INTRODUCING SHIFT-CONSTRAINED RADO NUMBERS

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Abstract

The 2-color Rado number for $ax + by = z$ with positive integers $1 \leq a \leq b$ is known; this is the least integer such that any 2-coloring of the integers from 1 to the Rado number must include a solution to the equation consisting of numbers that have been assigned the same color. We modify the requirements by introducing constraints on the colorings. These constraints are motivated by symbolic dynamics, specifically the golden mean shift and a version of the even shift, both over a 2-letter alphabet. We establish several initial results, offer some conjectures, and outline possible directions for further research in this new study of shift-constrained Rado numbers.

1. Background

We consider a problem in Ramsey theory on the positive integers informed by symbolic dynamics. We begin with background material for each of these two areas.

For a positive integer r, call a map $\Delta : \{1, \ldots, n\} \to \{0, 1, \ldots, r-1\}$ an r-coloring. A set of integers is monochromatic if Δ assigns the same color to each element of the set. In this article, we limit our attention to $r = 2$. Within the context of a specific coloring, we will write Δ_0 for the set of integers assigned the color zero and similarly Δ_1.

Issai Schur showed that, for any $r \geq 1$, there is a least positive integer $S(r)$ such that any r-coloring of $\{1, \ldots, S(r)\}$ includes a monochromatic solution to the equation $x + y = z$. For example, the second Schur number $S(2) = 5$. There are two steps to verifying such a value. First, one demonstrates that there is a 2-coloring on $\{1, 2, 3, 4\}$ that contains no monochromatic solutions—the coloring given by $\Delta_0 = \{1, 4\}$ and $\Delta_1 = \{2, 3\}$ satisfies this condition, along with the coloring that reverses the assignments to zero and one. Clearly, the same coloring restricted to $\{1, 2, 3\}$ is equally valid, likewise restricted to $\{1, 2\}$ and to $\{1\}$. Second, one argues that any 2-coloring of $\{1, 2, 3, 4, 5\}$ would produce a monochromatic solution: The coloring Δ given for $\{1, 2, 3, 4\}$ (and its reverse) would result in a monochromatic solution for either color assignment of 5 as both $(x, y, z) = (1, 4, 5)$ and $(2, 3, 5)$ are solutions to $x + y = z$, and any other coloring will already have a monochromatic solution on $\{1, 2, 3, 4\}$.

Richard Rado extended these ideas to systems of general linear equations and developed a criterion for which systems have a bound analogous to $S(r)$; these are known as Rado numbers. See Landman and Robertson [4, Chapter 9] for additional background. In this article, we restrict our attention to the following parameters, where the Rado numbers are known to be finite.

Definition 1. Given integers a, b with $1 \leq a \leq b$, the *(2-color) Rado number* is the least positive integer $R(a, b)$ such that any 2-coloring of $\{1, \ldots, R(a, b)\}$ includes a monochromatic solution of $ax + by = z$.

Guo and Sun [2] proved

$$R(a, b) = a(a + b)^2 + b, \tag{1}$$

confirming a conjecture of the current author and Schaal [3] (the full results allow arbitrarily many variables). Work continues on determining the 2-color Rado number for the fully general linear equation $a_1x_1 + \cdots + a_{m-1}x_{m-1} = a_mx_m + c$; see Thanatipanonda [6] for some recent results towards this goal. There are also related results involving more colors, selected systems and nonlinear equations, and numeric structures beyond the integers. Ron Graham's wide interests in Ramsey theory included an article in this journal with Alexeev and Fox on minimal colorings without rational monochromatic solutions to $x_1 + x_2 + x_3 = 4x_4$ [1].

Moving to our other ingredient, symbolic dynamics begins with a choice of what symbols and patterns are allowed in the underlying system. The full r-shift allows all sequences over the alphabet $\{0, 1, \ldots, r-1\}$. For various theoretical and applied reasons, often only parts of the full shift are used. Subsets of the full shift are called shifts, subshifts, or shift spaces. See Lind and Marcus [5] for additional background.

We focus on two simple binary shifts (so $r = 2$), the golden mean shift and a version of the even shift, described next.

The golden mean shift is defined by forbidding consecutive ones. Table 1 shows the short binary sequences that can arise in the golden mean shift; notice that their

length	allowed words	count
1	0, 1	2
2	00, 01, 10	3
3	000, 001, 010, 100, 101	5
4	0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010	8

Table 1: Allowed length n words in the golden mean shift for small n.

length	allowed words	count
1	$0, 1$	2
2	$00, 10, 11$	3
3	$000, 001, 100, 110, 111$	5
4	$0000, 0010, 0011, 1000, 1001, 1100, 1110, 1111$	8

Table 2: Allowed length n words in our even shift for small n.

counts are Fibonacci numbers. Another reason for the name comes from the fact that the entropy of the golden mean shift is $\log(\varphi)$ where $\varphi = (1+\sqrt{5})/2$; entropy is a useful statistic in symbolic dynamics that is invariant under various operations and measures the information capacity of a shift [5, Chapter 4].

The version of the even shift that we use requires that runs of zeros have even length with the possible exception of a terminal run of zeros, which can have any length. Table 2 shows the short binary sequences allowed by this shift. For instance, 110 is allowed since the length 1 run of zeros is at the end of the word, but 101 is not allowed since that odd length run of zeros is nonterminal. (In the terminology of symbolic dynamics, this is the follower set of 1 in the even shift [5, Example 3.2.7] or a particular one-sided even shift; we will simply call it the even shift.) Notice that the word counts are the same as for the golden mean shift, which implies that the even shift also has entropy $\log(\varphi)$, but we will see that the two shifts differ notably in other ways.

Both of these shifts are closed under taking certain subwords. Specifically, if $x_1 \dots x_n$ is an allowed word in the golden mean shift then, for all $1 \le i \le j \le n$, the word $x_i \dots x_j$ is also allowed. The situation for our even shift is not as strong, but the following holds. If $x_1 \dots x_n$ is an allowed word in the even shift then, for all $1 \le j \le n$, the word $x_1 \dots x_j$ is also allowed.

In Section 2, we combine these ideas to create a new class of Rado number problems. Section 3 presents results related to the shift spaces mentioned here, with proofs in Section 4. We conclude in Section 5 with some conjectures and ideas for further investigations.

2. Shift-constrained Rado Numbers

We combine Ramsey theory on the integers and symbolic dynamics by requiring that colorings satisfy the conditions of a shift space. We have seen that shifts can treat different symbols in different ways, thus the Rado numbers for a given equation can vary depending on the color assigned to 1. (In other words, the standard initial step "Without loss of generality, assume that $\Delta(1) = 0$" is no longer valid.)

Definition 2. Given integers a, b with $1 \leq a \leq b$ and a binary shift S, the *S-constrained Rado number* $R_0^S(a,b)$ is the least positive integer such that any 2-coloring Δ of $\{1,\ldots,R_0^S(a,b)\}$ includes a monochromatic solution to $ax+by=z$ where $\Delta(1)=0$ and Δ satisfies the constraints of the shift S. Define $R_1^S(a,b)$ analogously with $\Delta(1)=1$.

As a first example, consider $x+y=z$ with the golden mean shift constraint, i.e., there is no i such that $\Delta(i)=\Delta(i+1)=1$. Note that the coloring detailed above, $\Delta_0=\{1,4\}$ and $\Delta_1=\{2,3\}$, is no longer valid. Write $R_0^{\varphi}(1,1)$ and $R_1^{\varphi}(1,1)$ for these particular shift-constrained Rado numbers; we will call them *golden Rado numbers*.

Suppose $\Delta(1)=0$. The solution $(1,1,2)$ forces $\Delta(2)=1$. By the golden mean shift requirement, we must have $\Delta(3)=0$. Then either possible coloring of 4 would give a monochromatic solution, $(1,3,4)$ in color zero or $(2,2,4)$ in color one. It is easy to check that the 2-coloring on $\{1,2,3\}$ given by $\Delta_0=\{1,3\}$ and $\Delta_1=\{2\}$ has no monochromatic solutions to $x+y=z$. We conclude that $R_0^{\varphi}(1,1)=4$.

For the other possibility, $\Delta(1)=1$, the reverse of the first coloring does work as before, since $\Delta_0=\{2,3\}$ and $\Delta_1=\{1,4\}$ has no consecutive numbers assigned color one. The issues with assigning a color to 5 are the same as before, thus $R_1^{\varphi}(1,1)=5$.

For a second example, consider $x+y=z$ with the even shift described above and write $R_0^e(1,1)$, $R_1^e(1,1)$ for these shift-constrained Rado numbers. With $\Delta(1)=0$, the solution $(1,1,2)$ precludes $\Delta(2)=0$, while $\Delta(2)=1$ would make an odd length nonterminal run of zeros, so $R^e(1,1)=1$. For $\Delta(1)=1$, the now-familiar coloring $\Delta_0=\{2,3\}$ and $\Delta_1=\{1,4\}$ has only an even length run of zeros and no monochromatic solutions, while no coloring of $\{1,2,3,4,5\}$ works as before, thus $R_1^e(1,1)=5$.

In both examples, the closure of the shifts under taking subwords, as discussed in Section 1, means that a coloring of $\{1,\ldots,n\}$ avoiding monochromatic solutions while satisfying the shift constraints restricts to a similarly valid coloring of $\{1,\ldots,k\}$ for all $1\leq k\leq n-1$.

The examples are consistent with the following result.

Proposition 1. *Given positive integers a, b with $1\leq a\leq b$ and a binary shift S, the S-constrained Rado numbers satisfy*

$$R_0^S(a,b)\leq R(a,b) \text{ and } R_1^S(a,b)\leq R(a,b).$$

In the case of the full shift F, we have $R_0^F(a,b)=R_1^F(a,b)=R(a,b)$.

Proof. The full shift F introduces no constraints, so the standard Rado number results apply, where color assignments to zero and one are interchangeable. Adding the constraints of a binary shift S may affect colorings such that monochromatic solutions must occur at smaller values, decreasing the Rado number, but requiring additional structure cannot increase the Rado number. □

Note that adding a nontrivial shift constraint does not necessarily decrease the Rado number, as $R_1^{\varphi}(1,1) = R_1^{e}(1,1) = R(1,1) = 5$.

In the next sections, we establish some initial results on $R_0^{\varphi}(a,b)$, $R_1^{\varphi}(a,b)$, and $R_0^{e}(a,b)$, $R_1^{e}(a,b)$.

3. Some Golden Mean Shift and Even Shift Constrained Rado Numbers

We begin with equations of the form $x + by = z$ and 2-colorings that satisfy the golden mean shift, i.e., there is no i for which $\Delta(i) = \Delta(i+1) = 1$.

Theorem 1. *Given a positive integer b, the golden Rado numbers are*

$$R_0^{\varphi}(1,b) = \begin{cases} 2b+2 & \text{if } b \text{ is odd,} \\ 2b & \text{if } b \text{ is even;} \end{cases}$$

$$R_1^{\varphi}(1,b) = \begin{cases} 5 & \text{if } b = 1, \\ 3b+1 & \text{if } b \geq 2. \end{cases}$$

The golden Rado numbers for $ax + by = z$ with $a \geq 2$ are more complicated. We prove one result for this shift, the case of equal coefficients. In Section 5, we mention other patterns suggested by computational data.

Theorem 2. *Given a positive integer a, the golden Rado numbers are*

$$R_0^{\varphi}(a,a) = 4a^2,\ R_1^{\varphi}(a,a) = 4a^2 + a.$$

Before considering Rado numbers with colorings constrained by the even shift, it is helpful to recall a 2-coloring detailed by Hopkins and Schaal. The following proposition comes from the proof of [3, Theorem 2] applied to the current case of a three variable equation.

Proposition 2. *Given integers $1 \leq a \leq b$, there are no monochromatic solutions to the equation $ax + by = z$ in the coloring of $\{1, \ldots, a(a+b)^2 + b - 1\}$ specified by*

$$\Delta_0 = \{1, 2, \ldots, a+b-1, (a+b)^2, (a+b)^2+1, \ldots, a(a+b)^2+b-1\},$$
$$\Delta_1 = \{a+b, a+b+1, \ldots, (a+b)^2-1\}.$$

This coloring consists of length $a+b-1$ and length $(a^2+ab-b)(a+b-1)$ runs of zeros and a length $(a+b)(a+b-1)$ run of ones.

As specified in the next theorem, several even shift constrained Rado numbers, where nonterminal runs of zeroes must have even length, match standard Rado numbers.

Theorem 3. *Given integers $1 \le a \le b$, the even shift constrained Rado numbers with $\Delta(1) = 1$ match the standard Rado numbers, i.e.,*

$$R_1^e(a,b) = R(a,b) = a(a+b)^2 + b.$$

Also, if $a+b$ is odd, then the same equality holds for colorings with $\Delta(1) = 0$, i.e.,

$$R_0^e(a,b) = R(a,b) = a(a+b)^2 + b.$$

Our final result mirrors Theorem 1, determining Rado numbers for equations of the form $x + by = z$ and 2-colorings that satisfy the even shift constraint.

Theorem 4. *Given a positive integer b, the even shift constrained Rado numbers are*

$$R_0^e(1,b) = \begin{cases} 1 & \text{if } b = 1, \\ b^2 + 3b + 1 & \text{if } b \text{ is even}, \\ b^2 + 2b & \text{if } b \text{ is odd and } b \ge 3; \end{cases}$$
$$R_1^e(1,b) = b^2 + 3b + 1.$$

Note that several cases match the standard Rado number $R(1,b) = b^2 + 3b + 1$.

4. Proofs of Results

The proofs of the four theorems are elementary but sometimes a bit lengthy. When arguments are analogous to previous ones, we skip some details.

Proof of Theorem 1. We organize the proof into four cases; it is helpful to split the $R_1^{\varphi}(1,b)$ proof into cases for even b and odd b even though the conclusion is the same for both. Each case requires two things. First, we demonstrate a coloring from 1 to one less than the Rado number that satisfies the conditions of the golden mean shift and contains no monochromatic solutions to $x + by = z$. Second, we show that any coloring from 1 to the Rado number satisfying the shift constraint includes a monochromatic solution.

(a) Consider the case b odd and colorings with $\Delta(1) = 0$. Write $b = 2k - 1$. We show that the coloring of $\{1, \ldots, 4k-1\}$ specified by

$$\Delta_0 = \{1, 3, \ldots, 4k-1\},\ \Delta_1 = \{2, 4, \ldots, 4k-2\}$$

includes no monochromatic solutions to $x + (2k-1)y = z$; clearly it satisfies the golden mean shift condition. By parity arguments, any solution has exactly one or three of x, y, z even. Therefore there can be no monochromatic solution with $x, y, z \in \Delta_0$. Now suppose $x, y \in \Delta_1$. Since $x + (2k-1)y > 2 + (4k-2) = 4k$, beyond the range of the coloring, there is no monochromatic solution in color one. This valid coloring shows that $R_0^{\varphi}(1, 2k-1) \geq 4k = 2b+2$.

To complete this case, we show that any coloring of $\{1, \ldots, 4k\}$ with $\Delta(1) = 0$ and satisfying the golden mean shift condition includes a monochromatic solution to $x + (2k-1)y = z$.

If $\Delta(2k) = 0$, then $(1, 1, 2k)$ would be a monochromatic solution in color zero, so we may assume that $\Delta(2k) = 1$. By the golden mean shift constraint, $\Delta(2k-1) = 0$ and $\Delta(2k+1) = 0$.

If $\Delta(2) = 0$, then $(2, 1, 2k+1)$ would be a monochromatic solution, so we may assume that $\Delta(2) = 1$.

Now either color assignment for $4k$ gives a monochromatic solution. If $\Delta(4k) = 0$, then $(2k+1, 1, 4k)$ would be a monochromatic solution in color zero. If $\Delta(4k) = 1$, then $(2, 2, 4k)$ would be a monochromatic solution in color one. This shows that $R_0^{\varphi}(1, 2k-1) \leq 4k$.

Together with the bound from the valid coloring, we conclude $R_0^{\varphi}(1, 2k-1) = 4k$, i.e., $R_0^{\varphi}(1, b) = 2b+2$ for $b = 2k-1$.

(b) Consider the case b even and colorings with $\Delta(1) = 0$. Write $b = 2k$. We will show that the coloring of $\{1, \ldots, 4k-1\}$ specified by

$$\Delta_0 = \{1, 3, \ldots, 2k-1, 2k, 2k+2, \ldots, 4k-2\},$$
$$\Delta_1 = \{2, 4, \ldots, 2k-2, 2k+1, 2k+3, \ldots, 4k-1\}$$

includes no monochromatic solutions to $x + 2ky = z$; clearly it satisfies the golden mean shift condition. It is straightforward to verify that this coloring is valid: For a solution (x, y, z) to have $z < 4k$ requires $x < 2k$ and $y = 1$, and in that range parity arguments similar to those in (a) can be made, etc. The validity of this coloring also follows from the second part of this case.

We show that any coloring of $\{1, \ldots, 4k\}$ with $\Delta(1) = 0$ and satisfying the golden mean shift condition includes a monochromatic solution to $x + 2ky = z$.

If $\Delta(2k+1) = 0$, then $(1, 1, 2k+1)$ would be a monochromatic solution, so we may assume that $\Delta(2k+1) = 1$. By the golden mean shift constraint, $\Delta(2k) = 0$ and $\Delta(2k+2) = 0$.

If $\Delta(2) = 0$, then $(2, 1, 2k+2)$ would be a monochromatic solution, so we may assume that $\Delta(2) = 1$. It follows that $\Delta(3) = 0$.

If $\Delta(2k+3) = 0$, then $(3, 1, 2k+3)$ would be a monochromatic solution, so we may assume that $\Delta(2k+3) = 1$. It follows that $\Delta(2k+4) = 0$.

If $\Delta(4) = 0$, then $(4, 1, 2k+4)$ would be a monochromatic solution, so we may assume that $\Delta(4) = 1$. It follows that $\Delta(5) = 0$.

This bootstrapping continues through $\Delta(4k-2) = 0$ by the golden mean shift constraint.

If $\Delta(2k-2) = 0$, then $(2k-2, 1, 4k-2)$ would be a monochromatic solution, so we may assume that $\Delta(2k-2) = 1$. It follows that $\Delta(2k-1) = 0$.

If $\Delta(4k-1)=0$, then $(2k-1,1,4k-1)$ would a monochromatic solution, so we may assume that $\Delta(4k-1)=1$.

At this point, we have shown that the coloring described above for $\{1,\ldots,4b-1\}$ does not contain any monochromatic solutions to $x+2ky=z$. (It is the unique such coloring, as the color assignments have all been forced.)

Now either color assignment for $4k$ gives a monochromatic solution. If $\Delta(4k)=0$, then $(2k,1,4k)$ would be a monochromatic solution in color zero. If $\Delta(4k)=1$, then Δ would violate the golden mean shift constraint, as the valid coloring requires $\Delta(4k-1)=1$. We conclude that $R_0^{\varphi}(1,2k)=4k$, i.e., $R_0^{\varphi}(1,b)=2b$ for $b=2k$.

(c) Consider the case b even and colorings with $\Delta(1)=1$. Write $b=2k$. The coloring of $\{1,\ldots,6k\}$ specified by

$$\Delta_0=\{2,4,\ldots,2k,2k+1,2k+2,\ldots,4k+1,4k+3,\ldots,6k-1\},$$
$$\Delta_1=\{1,3,\ldots,2k-1,4k+2,4k+4,\ldots,6k\}$$

includes no monochromatic solutions to $x+2ky=z$; clearly it satisfies the golden mean shift condition. Starting with this case, we omit verifications that the given colorings are valid. As in (b), one can show that this is the unique valid coloring in this case.

We show that any coloring of $\{1,\ldots,6k+1\}$ with $\Delta(1)=1$ and satisfying the golden mean shift constraint includes a monochromatic solution to $x+2ky=z$.

Since $\Delta(1)=1$, the golden mean shift constraint requires $\Delta(2)=0$.

If $\Delta(2k+1)=1$, then $(1,1,2k+1)$ would be a monochromatic solution, so we may assume that $\Delta(2k+1)=0$.

If $\Delta(4k+2)=0$, then $(2,2,4k+2)$ would be a monochromatic solution, so we may assume that $\Delta(4k+2)=1$. It follows that $\Delta(4k+3)=0$.

If $\Delta(3)=0$, then $(3,2,4k+3)$ would be a monochromatic solution, so we may assume that $\Delta(3)=1$.

Now either color assignment for $6k+1$ gives a monochromatic solution. If $\Delta(6k+1)=0$, then $(2k+1,2,6k+1)$ would be a monochromatic solution in color zero. If $\Delta(6k+1)=1$, then $(1,3,6k+1)$ would be a monochromatic solution in color one. With the valid coloring above, we conclude that $R_1^{\varphi}(1,2k)=6k+1$, i.e., $R_1^{\varphi}(1,b)=3b+1$ for b even.

(d) Consider the case b odd and colorings with $\Delta(1)=1$. We established in Section 2 that $R_1^{\varphi}(1,1)=5$, so we assume $b\geq 3$. Write $b=2k-1$. The coloring of $\{1,\ldots,6k-3\}$ specified by

$$\Delta_0=\{2,4,\ldots,2k-2,2k,2k+1,\ldots,4k-1,4k+1,4k+3,\ldots,6k-3\},$$
$$\Delta_1=\{1,3,\ldots,2k-1,4k,4k+2,\ldots,6k-4\}$$

includes no monochromatic solutions to $x+(2k-1)y=z$; clearly it satisfies the golden mean shift condition. In fact, it is the unique valid coloring in this case.

The argument to show that any coloring of $\{1,\ldots,6k-2\}$, with $\Delta(1)=1$ and satisfying the golden mean shift condition, includes a monochromatic solution to $x+(2k-1)y=z$ is very similar to the steps of (c). One can show

$$2, 2k, 4k+1 \in \Delta_0,\ 1, 3, 4k \in \Delta_1$$

from which no color assignment for $6k-2$ is valid due to the solutions $(2k,2,6k-2)$ and $(1,3,6k-2)$. With the coloring above, we conclude that $R_1^{\varphi}(1,2k-1)=6k-2$, i.e., $R_1^{\varphi}(1,b)=3b+1$ for odd $b\geq 3$. □

Proof of Theorem 2. For the equation $ax+ay=z$, first consider colorings with $\Delta(1)=0$. The coloring of $\{1,\ldots,4a^2-1\}$ specified by

$$\begin{gathered}\Delta_0=\{1,2,\ldots,2a-1,2a+1,2a+2,\ldots,3a-1,3a+1,3a+2,\ldots,\\ 4a^2-a-1,4a^2-a+1,4a^2-a+2,\ldots,4a^2-1\},\\ \Delta_1=\{2a,3a,\ldots,4a^2-a\}\end{gathered}$$

includes no monochromatic solutions to $ax+ay=z$; clearly it satisfies the golden mean shift condition. Following the convention adapted in the previous proof, we do not verify that the coloring in valid. We mention in passing that the color assignments are forced except for the integers greater than $4a^2-a$ (although the golden mean shift constraint still applies).

We show that any coloring of $\{1,\ldots,4a^2\}$ with $\Delta(1)=0$ and satisfying the golden mean shift condition includes a monochromatic solution to $ax+ay=z$.

If $\Delta(2a)=0$, then $(1,1,2a)$ would be a monochromatic solution, so we may assume that $\Delta(2a)=1$. It follows that $\Delta(2a-1)=0$ and $\Delta(2a+1)=0$.

Already, either color assignment for $4a^2$ gives a monochromatic solution. If $\Delta(4a^2)=0$, then $(2a-1,2a+1,4a^2)$ would be a monochromatic solution in color zero. If $\Delta(4a^2)=1$, then $(2a,2a,4a^2)$ would be a monochromatic solution in color one. With the valid coloring above, we conclude that $R_0^{\varphi}(a,a)=4a^2$.

Second, consider colorings with $\Delta(1)=1$. The coloring of $\{1,\ldots,4a^2+a-1\}$ specified by

$$\begin{gathered}\Delta_0=\{2,3,\ldots,4a-1,4a+1,4a+2,\ldots,5a-1,5a+1,5a+2,\ldots,\\ 4a^2-1,4a^2+1,4a^2+2,\ldots,4a^2+a-1\},\\ \Delta_1=\{1,4a,5a,\ldots,4a^2\}\end{gathered}$$

includes no monochromatic solutions to $ax+ay=z$; clearly it satisfies the golden mean shift condition. The color assignments are forced except for the integers greater than $4a^2$ (although the golden mean shift constraint still applies).

The argument that any coloring of $\{1,\ldots,4a^2+a\}$, with $\Delta(1)=1$ and satisfying the golden mean shift condition, includes a monochromatic solution to $ax+ay=z$

is very similar to the $\Delta(1) = 0$ case. One can show

$$2, 2a, 4a - 1 \in \Delta_0,\ 1, 4a \in \Delta_1$$

from which no color assignment for $4a^2 + a$ would be valid due to the solutions $(2, 4a - 1, 4a^2 + a)$ and $(1, 4a, 4a^2 + a)$. With the valid coloring above, we conclude that $R_1^{\varphi}(a, a) = 4a^2 + a$. □

Proof of Theorem 3. For the equation $ax + by = z$, first consider colorings with $\Delta(1) = 0$ constrained by the even shift. In the cases that the coloring described in Proposition 2 has only even length nonterminal runs of zeros, that coloring shows $R_0^e(a, b) \geq R(a, b)$. That will complete the proof of this case since $R_0^e(a, b) \leq R(a, b)$ by Proposition 1.

As described after Proposition 2, the length of the initial run of zeros is $a+b-1$ which is even exactly when $a + b$ is odd. Thus $R_0^e(a, b) = R(a, b)$ when $a + b$ is odd and Equation (1) provides the formula.

Second, consider colorings with $\Delta(1) = 1$. The coloring described in Proposition 2 is for standard Rado numbers, so the assignments to zero and one are interchangeable. Reversing the zero and one colors to satisfy $\Delta(1) = 1$ results in one run of zeros, length $(a + b)(a + b - 1)$, which is always even. Therefore the reversed coloring satisfies the even shift constraint and, as before, we conclude $R_1^e(a, b) = R(a, b) = a(a + b)^2 + b$. □

Proof of Theorem 4. The $b = 1$ case was treated in Section 2. Theorem 3 applies to the remaining cases except $R_0^e(1, b)$ when b is odd, so we take $b \geq 3$.

Let $b = 2k - 1$ with $k \geq 2$ and consider colorings with $\Delta(1) = 0$. We want to show that the even shift constrained Rado number is $b^2 + 2b = 4k^2 - 1$. First, we show that the coloring of $\{1, \dots, 4k^2 - 2\}$ specified by

$$\Delta_0 = \{1, 2, \dots, 2k - 2, 4k^2 - 2k, 4k^2 - 2k + 1, \dots, 4k^2 - 2\},$$
$$\Delta_1 = \{2k - 1, 2k, \dots, 4k^2 - 2k - 1\}$$

includes no monochromatic solutions to $x + (2k - 1)y = z$. Since the only nonterminal run of zeros has length $2k - 2$, the coloring satisfies the even shift condition. The standard argument concerning monochromatic solutions is straightforward (i.e., $x, y \leq 2k - 2$ in Δ_0 give $z \in \Delta_1$, then $x, y \in \Delta_1$ give $z \geq 4k^2 - 2k$, etc.), but we will show that this is the unique valid coloring in the second part of the proof.

We show that any coloring of $\{1, \dots, 4k^2 - 1\}$ with $\Delta(1) = 0$ and satisfying the even shift condition would include a monochromatic solution to $x + by = z$.

Let c be the least integer such that $\Delta(c) = 1$. We show that $c = 2k - 1$. Since $\Delta(1) = 0$, we know $c \geq 2$. By the definition of c, we have $\Delta(c - 1) = 0$.

If $\Delta(2ck) = 1$, then $(c, c, 2ck)$ would be a monochromatic solution, so we may assume that $\Delta(2ck) = 0$.

If $\Delta(2k+c-1)=0$, then $(2k+c-1, c-1, 2ck)$ would be a monochromatic solution, so we may assume that $\Delta(2k+c-1)=1$.

Either color assignment for $(2c+2)k-1$ would give a monochromatic solution. If $\Delta((2c+2)k-1)=0$, then $(2ck, 1, (2c+2)k-1)$ would be a monochromatic solution in color zero. If $\Delta((2c+2)k-1)=1$, then $(2k+c-1, c, (2c+2)k-1)$ would be a monochromatic solution in color one.

To avoid this problematic $(2c+2)k-1$ in the range of integers of the valid coloring, we need $(2c+2)k-1 > 4k^2-2$, equivalently $c > 2k-1-1/(2k)$. By the solution $(1,1,2k)$, we must have $\Delta(2k)=1$. Thus the range for c not leading to a monochromatic solution is

$$2k-1-\frac{1}{2k} < c \leq 2k.$$

Of the two possible integer values, it must be that $c=2k-1$ since the initial run of zeros needs to have even length to satisfy the shift constraint. Keeping track of the color assignments, so far we have

$$1, \ldots, 2k-2 \in \Delta_0,\ 2k-1, 2k \in \Delta_1.$$

In addition to $2k-1, 2k \in \Delta_1$, the initial run of zeros forces many integers to be assigned color one. Specifically, by the solution $(2,1,2k+1)$ we must have $\Delta(2k+1)=1, \ldots$, by $(2k-2, 1, 4k-3)$ we must have $\Delta(4k-3)=1$. By the solutions $(1,2,4k-1)$ through $(2k-2, 2, 6k-4)$, we must have $4k-1, \ldots, 6k-4 \in \Delta_1$. Note that $4k-2$ has not been assigned a color. If $\Delta(4k-2)=0$, then there would be a length one nonterminal run of zeros, so $\Delta(4k-2)=1$ by the even shift constraint. All of this continues through the solution $(2k-2, 2k-2, 4k^2-4k)$ implying $\Delta(4k^2-4k)=1$. That is,

$$2k+1, \ldots, 4k^2-4k \in \Delta_1.$$

Next, we show the color zero assignments that follow from this run of ones. By the solution $(2k-1, 2k-1, 4k^2-2k)$, we must have $\Delta(4k^2-2k)=0$. By the solutions $(2k, 2k-1, 4k^2-2k+1)$ through $(4k-3, 2k-1, 4k^2-2)$, we have the length $2k-1$ run of zeros

$$4k^2-2k, \ldots, 4k^2-2 \in \Delta_0. \tag{2}$$

These additional color zero assignments allow us to extend the run of ones. If $\Delta(4k^2-4k+1)=0$, then $(4k^2-4k+1, 1, 4k^2-4k)$ would be a monochromatic solution in color zero, so we may assume that $\Delta(4k^2-4k+1)=1$. This continues through the solution $(4k^2-2k-1, 1, 4k^2-2)$, so that

$$4k^2-4k+1, \ldots, 4k^2-2k-1 \in \Delta_1.$$

This establishes the validity and uniqueness of the coloring given at the beginning of the proof.

To complete the proof, we show that neither color assignment for $4k^2-1$ is valid. If $\Delta(4k^2-1)=0$, then $(4k^2-2k,1,4k^2-1)$ would be a monochromatic solution in color zero. If $\Delta(4k^2-1)=1$, then Equation (2) would describe an odd length nonterminal run of zeros. (Also, $(4k-2,2k-1,4k^2-1)$ would be a monochromatic solution in color one.)

We conclude that $R_0^e(1,2k-1)=4k^2-1$ for $k\geq 2$, i.e., $R_0^e(1,b)=b^2+2b$ for odd $b\geq 3$. □

5. Conjectures and Other Ideas for Further Study

Here are some possible next investigations in this new study of shift-constrained Rado numbers.

For the golden mean shift, computation data suggest additional identities, two of which we include here as conjectures.

Conjecture 1. Given positive integers a and ℓ, the golden Rado number for the case $\Delta(1)=0$ is $R_0^\varphi(a,\ell a)=(\ell+1)^2a^2$.

Theorem 2 confirms the $\ell=1$ case of Conjecture 1.

By Theorem 1, we know $R_0^\varphi(1,b)<R_1^\varphi(1,b)$ for all positive integers b. Also, by Theorem 2, $R_0^\varphi(a,a)<R_1^\varphi(a,a)$ for all positive integers a. In general, though, the relation between $R_0^\varphi(a,b)$ and $R_1^\varphi(a,b)$ is not clear. That is, which color assignment for 1 has the greater effect on lowering the standard Rado number when applying the golden mean shift constraint? Computational data support the following claim.

Conjecture 2. Given integers $2\leq a\leq b$, the golden mean shift constrained Rado numbers satisfy $R_0^\varphi(a,b)<R_1^\varphi(a,b)$ except when $b=\ell a$ for some integer $\ell\geq 2$, in which case $R_0^\varphi(a,\ell a)>R_1^\varphi(a,\ell a)$.

Of course, a full understanding of $R_0^\varphi(a,b), R_1^\varphi(a,b)$ is desired. Similarly for the even shift, where we have not determined $R_0^e(a,b)$ for $a+b$ even with $a>1$. (The analogue of Conjecture 2 for the even shift constraint is clear, as we know from Proposition 1 and Theorem 3 that $R_0^e(a,b)\leq R_1^e(a,b)=R(a,b)$.)

Following developments in the study of Rado numbers, one can generalize from $ax+by=z$ to equations with arbitrarily many variables, including a constant term, requiring $x\neq y$, etc. In symbolic dynamics, there are many interesting shifts to explore, including run-length limited shifts, charge constrained shifts, and generalized Morse shifts, along with larger alphabets corresponding to more colors.

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