in C^2$.

Now assume that $B = \begin{pmatrix} c & a \\ c & b \end{pmatrix}$ for some $a, b \in \omega$ with $a < b$. Pick an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\{x_n - x_m : m < n \text{ in } \mathbb{N}\} \subseteq C$. By thinning the sequence we may assume that for all m and n, $x_n \equiv x_m \pmod{c(b-a)}$. Pick $n \in \mathbb{N} \setminus \{1, 2\}$ such that $(x_n - x_2 > \dfrac{a(x_2 - x_1)}{b - a}$. Let $y_1 = \dfrac{x_n - x_2}{c} - \dfrac{a(x_2 - x_1)}{c(b-a)}$ and let $y_2 = \dfrac{x_2 - x_1}{b - a}$. Then $\vec{y} \in \mathbb{N}^2$ and $B\vec{y} = \begin{pmatrix} x_n - x_2 \\ x_n - x_1 \end{pmatrix} \in C^2$. $\square$

Theorem 4.7. *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, let $C = \{2^{2t} - 2^{2s} : s < t \text{ in } \mathbb{N}\}$, and let $B = \{\vec{y} \in \mathbb{N}^2 : A\vec{y} \in C^2\}$. There do not exist $\vec{y}$ and $\vec{z}$ in B such that $\vec{y} + \vec{z} \in B$. In particular B is not a Q-set.*

Proof. For the "in particular" assertion note that if $\vec{y}\,(1)$, $\vec{y}\,(2)$, and $\vec{y}\,(3)$ are in $\mathbb{N}^2$, then $(\vec{y}\,(3) - \vec{y}\,(2)) + (\vec{y}\,(2) - \vec{y}\,(1)) = (\vec{y}\,(3) - \vec{y}\,(1))$.

For $x \in \mathbb{N}$, let $\phi(x) = \max(\{i \in \omega : 2^i \leq x\})$. Observe that, for every $x, y \in \mathbb{N}$ for which $\phi(x) = \phi(y)$, $\phi(x + y) = \phi(x) + 1$. Observe also that $\phi(x)$ is odd if $x \in C$.

We claim that, if $a, b, a + b \in C$ with $a > b$, there exist $s, t, p \in \mathbb{N}$ with $t > s > p$, such that $a = 2^{2t} - 2^{2s}$ and $b = 2^{2s} - 2^{2p}$. To see this, suppose that $a = 2^{2t} - 2^{2s}$ and $b = 2^{2r} - 2^{2p}$, where $p, r, s, t \in \mathbb{N}$, $t > s$ and $r > p$. Observe that $a + b \in C$ implies that $t > r$, because $\phi(a)$, $\phi(b)$ and $\phi(a + b)$ are odd. Since $2^{2t} - 2^{2s} + 2^{2r} - 2^{2p} = 2^{2n} - 2^{2m}$ for some $m, n \in \mathbb{N}$ with $n > m$, $2^{2t} + 2^{2r} + 2^{2m} = 2^{2s} + 2^{2p} + 2^{2n}$. Now $t > s$ and $t > r > p$. So $t = n$, and hence $2^{2r} + 2^{2m} = 2^{2s} + 2^{2p}$. Since $r > p$, it follows that $r = s$.

Suppose we have $\vec{y}$ and $\vec{z}$ in B such that $\vec{y} + \vec{z} \in B$. Observe that $A^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Let $A\vec{y} = \vec{w}$ and $A\vec{z} = \vec{x}$. Then $\vec{w}$ and $\vec{x}$ are in C^2, and $A^{-1}\vec{w}$ and $A^{-1}\vec{x}$ are in $\mathbb{N}^2$. So x_1 and x_2 are in C, and $(1/2)x_2 < x_1 < x_2$. Similarly, w_1 and w_2 are in C, and $(1/2)w_2 < w_1 < w_2$. Now $2^{\phi(x_2)} \leq x_2 < 2^{\phi(x_2)+1}$ so $2^{\phi(x_2)-1} \leq (1/2)x_2 < x_1 < x_2 < 2^{\phi(x_2)+1}$ and thus $\phi(x_1) = \phi(x_2) - 1$ or $\phi(x_1) = \phi(x_2)$. Since $\phi(x_1)$ and $\phi(x_2)$ are odd, $\phi(x_1) = \phi(x_2)$. Also $\phi(w_1) = \phi(w_2)$. This implies that $x_1 = 2^{2t} - 2^{2s}$ and $x_2 = 2^{2t} - 2^{2r}$ for some $s, t, r \in \mathbb{N}$, and $w_1 = 2^{2p} - 2^{2n}$ and $w_2 = 2^{2p} - 2^{2m}$ for some m, n, p in $\mathbb{N}$. Since $A(\vec{y} + \vec{z}) \in C^2$, $\vec{w} + \vec{x} \in C^2$. So $w_1 + x_1$ and $w_2 + x_2$ are in C, and thus $p = s = r$ and $x_1 = x_2$, a contradiction. $\square$

To conclude our discussion of [10, Question 4.9], we show that the matrix of Theorem 4.2 is a strong counterexample to part (2) of that question.

Theorem 4.8. *Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and let $C = \{2^{4t} - 2^{4s} : s < t \text{ in } \mathbb{N}\}$. Then $\{\vec{y} \in \mathbb{N}^3 : A\vec{y} \in C^3\} = \emptyset$.*

Proof. Recall that $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 2 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}$.

Let $\alpha_1 = 2^{4t} - 2^{4s}$, $\alpha_2 = 2^{4l} - 2^{4k}$, and $\alpha_3 = 2^{4m} - 2^{4r}$.

Suppose we have $\vec{y} \in \mathbb{N}^3$ such that $A\vec{y} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$. Then multiplying by A^{-1} we see that $\alpha_2 > 0$, $4\alpha_3 > \alpha_1 - \alpha_2$ and $\alpha_1 - \alpha_2 > 2\alpha_3$. That is, $2^{4m+2} - 2^{4r+2} > 2^{4t} - 2^{4s} + 2^{4k} - 2^{4l} > 2^{4m+1} - 2^{4r+1}$.

Since $\alpha_1 > \alpha_2$, either both $t = l$ and $k > s$ or $t > l$.

Case 1. $t = l$ and $k > s$. Then $2^{4m+2} > 2^{4m+2} - 2^{4r+2} > 2^{4k} - 2^{4s} > 2^{4m+1} - 2^{4r+1}$ Then $2^{4m+2} + 2^{4s} > 2^{4k}$. The highest power on the left is at least as big as $4k$ and $s < k$ so $4m + 2 \geq 4k$ and thus $m \geq k$.

Also $2^{4k} > 2^{4k} - 2^{4s} > 2^{4m+1} - 2^{4r+1}$ so $2^{4k} + 2^{4r+1} > 2^{4m+1}$. Since $r < m$ we must have $4k \geq 4m + 1$ so $k > m$, a contradiction.

Case 2. $t > l$. Then $2^{4m+2} > 2^{4m+2} - 2^{4r+2} > 2^{4t} - 2^{4s} + 2^{4k} - 2^{4l}$ so $2^{4m+2} + 2^{4s} + 2^{4l} > 2^{4t} + 2^{4k}$. Now $k < l < t$ so the highest power on the right is $4t$. Also $s < t$ and $l < t$ so $4s < 4t$ and $4l < 4t$ and (if $s = l$) $4s + 1 < 4t$ so we must have that $4m + 2 \geq 4t$ and therefore $m \geq t$.

Also $2^{4t} + 2^{4k} > 2^{4t} - 2^{4s} + 2^{4k} - 2^{4l} > 2^{4m+1} - 2^{4r+1}$ so $2^{4t} + 2^{4k} + 2^{4r+1} > 2^{4m+1}$. Now $r < m$ and $k < l < t$ so we must have $4t \geq 4m+1$ so $t > m$, a contradiction. □

It is easy to take a matrix which is not SIPR/$\mathbb{N}$ and make it SIPR/$\mathbb{N}$ by adding a column. For example, $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ is not even IPR/$\mathbb{N}$ but $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ is SIPR/$\mathbb{N}$.

Question 4.9. *Let $u, v \in \mathbb{N}$ with $u < v$ and let A be a $u \times v$ matrix with rational entries and* $\operatorname{rank}(A) = u$ *such that A is SIPR/$\mathbb{N}$. Must there exist u columns of A that form an SIPR/$\mathbb{N}$ matrix with rank u?*

5. Infinite Strongly Image Partition Regular Matrices

In this section we allow infinite matrices, so some earlier definitions must be modified. (If u and v are in $\mathbb{N}$, nothing changes.)

Definition 5.1. Let S be a commutative semigroup, let $u, v \in \mathbb{N} \cup \{\omega\}$, and let A be a $u \times v$ matrix. If $S = \mathbb{N}$, then A is *appropriate for S* provided no row of A is zero, the number of nonzero entries in each row is finite, and the entries of A come from $\mathbb{Q}$. If $S \neq \mathbb{N}$ and S is cancellative and therefore embeddable in a group, then A is *appropriate for S* provided no row of A is zero, the number of nonzero entries in each row is finite, and the entries of A come from $\mathbb{Z}$. If S is not cancellative, then

A is *appropriate for* S provided no row of A is zero, the number of nonzero entries in each row is finite, and the entries of A come from ω.

Except for the fact that the matrix in question is allowed to be infinite, the definitions of IPR/S and SIPR/S remain verbatim the same.

If $S \setminus \{0\}$ is not an IP-set, then any finite matrix which is appropriate for S is vacuously SIPR/S. If $S \setminus \{0\}$ is an IP-set, then any finite identity matrix is SIPR/S. So the number of finite matrices that are SIPR/S is infinite. Since the number of finite matrices with entries from $\mathbb{Q}$ is countable, one can enumerate the finite matrices that are SIPR/S.

We set out to produce an infinite matrix which is SIPR/S. It is based on the results of [3].

Definition 5.2. Let $(S, +)$ be a commutative semigroup. For each $n \in \mathbb{N}$ let $Y_n \in \mathcal{P}_f(S)$. Then

$$FS(\langle Y_n \rangle_{n=1}^{\infty}) = \{\textstyle\sum_{n\in F} x_n : F \in \mathcal{P}_f(\omega) \text{ and for each } n \in F\,,\ x_n \in Y_n\}\,.$$

Thus $FS(\langle Y_n \rangle_{n=1}^{\infty})$ is all finite sums choosing at most one term from each Y_n.

The following theorem can be proved using the algebra of βS copying the proof of [8, Theorem 6.16] almost verbatim. We present an elementary proof because it is so simple.

Theorem 5.3. *Let $(S, +)$ be a commutative semigroup such that $S \setminus \{0\}$ is an IP-set. Let $\langle A(n) \rangle_{n=1}^{\infty}$ enumerate the finite matrices that are (appropriate for S and) SIPR/S where each $A(n)$ is a $u(n) \times v(n)$ matrix. Let C be an IP-set contained in $S \setminus \{0\}$. There exists for each $n \in \mathbb{N}$, a choice of $\vec{x}(n) \in (S \setminus \{0\})^{v(n)}$ such that if Y_n is the set of entries of $A(n)\vec{x}(n)$, then $FS(\langle Y_n \rangle_{n=1}^{\infty}) \subseteq C$.*

Proof. Pick a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq C$. Pick $\vec{x}(1) \in (S \setminus \{0\})^{v(1)}$ such that $A(1)\vec{x}(1) \in \big(FS(\langle y_n \rangle_{n=1}^{\infty})\big)^{u(1)}$. Pick $m(1)$ such that all entries of $A(1)\vec{x}(1)$ are in $FS(\langle y_n \rangle_{n=1}^{m(1)}$. Inductively, let $k \in \mathbb{N}$ and assume we have chosen $\vec{x}(k)$ and $m(k)$. Pick $\vec{x}(k+1) \in (S \setminus \{0\})^{v(k+1)}$ such that $A(k+1)\vec{x}(k+1) \in \big(FS(\langle y_n \rangle_{n=m(k)+1}^{\infty})\big)^{u(k+1)}$. Pick $m(k+1)$ such that all entries of $A(k+1)\vec{x}(k+1)$ are in $FS(\langle y_n \rangle_{n=m(k)+1}^{m(k+1)})$. For each $k \in \mathbb{N}$ let Y_k be the set of entries of $A(k)\vec{x}(k)$. Then $FS(\langle Y_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq C$. □

Corollary 5.4. *Let $(S, +)$ be a commutative semigroup such that $S \setminus \{0\}$ is an IP-set and let $u, v, w \in \mathbb{N}$. Assume that A ia a $u \times v$ matrix which is SIPR/S and B is a $u \times w$ matrix which is SIPR/S. Then the matrix $\begin{pmatrix} A & B \end{pmatrix}$ is SIPR/S.*

Proof. Let $\langle A(n) \rangle_{n=1}^{\infty}$ be the enumeration in Theorem 5.3 and pick $n, m \in \mathbb{N}$ such that $A = A(n)$ and $B = A(m)$. Let C be an IP-set contained in $S \setminus \{0\}$. Then all entries of $\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \vec{x}(n) \\ \vec{x}(m) \end{pmatrix}$ are in $Y_n + Y_m \subseteq C$. □

The matrix in the following definition is based on the construction of a DH-matrix in [7] which started with an enumeration of all finite matrices with rational entries that are IPR/$\mathbb{N}$.

Definition 5.5. Let $(S,+)$ be a commutative semigroup such that $S\setminus\{0\}$ is an IP-set. A *Strong DH-matrix for* S is an $\omega\times\omega$ matrix $\mathbf{SD}$ defined as follows. Let $K=\mathbb{Q}$ if $S=\mathbb{N}$, let $K=\mathbb{Z}$ if $S\neq\mathbb{N}$ and S is cancellative, and otherwise let $K=\omega$. First fix an enumeration $\langle A(n)\rangle_{n=0}^{\infty}$ of the finite matrices with entries from K that are SIPR/$\mathbb{N}$. For each n, assume that $A(n)$ is a $u(n)\times v(n)$ matrix. For each $i\in\mathbb{N}$, let $\vec{0}_i$ be the 0 row vector with i entries. Let $\mathbf{SD}$ be an $\omega\times\omega$ matrix with all rows of the form $\vec{r}_1{}^\frown\vec{r}_2{}^\frown\vec{r}_3{}^\frown\ldots$ where each $\vec{r}_i$ is either $\vec{0}_{v(i)}$ or is a row of $A(i)$, at least one $\vec{r}_i$ is a row of $A(i)$ and for all but finitely many $i\in\mathbb{N}$, $\vec{r}_i=\vec{0}_{v(i)}$.

Corollary 5.6. *Let $(S,+)$ be a commutative semigroup such that $S\setminus\{0\}$ is an IP-set and let* $\mathbf{SD}$ *be a Strong DH-matrix for S. Then* $\mathbf{SD}$ *is SIPR/S.*

Proof. Let C be an IP-set contained in $S\setminus\{0\}$. For each $n\in\mathbb{N}$ pick $\vec{x}(n)$ as guaranteed by Theorem 5.3. Then $\mathbf{SD}\begin{pmatrix}\vec{x}(1)\\ \vec{x}(2)\\ \vdots\end{pmatrix}\in C^{\omega}$. □

We remark that the property of being SIPR/S can be very different for different semigroups S. It follows from the definition of an IP-set, that every matrix (finite or infinite) with entries in $\{0,1\}$, which has no row whose entries are all zero and finitely many nonzero entries in each row is SIPR/S for every commutative semigroup S. We will show that these are the only matrices with entries in ω which have this universal property by considering the semigroup $(\mathbb{N},\cdot)$.

Since the operation is written multiplicatively, some adjustment in notation is required. A set C is an IP-set in $(\mathbb{N},\cdot)$ provided there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FP(\langle x_n\rangle_{n=1}^{\infty})\subseteq C$ where $FP(\langle x_n\rangle_{n=1}^{\infty})=\{\prod_{n\in F}x_n : F\in\mathcal{P}_f(\mathbb{N})\}$. The assertion that the $u\times v$ matrix A is SIPR/$(\mathbb{N}\cdot)$ says that whenever C is an IP-set in $(\mathbb{N}\setminus\{1\},\cdot)$ there exists $\vec{x}\in(\mathbb{N}\setminus\{1\})^v$ such that $\vec{x}^A\in C^u$ where the entry in row i of $\vec{x}^A$ is $\prod_{j=1}^{v}x^{a_{i,j}}$.

Assume that A is a $u\times v$ matrix with entries from ω, has no row equal to $\vec{0}$, and has finitely many nonzero entries in each row. Assume that A has some entry $a_{i,j}\in\omega\setminus\{0,1\}$. Let $\langle p_n\rangle_{n=1}^{\infty}$ be the sequence of primes. If $\vec{x}\in(\mathbb{N}\setminus\{1\})^v$, then entry i of $\vec{x}^A$ has a repeated prime factor, so is not in $FP(\langle p_n\rangle_{n=1}^{\infty})$.

The situation is more complicated for matrices with entries in $\mathbb{Z}$. For example, if $A=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$, then A is SIPR/S for every commutative cancellative semigroup S because $A\begin{pmatrix}x_1+x_2\\ x_2\end{pmatrix}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$. It would be interesting to characterize the matrices with entries in $\mathbb{Z}$ which have this property.

If S is a Boolean group, then every finite or infinite matrix with entries in $\mathbb{Z}$, is SIPR/S if and only if it has an odd entry in every row. To see this, let B denote the matrix obtained from A by replacing every even entry by 0 and every odd entry by 1. Then, for every column vector $\vec{x}$ with entries in S which has the same number of entries as A has columns, $A\vec{x} = B\vec{x}$.

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