MULTIDIMENSIONAL PADÉ APPROXIMATION OF BINOMIAL FUNCTIONS: EQUALITIES

Michael A. Bennett[1]
Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada
bennett@math.ubc.ca

Greg Martin[2]
Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada
gerg@math.ubc.ca

Kevin O'Bryant[3]
Department of Mathematics, City University of New York, The College of Staten Island and The Graduate Center, Staten Island, New York
kevin.obryant@csi.cuny.edu

Received: 1/9/21, Revised: 8/3/21, Accepted: 8/12/21, Published: 8/23/21

Abstract

Let $\omega_0, \dots, \omega_M$ be complex numbers. If $H_0, \dots, H_M$ are polynomials of degree at most $\rho_0, \dots, \rho_M$, and $G(z) = \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}$ has a zero at $z = 0$ of maximal order (for the given ω_m, ρ_m), we say that $H_0, \dots, H_M$ are a *multidimensional Padé approximation of binomial functions*, and call G the Padé remainder. We collect here with proof all of the known expressions for G and H_m, including a new one: the Taylor series of G. We also give a new criterion for systems of Padé approximations of binomial functions to be perfect (a specific sort of independence used in applications).

1. Introduction

Fix complex functions $f_0, f_1, \dots, f_M$ (all analytic in a neighborhood of 0) and nonnegative integers $\rho_0, \dots, \rho_M$. The set of functions

$$X := \left\{ \sum_{m=0}^{M} H_m(z) f_m(z) : H_m \in \mathbb{C}[z], \deg(H_m) \le \rho_m \right\}$$

[1]Supported in part by a Natural Sciences and Engineering Research Council of Canada Discovery grant
[2]Supported in part by a Natural Sciences and Engineering Research Council of Canada Discovery grant
[3]Support for this project was provided by a PSC-CUNY Award, jointly funded by The Professional Staff Congress and The City University of New York.

forms a finite dimensional vector space, and the subsets of functions

$$X_s := \{G \in X : \mathrm{ord}_{z=0}(G) \geq s\}$$

with a zero at $z = 0$ of order at least s are subspaces. Trivially $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$. Let σ be the least integer with X_σ having dimension 0, if such σ exists. Then $X_{\sigma-1}$ has positive dimension, and the functions in $X_{\sigma-1}$ are of particular interest, and are called the *Padé remainders of* $f_0, \ldots, f_M$.

The $M = 1$ case is the standard tool in numerical analysis known as Padé approximation [2], which generalizes Taylor Series. In particular, if $f_0(z) = -1$ identically, and $\rho_1 = 0$, then

$$X = \{-H_0(z) + H_1 \cdot f_1(z) : H_0 \in \mathbb{C}[z], \deg(H_0) \leq \rho_0, H_1 \in \mathbb{C}\}.$$

Taking $H_0(z)/H_1$ to be the ρ_0-th Taylor polynomial of $f_1(z)$, we find that the Padé remainders are the constant multiples of the Taylor polynomial remainder. Letting $\rho_1 \geq 0$ leads to rational functions $H_0(z)/H_1(z)$ that approximate $f_1(z)$ at least as well as Taylor polynomials. If f_1 has poles near 0, then this rational approximation is typically much sharper than the Taylor's polynomial approximation.

When $M > 1$, we include the adjective "multidimensional". This setting has not been exploited as systematically as the $M = 1$ case. For a few particular choices of $f_0, \ldots, f_M$, there is enough structure that we can work out explicit formulae for the Padé remainders and for the system of Padé approximants, i.e., generating polynomials $H_0, \ldots, H_M$. In this paper, we take the binomials $f_m(z) := (1-z)^{\omega_m}$ for complex numbers $\omega_0, \ldots, \omega_M$, no pair of which has an integer difference. The resulting system of equations was studied by Riemann [10], Thue [13], Siegel [11], Mahler [7], Baker [1], Chudnovsky [4], Bennett [3], and many others, and the use of these Padé approximations for Diophantine analysis is known as the method of Thue-Siegel.

We present in this article our exposition of these classic results on multidimensional Padé approximation of binomial functions. We combine, and in some cases simplify, the work of Mahler and Jager [6, 7]. While there are some original results here, e.g., Theorem 4((iv)) and some cases of Theorem 6, we see the main value of this work as collating the work of many people over many years with common notation, complete proofs, and specialization to the choice $f_m(z)$. The results presented in this work are equalities, and so as a check against off-by-one errors, one can implement the various forms given and directly check the equations for randomly chosen parameters. We have done so in Mathematica; a notebook containing these calculations is on the arXiv.

The current work focuses on various expressions for the Padé remainders and approximants. In subsequent works, we will provide new bounds, both archimedean and non-archimedean, on the size of the approximant polynomials $H_0, \ldots, H_M$ and on the Padé remainder, and will exploit those bounds to give new irrationality measures for some numbers of the form $(a/b)^{s/n}$.

2. Statement of Results

Let M be a nonnegative integer. Consider $\vec{\omega} \coloneqq \langle \omega_0, \omega_1, \dots, \omega_M \rangle$, a vector of $M+1$ distinct complex numbers, no pair of which has a difference that is an integer, and $\vec{\rho} \coloneqq \langle \rho_0, \dots, \rho_M \rangle$, a vector of $M+1$ nonnegative integers (typically not distinct). We index the vectors $\vec{\omega} \in \mathbb{C}^{M+1}, \vec{\rho} \in \mathbb{N}^{M+1}$ with $0, 1, \dots, M$; for example, the 0-th coordinate of $\vec{\rho}$ is ρ_0 and the M-th coordinate is ρ_M. We will only consider M, $\vec{\omega}$, $\vec{\rho}$ satisfying these constraints. Two fundamental parameters are

$$\sigma = \sigma(\vec{\rho}) \coloneqq \sum_{m=0}^{M} (\rho_m + 1), \quad \text{and} \quad \vec{\rho}! \coloneqq \prod_{m=0}^{M} \rho_m!.$$

Some notation used in Theorem 1 is both standard and uncommon; we give definitions in the next section. When we add a scalar to a vector, we mean that the scalar is added to each coordinate, such as $\vec{\rho}+1 = \langle \rho_0+1, \rho_2+1, \dots, \rho_M+1 \rangle$. When we delete the m-th coordinate, reducing the length of the vector by 1, we use a "$\star m$" exponent, such as

$$\vec{\omega}^{\star m} = \langle \omega_0, \dots, \omega_{m-1}, \omega_{m+1}, \dots, \omega_M \rangle.$$

The standard basis vectors are denoted $\vec{e}_0, \vec{e}_1, \dots, \vec{e}_M$.

Theorem 1. *Let $\vec{\rho}$ and $\vec{\omega}$ be fixed vectors as above.*

(i) (Existence) There are polynomials H_m in z of degree at most ρ_m, with at least one H_m not identically 0, and with

$$G(z) \coloneqq \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}$$

having a zero of order at least $\sigma - 1$ at $z = 0$.

(ii) (Uniqueness) For such $G(z)$, the function $G(z)$ necessarily has a zero of order exactly $\sigma - 1$ at $z = 0$, and furthermore the polynomials $H_m(z)$ are uniquely determined given the additional constraint that

$$\lim_{z \to 0} \frac{G(z)}{z^{\sigma-1}} = \frac{1}{(\sigma-1)!}.$$

Each $H_m(z)$ has degree exactly ρ_m. There is no $\alpha \in \mathbb{C}$ with

$$H_0(\alpha) = \cdots = H_M(\alpha) = 0.$$

(iii) (Domain) $G(z)$ is analytic on $\mathbb{C} \setminus [1, \infty)$.

Theorem 1 allows us to make the following definition of Padé approximants and remainders.

Definition 2. Let $\vec{\rho}$ and $\vec{\omega}$ be fixed vectors as above. The $M+1$ *Padé approximants* $\textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ (with $0 \le m \le M$) are the polynomials with degrees ρ_m, and with *Padé remainder*

$$\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) := \sum_{m=0}^{M} \textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)\,(1-z)^{\omega_m}$$

both having a zero of order $\sigma - 1$ at $z = 0$, and satisfying

$$\lim_{z\to 0} \frac{\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)}{z^{\sigma-1}} = \frac{1}{(\sigma-1)!}.$$

In Proposition 3, we draw attention to some obvious symmetries, immediate from Theorem 1, whose proofs we do not spell out.

Proposition 3 (Permutation and Shift Symmetry)**.** *If π is any permutation of $0, 1, \ldots, M$, then*

$$\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \textsc{Rem}\big(z \,\big|\, {\langle \omega_0,\omega_1,\ldots,\omega_M\rangle \atop \langle \rho_0,\rho_1,\ldots,\rho_M\rangle}\big) = \textsc{Rem}\big(z \,\big|\, {\langle \omega_{\pi(0)},\omega_{\pi(1)},\ldots,\omega_{\pi(M)}\rangle \atop \langle \rho_{\pi(0)},\rho_{\pi(1)},\ldots,\rho_{\pi(M)}\rangle}\big)$$

and

$$\textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \textsc{Poly}_m\big(z \,\big|\, {\langle \omega_0,\omega_1,\ldots,\omega_M\rangle \atop \langle \rho_0,\rho_1,\ldots,\rho_M\rangle}\big) = \textsc{Poly}_{\pi^{-1}(m)}\big(z \,\big|\, {\langle \omega_{\pi(0)},\omega_{\pi(1)},\ldots,\omega_{\pi(M)}\rangle \atop \langle \rho_{\pi(0)},\rho_{\pi(1)},\ldots,\rho_{\pi(M)}\rangle}\big).$$

For any α, we have

$$(1-z)^{\alpha}\,\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \textsc{Rem}\big(z \,\big|\, {\alpha+\vec{\omega} \atop \vec{\rho}}\big) \quad \textit{and} \quad \textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \textsc{Poly}_m\big(z \,\big|\, {\alpha+\vec{\omega} \atop \vec{\rho}}\big).$$

The purpose of the current work is to collect together various explicit formulae for the Padé remainder $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ and the Padé approximants $\textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$, in a common notation, and with complete proofs. Formulae for the Padé remainder are given in Theorem 4, and formulae for the Padé approximants are given in Theorem 5.

Theorem 4 (Forms for the Padé Remainder)**.** *The following five expressions give* $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$.

(i) The Padé remainder $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ is given by the iterated integral

$$\frac{(1-z)^{\omega_0}}{\vec{\rho}\,!} \int_0^z \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{M-1}} \mathcal{G}(z, t_1, t_2, \ldots, t_M)\, dt_M \cdots dt_3\, dt_2\, dt_1,$$

where

$$\mathcal{G}(t_0, t_1, \ldots, t_M) = t_M^{\rho_M} \left(\prod_{h=1}^{M} \left(\frac{t_{h-1} - t_h}{1 - t_h} \right)^{\rho_{h-1}} \right) \left(\prod_{h=1}^{M} (1 - t_h)^{\omega_h - \omega_{h-1} - 1} \right).$$

(ii) The Padé remainder $\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ *is given by the M-dimensional integral*

$$z^{\sigma-1}\frac{(1-z)^{\omega_0}}{\vec\rho\,!}\int_{[0,1]^M} U_M^{-1}\prod_{h=1}^{M}\frac{U_h^{1+\rho_h}}{(1-zU_h)^{1-\omega_h+\omega_{h-1}}}\left(\frac{1-u_h}{1-zU_h}\right)^{\rho_{h-1}} d\vec u,$$

where $U_m=\prod_{h=1}^{m}u_h$*.*

(iii) The Padé remainder $\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ *is the contour integral*

$$\frac{(-1)^{\sigma-1}}{2\pi i}\int_\gamma (1-z)^{\xi}\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi,$$

where γ *is any simple positively oriented contour enclosing all* σ *of the complex numbers* ω_m+r $(0\le m\le M, 0\le r\le\rho_m)$*.*

(iv) The Maclaurin series for $\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ *is*

$$\sum_{n=0}^{\infty}(-1)^n\sum_{m=0}^{M}\frac{1}{\rho_m!}\sum_{r=0}^{\rho_m}\binom{\rho_m}{r}\frac{(-1)^r(\omega_m+r)^{\underline{n}}}{\prod_{\substack{k=0\\k\ne m}}^{M}(\omega_k-\omega_m-r)^{\overline{\rho_k+1}}}\,\frac{z^n}{n!},$$

which converges for $|z|<1$*.*

(v) Finally, $\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ *is the special value of Meijer's G-function given by*

$$G^{M+1,\ 0}_{M+1,\ M+1}\left(1-z\,\middle|\,\begin{matrix}\vec\omega+\vec\rho+1\\ \vec\omega\end{matrix}\right).$$

In addition to the formulae for $\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ given in Theorem 4, we note that

$$\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)=\sum_{m=0}^{M}\mathrm{Poly}_m\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)\,(1-z)^{\omega_m},$$

and so any formula for $\mathrm{Poly}_m\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ generates a formula for $\mathrm{Rem}\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$. Theorem 5 gives a number of useful representations of $\mathrm{Poly}_m\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$.

Theorem 5 (Forms for the Padé Approximants)**.** *The following five expressions give* $\mathrm{Poly}_m\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$*.*

(i) Let γ_m *be a simple positively oriented contour enclosing all* ρ_m+1 *of the complex numbers* ω_m+r $(0\le r\le\rho_m)$ *and none of* ω_k+r $(0\le k\le M, k\ne m, 0\le r\le\rho_k)$*. Then* $\mathrm{Poly}_m\big(z \,\big|\, \substack{\vec\omega\\ \vec\rho}\big)$ *is given by*

$$\frac{(-1)^{\sigma-1}}{2\pi i}\int_{\gamma_m}(1-z)^{\xi-\omega_m}\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi.$$

(ii) The Padé approximant $\mathrm{POLY}_m\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ *is equal to*

$$\frac{1}{\rho_m!}\sum_{r=0}^{\rho_m}(z-1)^r\binom{\rho_m}{r}\prod_{\substack{k=0\\ k\neq m}}^{M}\frac{1}{(\omega_k-\omega_m-r)^{\overline{\rho_k+1}}}.$$

(iii) For $M \geq 1$*, the Padé approximant* $\mathrm{POLY}_m\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ *is the* M*-fold iterated integral*

$$\frac{Q_m}{\vec{\rho}!}\int_{(G)} T_m^{-\omega_m-1}\Bigg(\prod_{\substack{k=0\\ k\neq m}}^{M} t_k^{\omega_k}(1+t_k)^{\rho_k}\Bigg)\left(1-(-1)^M\,\frac{1-z}{T_m}\right)^{\rho_m}\,d\vec{t},$$

where $\int_{(G)}\cdots d\vec{t}$ *integrates each of* $t_0,\ldots,t_M$ *(except* t_m*) counterclockwise on the unit circle from* $-\pi$ *radians to* π *radians (i.e., the principal value),*

$$Q_m \coloneqq \prod_{\substack{k=0\\ k\neq m}}^{M}\frac{1}{2i\sin(\pi(\omega_k-\omega_m))}, \quad \textit{and} \quad T_m \coloneqq \prod_{\substack{k=0\\ k\neq m}}^{M} t_k.$$

(iv) The Padé approximant is a scaled generalized hypergeometric function:

$$\frac{1}{\rho_m!}\Bigg(\prod_{\substack{k=0\\ k\neq m}}^{M}\frac{1}{(\omega_k-\omega_m)^{\overline{\rho_k+1}}}\Bigg)\,{}_{M+1}F_M\left[\begin{matrix}\omega_m-\vec{\omega}-\vec{\rho}\\ (1+\omega_m-\vec{\omega})^{\star m}\end{matrix};1-z\right].$$

(v) Set $W \coloneqq W(m,k) = \omega_k - \omega_m$*, and define* $C_{m,k,r}$ *by*

$$C_{m,k,r} \coloneqq \binom{\rho_k}{r},$$

if $m = k$*, by*

$$C_{m,k,r} \coloneqq (-1)^{\rho_k+1}\binom{r}{\rho_k}^{-1}\frac{\Gamma(r+1)}{\Gamma(r+1-W)}\frac{\Gamma(r-\rho_k-W)}{\Gamma(r-\rho_k+1)}$$

if $m \neq k$ *and* $\rho_k < r$*, and by*

$$C_{m,k,r} \coloneqq (-1)^r\binom{\rho_k}{r}\frac{\Gamma(r+1)}{\Gamma(r+1-W)}\frac{\Gamma(\rho_k-r+1)}{\Gamma(\rho_k-r+1+W)}\frac{\pi}{\sin(\pi W)}$$

if $m \neq k$ *and* $\rho_k \geq r$*. Then, we have*

$$\mathrm{POLY}_m\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big) = \frac{1}{\vec{\rho}!}\sum_{r=0}^{\rho_m}(z-1)^r\prod_{k=0}^{M}C_{m,k,r}.$$

Theorem 6 precisely states that notion that the approximants for nearby $\vec{\rho}$ are independent. This property is referred to as "perfect approximation", and relies mostly on $\deg(\textsc{Poly}_m(z \,|\, {\vec{\omega} \atop \vec{\rho}})) = \rho_m$ and $\mathrm{ord}_{z=0}(\textsc{Rem}(z \,|\, {\vec{\omega} \atop \vec{\rho}})) = \sigma - 1$. Recall that our $M+1$ dimensional vectors have coordinates indexed from 0 through M.

Theorem 6 (Approximants are Perfect)**.** *Fix $\vec{\rho} \in \mathbb{N}^{M+1}$ and $\vec{\epsilon}_0, \vec{\epsilon}_1, \dots, \vec{\epsilon}_M \in \mathbb{Z}^{M+1}$ with each $\vec{\rho} + \vec{\epsilon}_k$ having nonnegative coordinates, and denote the j-th coordinate of $\vec{\epsilon}_i$ as $\vec{\epsilon}_{i,j}$. Let S be maximum of $\sum_{i=0}^{M} \vec{\epsilon}_{i,\beta(i)}$ taken over all permutations β of $0, 1, \dots, M$, and let T be the minimum of $\sum_{j=0}^{M} \vec{\epsilon}_{i,j}$ taken over $0 \le i \le M$. Suppose the following two conditions are satisfied:*

(i) There is a unique permutation α of $0, 1, \dots, M$ with $S = \sum_{i=0}^{M} \vec{\epsilon}_{i,\alpha(i)}$;

(ii) $T + M = S$.

Then the $(M+1) \times (M+1)$ matrix whose (k, m) coordinate is the polynomial $\textsc{Poly}_m(z \,|\, {\vec{\omega} \atop \vec{\rho}+\vec{\epsilon}_k})$ has determinant

$$Cz^{\sigma(\vec{\rho})+T-1},$$

where C does not depend on z.

The most startling aspect of Theorem 6 is that $\vec{\omega}$ plays no role in the hypotheses nor in the conclusion.

We note that (in Theorem 6) with $\vec{\epsilon}_k = \vec{e}_k$ one has $T = 1, S = M + 1$, and the conditions in Theorem 6 are satisfied. This recovers a result stated and used by Mahler, Chudnovsky and Bennett [3, 4, 7]. If one takes $I_k \subseteq \{0, 1, \dots, k-1\}$ and $\vec{\epsilon}_k = \vec{e}_k + \sum_{i \in I_k} \vec{e}_i$, one recovers a result of Jager [6]. Our result covers many more examples than we found in the literature, but it is not exhaustive.

3. More Notation

We denote the rising and falling factorials as

$$x^{\overline{r}} := x \cdot (x+1)^{\overline{r-1}} = x \cdot (x+1) \cdot (x+2) \cdots (x+r-1),$$
$$x^{\underline{r}} := x \cdot (x-1)^{\underline{r-1}} = x \cdot (x-1) \cdot (x-2) \cdots (x-r+1),$$

for positive integers r, and define $x^{\overline{0}} = x^{\underline{0}} = 1$. We use the following trivial identities without comment (provided $x - r + 1 \notin \{0, -1, -2, \dots\}$):

$$x^{\underline{r}} = \frac{\Gamma(x+1)}{\Gamma(x-r+1)}, \quad x^{\underline{r}} = (x-r+1)^{\overline{r}} = (-1)^r(-x)^{\overline{r}},$$

and typically choose to eliminate ratios of Γ functions in preference for the more computationally friendly rising and falling factorials. All of our functions will be analytic in a complex neighborhood of $z = 0$. We use $\deg(f(z))$ to be the degree of f, which is ∞ if f is not a polynomial. We use $\mathrm{ord}_{z=0}(f(z))$ to denote the order of the zero of f at $z = 0$, and we use $O(z^k)$ to denote a function that has a zero at $z = 0$ of order at least k.

We shall briefly encounter the generalized hypergeometric function (defined for $|z| < 1$, $q < p$, and appropriate integers a_i, b_i)

$$ {}_pF_q\left[\begin{matrix} a_1,\ a_2,\ \ldots,\ a_p \\ b_1,\ b_2,\ \ldots,\ b_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{a_1^{\overline{n}} a_2^{\overline{n}} \cdots a_p^{\overline{n}}}{b_1^{\overline{n}} b_2^{\overline{n}} \cdots b_q^{\overline{n}}} \frac{z^n}{n!}, $$

and also the Meijer G-function [8]

$$ G_{p,\,q}^{m,\,n}\left(z \,\middle|\, \begin{matrix} a_1,\ a_2,\ \ldots,\ a_p \\ b_1,\ b_2,\ \ldots,\ b_q \end{matrix}\right) $$

(defined for natural numbers m, n, p, q, provided $m \le q$ and $n \le p$, although we only encounter it in this work with $n = 0, m = p = q = M + 1$), defined by

$$ \frac{1}{2\pi i} \int_C \frac{\prod_{k=1}^{m} \Gamma(s + b_k) \prod_{k=1}^{n} \Gamma(1 - a_k - s)}{\prod_{k=n+1}^{p} \Gamma(s + a_k) \prod_{k=m+1}^{q} \Gamma(1 - b_k - s)} z^{-s}\, ds, $$

where C is an infinite contour that separates the poles of $\Gamma(1 - a_k - s)$ from those of $\Gamma(b_k + s)$; the particular contour required for convergence varies depending on m, n, p, q, z.

4. Claims and Proofs

It is at least plausible that there are polynomials $H_0, \ldots, H_M$ with degrees $\rho_0, \ldots, \rho_M$ and

$$ G(z) \coloneqq \sum_{m=0}^{M} H_m(z)(1 - z)^{\omega_m} = \frac{z^{\sigma-1}}{(\sigma - 1)!} + O(z^{\sigma}), \tag{1} $$

where $O(z^\sigma)$ refers to $z \to 0$. After all, the polynomials have a total of σ coefficients, and we may choose them so that $G(z)$ has a zero at $z = 0$ of order $\sigma - 1$, and the first nonzero coefficient in the power series expansion of $G(z)$ is according to our choosing. Establishing this rigorously is the point to our first claims.

In all the Claims in this section, we assume that M is a nonnegative integer, and that $0 \le m \le M$. We assume that $\vec{\rho} = \langle \rho_0, \ldots, \rho_M \rangle$ is vector of $M + 1$ nonnegative integers, and that $\vec{\omega} = \langle \omega_0, \ldots, \omega_M \rangle$ is a vector of $M + 1$ distinct complex numbers, no two of which have a difference that is an integer. Both $\vec{\rho}$ and $\vec{\omega}$ (and vectors derived from them) are indexed 0 through M.

4.1. Existence and Uniqueness

The following claim is used implicitly frequently throughout this work.

Claim 7. *For any polynomials $H_m(z)$ (not all zero), the sum*

$$G(z) \coloneqq \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}$$

is not identically 0.

Proof. Since no two ω_i have difference that is an integer, there is a unique k with

$$\omega_k + \deg(H_k) = \max\{\omega_i + \deg(H_i) : H_i \neq 0\}.$$

Then

$$\lim_{z\to-\infty} \frac{G(z)}{H_k(z)(1-z)^{\omega_k}} = 1 + \sum_{\substack{m=0\\m\neq k}}^{M} \lim_{z\to-\infty} \frac{H_m(z)(1-z)^{\omega_m}}{H_k(z)(1-z)^{\omega_k}} = 1.$$

Consequently, G cannot be identically 0. □

Claim 8. *There are polynomials $H_0(z), \ldots, H_M(z)$ of degrees at most $\rho_0, \ldots, \rho_M$, respectively, not all identically 0, such that*

$$\mathrm{ord}_{\mathrm{z}=0}\left(\sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}\right) \geq \sigma - 1.$$

Proof. Consider polynomials $H_0(z), \ldots, H_M(z)$ of degrees $\rho_0, \ldots, \rho_M$ with unknown coefficients, a total of σ unknowns. Recall Newton's Binomial Theorem: for $|z| < 1$ and any complex ω, we have

$$(1-z)^{\omega} = \sum_{i=0}^{\infty} (-1)^i \frac{\omega^{\underline{i}}}{i!} z^i.$$

Considering the coefficient of z^j, for $0 \leq j \leq \sigma - 2$, on both sides of the desired equality

$$\sum_{m=0}^{M} H_m(z) \sum_{i\geq 0} (-1)^i \frac{\omega_m{}^{\underline{i}}}{i!} z^i = O(z^{\sigma-1})$$

yields a homogeneous linear equation in the unknowns, a total of $\sigma - 1$ equations. By linear algebra, there is a choice of the σ unknowns, not all zero, which satisfies all of the equations. In other words, there are polynomials $H_0(z), \ldots, H_M(z)$ (not all zero) with degrees at most $\rho_0, \ldots, \rho_M$, such that

$$\sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}$$

has a zero of order at least $\sigma - 1$ at $z = 0$. □

Claim 8 establishes Theorem 1(i).

The next claim is slightly stronger than the $M = 0$ case of Theorem 1, in that explicit formulae are given, and is used as a base case for subsequent induction arguments.

Claim 9. *The $M = 0$ Padé approximant and remainder are given by the formulae* $\textsc{Poly}_0\big(z \,\big|\, {\langle\omega_0\rangle \atop \langle\rho_0\rangle}\big) = \dfrac{z^{\rho_0}}{\rho_0!}$, *and* $\textsc{Rem}\big(z \,\big|\, {\langle\omega_0\rangle \atop \langle\rho_0\rangle}\big) = \dfrac{z^{\rho_0}}{\rho_0!}(1-z)^{\omega_0}$.

Proof. We need to show that the only nonzero polynomials H_0 with $\mathrm{ord}_{z=0}(H_0(z)(1-z)^{\omega_0}) \geq \sigma - 1$ and degree at most ρ_0 are $H_0(z) = Cz^{\rho_0}$. First, observe that $\sigma = \rho_0 + 1$. As $\mathrm{ord}_{z=0}((1-z)^{\omega_0}) = 0$, we know that $\mathrm{ord}_{z=0}(H_0(z)(1-z)^{\omega_0}) = \mathrm{ord}_{z=0}(H_0)$. That is, H_0 must be a nonzero polynomial with $\mathrm{ord}_{z=0}(H_0) \geq \rho_0$ and $\deg(H_0) \leq \rho_0$. The only candidates are $\textsc{Poly}_0\big(z \,\big|\, {\langle\omega_0\rangle \atop \langle\rho_0\rangle}\big) = Cz^{\rho_0}$ and $\textsc{Rem}\big(z \,\big|\, {\langle\omega_0\rangle \atop \langle\rho_0\rangle}\big) = Cz^{\rho_0}(1-z)^{\omega_0}$.

Now, observe that

$$C = \lim_{z\to 0} \frac{Cz^{\rho_0}(1-z)^{\omega_0}}{z^{\rho_0}} = \frac{1}{(\sigma-1)!} = \frac{1}{\rho_0!}.$$

Thus, Theorem 1(ii) is proved in the $M = 0$ case, and the values of $\textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ and $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ are as claimed here. □

Claim 10. *If* $\deg(H_m(z)) \leq \rho_m$, *and some* $H_m \neq 0$, *then*

$$\mathrm{ord}_{z=0}\left(\sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}\right) \leq \sigma - 1.$$

Proof. Suppose $M = 0$. With H_0 a nonzero polynomial with degree at most ρ_0, we have

$$\mathrm{ord}_{z=0}\left(H_0(z)(1-z)^{\omega_0}\right) = \mathrm{ord}_{z=0}\left(H_0(z)\right) \leq \deg(H_0) \leq \rho_0 = \sigma - 1.$$

So, the claim holds for $M = 0$.

Assume the claim is false, and let M be the smallest positive integer for which this claim does not hold, and let ρ_0 correspond to the first counterexample: that is, for any $\vec{\omega}, \vec{\rho}$ that has a smaller M, or the same M but smaller ρ_0, the claim holds. Let

$$G(z) := \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}$$

be a counterexample, i.e., $\mathrm{ord}_{z=0}(G) \geq \sigma$. As multiplying by $(1-z)^{-\omega_0}$ does not change $\mathrm{ord}_{z=0}(G(z))$, we may assume that $\omega_0 = 0$.

If $\rho_0 = 0$, so that $H_0(z)$ is a constant, we have

$$\begin{aligned}\frac{d}{dz}G(z) &= \tfrac{d}{dz}H_0(z) + \sum_{m=1}^{M} \tfrac{d}{dz}H_m(z)(1-z)^{\omega_m} \\ &= \sum_{m=1}^{M} \left(H_m'(z)(1-z) - H_m(z)\omega_m\right)(1-z)^{\omega_m - 1}.\end{aligned}$$

Note that $\deg(H_m'(z)(1-z) - H_m(z)\omega_m) \le \deg(H_m) \le \rho_m$, for $1 \le m \le M$. Thus $\frac{d}{dz}G(z)$ has a smaller M and the same ρ_m. By assumption on $G(z)$,

$$\mathrm{ord}_{z=0}(\tfrac{d}{dz}G(z)) \ge \sigma - 1,$$

but by our assumption of the minimality of $G(z)$, we know that

$$\mathrm{ord}_{z=0}\left(\tfrac{d}{dz}G(z)\right) \le \sigma - 2.$$

This contradiction shows that $\rho_0 \ne 0$. But even in the case that $\rho_0 > 0$,

$$\begin{aligned}\frac{d}{dz}G(z) &= H_0'(z) + \sum_{m=1}^{M} \tfrac{d}{dz}H_m(z)(1-z)^{\omega_m} \\ &= H_0'(z) + \sum_{m=0}^{M} \left(H_m'(z)(1-z) - H_m(z)\omega_m\right)(1-z)^{\omega_m - 1}.\end{aligned}$$

As above, our assumption on the minimality of ρ_0, as $\deg(H_0') = \deg(H_0) - 1$, implies a contradiction. □

The proof of the next claim establishes the rest of Theorem 1(ii), and justifies Definition 2.

Claim 11. *Suppose that H_m (with $0 \le m \le M$) are polynomials with degree at most ρ_m, and that $G(z) := \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m}$ has a zero of order at least $\sigma - 1$ at $z = 0$. Then $G(z)$ has an order of exactly $\sigma - 1$ at $z = 0$. Suppose further that*

$$\lim_{z\to 0} \frac{G(z)}{z^{\sigma-1}} = \frac{1}{(\sigma-1)!}.$$

Then G and H_m are uniquely determined by these constraints. The polynomial $H_m(z)$ has degree exactly ρ_m, and there is no $\alpha \in \mathbb{C}$ with $H_0(\alpha) = \cdots = H_m(\alpha) = 0$.

Proof. By Claims 8 and 10, we can take $\mathrm{ord}_{z=0}(G(z))$ to be at least $\sigma - 1$, and can never have it be larger than $\sigma - 1$, so there are polynomials H_m with

$$G(z) = \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m} = Cz^{\sigma-1} + O(z^{\sigma}).$$

By multiplying through by a constant, we can take

$$C = \frac{1}{(\sigma-1)!}.$$

If both $G_1(z)$ and $G_2(z)$ have this form, then their difference would have a zero of order greater than $\sigma-1$, and by Claim 10 this is not possible unless $G_1(z)-G_2(z)$ is identically 0. By Claim 7, however, this is only possible if all of the polynomials are identically 0. That is, only if $G_1(z)=G_2(z)$. Thus, G is uniquely defined and the definition of $\text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ is justified.

If

$$\text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \sum_{m=0}^{M} H_m(z)(1-z)^{\omega_m} = \sum_{m=0}^{M} B_m(z)(1-z)^{\omega_m}$$

for polynomials H_m, B_m of degree at most ρ_m, then

$$0 = \sum_{m=0}^{M} (H_m(z) - B_m(z))(1-z)^{\omega_m}.$$

But by Claim 7, this implies that $H_m(z) = B_m(z)$. Thus, H_m is uniquely defined and the definition of $\text{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ is justified.

Suppose that $\text{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ has degree strictly less than ρ_m, which in particular means that $\rho_m \geq 1$. Let $\vec{e}_m$ be the $M+1$-dimensional unit vector in the m-th coordinate direction. Then $\text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}-\vec{e}_m}\big)$ is a constant multiple of $\text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$, which has a zero of order $\sigma(\vec{\rho})-1 > \sigma(\vec{\rho}-\vec{e}_m)-1$, contradicting Claim 10.

Finally, if $H_m(\alpha) = 0$ for $0 \leq m \leq M$, then $H_m(z)/(z-\alpha)$ are polynomials with degree $\rho_m - 1$, and $G(z)/(z-\alpha)$ has a zero of order $\sigma(\vec{\rho})-1$ at $z=0$, contradicting the uniqueness of G. □

4.2. Respectful Differential Operators

Claim 12 (Differentiation To Reduce ρ)**.** *Define the operators*

$$d_\omega := (1-z)^{\omega+1}\left(\frac{d}{dz}\right)(1-z)^{-\omega}.$$

If $\rho_i > 0$, then d_{ω_i} reduces ρ_i by 1 and increases ω_i by 1, i.e., if $\rho_i > 0$, then

$$d_{\omega_i}\, \text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \text{Rem}\big(z \,\big|\, {\vec{\omega}+\vec{e}_i \atop \vec{\rho}-\vec{e}_i}\big).$$

If $\rho_i = 0$, then d_{ω_i} eliminates the i-th coordinates of $\vec{\omega}$ and $\vec{\rho}$, i.e., if $\rho_i = 0$, then

$$d_{\omega_0}\, \text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \text{Rem}\big(z \,\big|\, {\vec{\omega}^{\star i} \atop \vec{\rho}^{\star i}}\big).$$

Consequently, for any ρ_0,

$$(1-z)^{\omega_0+\rho_0+1}\left(\frac{d}{dz}\right)^{\rho_0+1}(1-z)^{-\omega_0}\, \text{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \text{Rem}\big(z \,\big|\, {\langle \omega_1,\ldots,\omega_M \rangle \atop \langle \rho_1,\ldots,\rho_M \rangle}\big).$$

Proof. As d_ω is linear and

$$\textsc{Rem}\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big) = \textsc{Poly}_i\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)(1-z)^{\omega_i} + \sum_{\substack{m=0\\ m\neq i}}^{M} \textsc{Poly}_m\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)(1-z)^{\omega_m},$$

we can assess the impact of d_{ω_i} on the two pieces separately. First,

$$\begin{aligned} d_{\omega_i} \textsc{Poly}_i\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)(1-z)^{\omega_i} &= (1-z)^{\omega_i+1}\left(\frac{d}{dz}\right)(1-z)^{-\omega_i}\cdot \textsc{Poly}_i\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)(1-z)^{\omega_i} \\ &= \left(\frac{d}{dz}\textsc{Poly}_i\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)\right)(1-z)^{\omega_i+1}. \end{aligned}$$

This is 0 if $\rho_i = 0$, and if $\rho_i > 0$ it has the form $P_i(z)(1-z)^{\omega_i+1}$ with P_i a polynomial of degree $\rho_i - 1$. The other piece is more involved (for the sake of the margins, we let $H(z) := \textsc{Poly}_m\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)$ in the following displayed equations):

$$\begin{aligned} & d_{\omega_i} \sum_{\substack{m=0\\ m\neq i}}^{M} \textsc{Poly}_m\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)\,(1-z)^{\omega_m} \\ &\quad = (1-z)^{\omega_i+1}\left(\frac{d}{dz}\right)(1-z)^{-\omega_i}\sum_{\substack{m=0\\ m\neq i}}^{M} H(z)(1-z)^{\omega_m} \\ &\quad = (1-z)^{\omega_i+1}\sum_{\substack{m=0\\ m\neq i}}^{M}\frac{d}{dz}H(z)(1-z)^{\omega_m-\omega_i} \\ &\quad = (1-z)^{\omega_i+1}\sum_{\substack{m=0\\ m\neq i}}^{M}(1-z)^{\omega_m-\omega_i}\tfrac{d}{dz}H(z) - H(z)(\omega_m-\omega_i)(1-z)^{\omega_m-\omega_i-1} \\ &\quad = (1-z)^{\omega_i+1}\sum_{\substack{m=0\\ m\neq i}}^{M}\big((1-z)\tfrac{d}{dz}H(z) - (\omega_m-\omega_i)H(z)\big)\,(1-z)^{\omega_m-\omega_i-1} \\ &\quad = \sum_{\substack{m=0\\ m\neq i}}^{M}\big((1-z)\tfrac{d}{dz}\textsc{Poly}_m\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big) - (\omega_m-\omega_i)\,\textsc{Poly}_m\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)\big)\,(1-z)^{\omega_m}. \end{aligned}$$

This has the form

$$\sum_{\substack{m=0\\ m\neq i}}^{M} P_m(z)(1-z)^{\omega_m}$$

with P_m a polynomial of degree at most ρ_m. To wit, $d_{\omega_i}\,\textsc{Rem}\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)$ has the correct form to be $\textsc{Rem}\big(z \,\big|\, \tfrac{\vec{\omega}+\vec{e}_i}{\vec{\rho}-\vec{e}_i}\big)$ if $\rho_i > 0$, and the correct form to be $\textsc{Rem}\big(z \,\big|\, \tfrac{\vec{\omega}^{\star i}}{\vec{\rho}^{\star i}}\big)$ if $\rho_i = 0$.

By our earlier uniqueness result, it remains only to check that $d_{\omega_i}\,\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ has a zero (at $z = 0$) of order one less than $\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ and the correct scaling. These are both clear, as $(1-z)^{\omega_i+1}$ and $(1-z)^{-\omega_i}$ have no zero at $z = 0$, and the $\frac{d}{dz}$ reduces the order of the zero by one and the scaling coefficient is multiplied by $\sigma - 1$.

The last sentence of Claim 12 is now immediate, as the product of operators telescopes

$$d_{\omega_0+\rho_0} \cdots d_{\omega_0+1} d_{\omega_0} = (1-z)^{\omega_0+\rho_0+1} \left(\frac{d}{dz}\right)^{\rho_0+1} (1-z)^{-\omega_0}.$$

□

The previous claim establishes $\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ as the solution of a differential equation (henceforth DE), which we make explicit next. Then, we solve the DE to express $\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ as an M-fold iterated integral.

Claim 13. *Let $D_0, \ldots, D_M$ be the operators*

$$D_i \coloneqq (1-z)^{\omega_i+\rho_i+1} \left(\frac{d}{dz}\right)^{\rho_i+1} (1-z)^{-\omega_i}.$$

With this notation, $G(z) = \mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ is the unique analytic solution to the differential equation

$$D_{M-1} \cdots D_1 D_0 G(z) = \frac{z^{\rho_M}}{\rho_M!} (1-z)^{\omega_M}$$

with initial conditions

$$G^{(\sigma-1)}(0) = 1, \qquad G^{(m)}(0) = 0, \qquad (0 \le m \le \sigma - 2).$$

Proof. That $\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$ satisfies the DE is a consequence of Claim 12, and the initial conditions are part of the definition of $\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$.

As for uniqueness, suppose that $G(z) = \sum_{i=0}^{\infty} g_i z^i$. Observe that the initial conditions force

$$g_{\sigma-1} = \frac{1}{(\sigma-1)!}, \qquad g_i = 0, \quad (0 \le i \le \sigma - 2).$$

The DE then forces the value of g_i for $i \ge \sigma$. □

It would be interesting to use the proof of the above claim to work out the full power series of $\mathrm{Rem}\big(z \,\big|\, \substack{\vec{\omega}\\ \vec{\rho}}\big)$.

4.3. Iterated Integrals

Claim 14 is Theorem 4(i), and Claim 15 is Theorem 4(ii).

Claim 14 (Mahler)**.** *We can represent* $\text{REM}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$ *as an* M*-fold integral as*

$$\vec{\rho}! \cdot \text{REM}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big) = \\ (1-z)^{\omega_0} \int_0^z \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{M-1}} \mathcal{G}(z, t_1, t_2, \ldots, t_M)\, dt_M \cdots dt_3\, dt_2\, dt_1,$$

where

$$\mathcal{G}(t_0, t_1, \ldots, t_M) = t_M^{\rho_M} \left(\prod_{h=1}^{M} \left(\frac{t_{h-1} - t_h}{1 - t_h} \right)^{\rho_{h-1}} \right) \left(\prod_{h=1}^{M} (1 - t_h)^{\omega_h - \omega_{h-1} - 1} \right).$$

Mahler's Proof [7]. "This [Claim 13] can easily be brought to the following form." □

This result allows one to produce to an efficient bound for $\text{REM}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$, and is thereby a lynchpin in applications. Other authors cite Mahler, or cite authors who cite Mahler. We did not find it easy, and hence indicate in some detail how to arrive at Mahler's conclusion.

Proof. We begin with the differential equation given in Claim 12:

$$(1-z)^{\omega_0 + \rho_0 + 1} \left(\frac{d}{dz} \right)^{\rho_0 + 1} (1-z)^{-\omega_0}\, \text{REM}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big) = \text{REM}\big(z \,\big|\, \substack{\langle \omega_1, \ldots, \omega_M \rangle \\ \langle \rho_1, \ldots, \rho_M \rangle}\big).$$

Hence,

$$\left(\frac{d}{dz} \right)^{\rho_0 + 1} (1-z)^{-\omega_0}\, \text{REM}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big) = (1-z)^{-(\omega_0 + \rho_0 + 1)}\, \text{REM}\big(z \,\big|\, \substack{\langle \omega_1, \ldots, \omega_M \rangle \\ \langle \rho_1, \ldots, \rho_M \rangle}\big) \\ = \text{REM}\big(z \,\big|\, \substack{\langle \omega_1, \ldots, \omega_M \rangle - \omega_0 - \rho_0 - 1 \\ \langle \rho_1, \ldots, \rho_M \rangle}\big),$$

where the second equality follows from Proposition 3.

We observe that

$$\frac{d}{dz} \int_0^z \frac{(z-t)^k}{k!} f(t)\, dt = \begin{cases} f(z), & k = 0; \\ \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} f(t)\, dt, & k > 0. \end{cases}$$

It then follows by repetition that

$$\left(\frac{d}{dz} \right)^{\rho_0 + 1} \int_0^z \frac{(z-t)^{\rho_0}}{\rho_0!} f(t)\, dt = f(z).$$

Thus $(1-z)^{-\omega_0}\, \text{REM}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$ and $\int_0^z \frac{(z-t)^{\rho_0}}{\rho_0!}\, \text{REM}\big(t \,\big|\, \substack{\langle \omega_1, \ldots, \omega_M \rangle - \omega_0 - \rho_0 - 1 \\ \langle \rho_1, \ldots, \rho_M \rangle}\big)\, dt$ have the same $(\rho_0 + 1)$-th derivative. Therefore, they differ by a polynomial with degree at most ρ_0.

From the definition of $\operatorname{Rem}\big(z \,\big|\, \substack{\vec\omega \\ \vec\rho}\big)$, we have

$$\operatorname{ord}_{z=0}\left(\operatorname{Rem}\big(z \,\big|\, \substack{\vec\omega \\ \vec\rho}\big)\right) = \rho_0 + 1 + \operatorname{ord}_{z=0}\left(\operatorname{Rem}\big(t \,\big|\, \substack{\langle \omega_1,\ldots,\omega_M\rangle - \omega_0 - \rho_0 - 1 \\ \langle \rho_1,\ldots,\rho_M\rangle}\big)\right),$$

which dictates that the degree-at-most-ρ_0 polynomial is identically 0. We can thus undo the differential operators as

$$(1-z)^{-\omega_0} \operatorname{Rem}\big(z \,\big|\, \substack{\vec\omega \\ \vec\rho}\big) = \int_0^z \frac{(z-t)^{\rho_0}}{\rho_0!} \operatorname{Rem}\big(t \,\big|\, \substack{\langle \omega_1,\ldots,\omega_M\rangle - \omega_0 - \rho_0 - 1 \\ \langle \rho_1,\ldots,\rho_M\rangle}\big)\, dt,$$

whence

$$\operatorname{Rem}\big(z \,\big|\, \substack{\vec\omega \\ \vec\rho}\big) = \frac{(1-z)^{\omega_0}}{\rho_0!} \int_0^z (z-t)^{\rho_0} \operatorname{Rem}\big(t \,\big|\, \substack{\langle \omega_1,\ldots,\omega_M\rangle - \omega_0 - \rho_0 - 1 \\ \langle \rho_1,\ldots,\rho_M\rangle}\big)\, dt. \tag{2}$$

We wish to apply equation (2) inductively to express $\operatorname{Rem}\big(z \,\big|\, \substack{\vec\omega \\ \vec\rho}\big)$ as an iterated integral. So that the notation will fit on the page, we define for $1 \le i \le M$

$$\begin{aligned} S_0 &\coloneqq 0, \\ S_i &\coloneqq \omega_{i-1} + \rho_{i-1} + 1, \\ G_0(z) &\coloneqq \operatorname{Rem}\big(z \,\big|\, \substack{\vec\omega \\ \vec\rho}\big), \\ G_i(z) &\coloneqq \operatorname{Rem}\big(z \,\big|\, \substack{\langle \omega_i,\omega_{i+1},\ldots,\omega_M\rangle - S_i \\ \langle \rho_i,\ldots,\rho_M\rangle}\big). \end{aligned}$$

Note that $G_M(z) = \frac{z^{\rho_M}}{\rho_M!}(1-z)^{\omega_M - S_M}$ by Claim 9, while equation (2) gives

$$G_i(z) = \frac{(1-z)^{\omega_i - S_i}}{\rho_i!} \int_0^z (z-t)^{\rho_i} G_{i+1}(t)\, dt$$

for $0 \le i < M$. Now, iterating equation (2) gives

$$\begin{aligned} \operatorname{Rem}\big(t_0 \,\big|\, \substack{\vec\omega \\ \vec\rho}\big) &= G_0(t_0) \\ &= \frac{(1-t_0)^{\omega_0}}{\rho_0!} \int_0^{t_0} (t_0 - t_1)^{\rho_0} G_1(t_1)\, dt_1 \\ &= \frac{(1-t_0)^{\omega_0}}{\rho_0!\rho_1!} \int_0^{t_0} (t_0 - t_1)^{\rho_0} \cdot (1-t_1)^{\omega_1 - S_1} \int_0^{t_1} (t_1 - t_2)^{\rho_1} G_2(t_2)\, dt_2\, dt_1 \\ &\vdots \\ &= \frac{(1-t_0)^{\omega_0}}{\rho_0! \cdots \rho_M!} \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{M-1}} \mathcal{G}(t_0, t_1, \ldots, t_M)\, dt_M \cdots dt_2\, dt_1, \end{aligned}$$

where

$$\begin{aligned} \mathcal{G}(t_0, t_1, \ldots, t_M) &= \left(\prod_{h=0}^{M-1} (t_h - t_{h+1})^{\rho_h}\right) \left(\prod_{h=1}^{M-1} (1-t_h)^{\omega_h - S_h}\right) t_M^{\rho_M} (1-t_M)^{\omega_M - S_M} \\ &= t_M^{\rho_M} \left(\prod_{h=1}^{M} \left(\frac{t_{h-1} - t_h}{1 - t_h}\right)^{\rho_{h-1}}\right) \left(\prod_{h=1}^{M} (1-t_h)^{\omega_h - \omega_{h-1} - 1}\right), \end{aligned}$$

as claimed. □

Claim 15. *The Padé remainder* $\operatorname{REM}\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right)$ *is given by the* M*-dimensional integral*

$$z^{\sigma-1}\,\frac{(1-z)^{\omega_0}}{\vec{\rho}\,!}\int_{[0,1]^M}\left(U_M^{-1}\prod_{h=1}^{M}U_h^{1+\rho_h}\left(\frac{1-u_h}{1-zU_h}\right)^{\rho_{h-1}}(1-zU_h)^{\omega_h-\omega_{h-1}-1}\right)d\vec{u},$$

where $U_m=\prod_{h=1}^{m}u_h$.

Proof. This follows from the previous claim upon the substitutions

$$t_h=z\prod_{i=1}^{h}u_i=z\,U_h,\qquad dt_M dt_{M-1}\cdots dt_2 dt_1=z^M\prod_{h=1}^{M-1}U_h d\vec{u},$$

and the obvious algebraic manipulations. □

4.4. Contour Integrals and Derived Expressions

Claim 16 is Theorem 4(iii). Claim 17 is Theorem 5(i). Claim 18 is Theorem 5(ii). Claim 19 is Theorem 5(v). Claim 20 is Theorem 5(iii).

Claim 16. *Let* γ *be a simple positively oriented contour enclosing all* σ *of the complex numbers* ω_m+r $(0\le m\le M, 0\le r\le\rho_m)$*. Then*

$$\operatorname{REM}\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right)=\frac{(-1)^{\sigma-1}}{2\pi i}\int_\gamma(1-z)^\xi\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi.$$

Proof. Set

$$I\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right):=\frac{(-1)^{\sigma-1}}{2\pi i}\int_\gamma(1-z)^\xi\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi,$$

and, as in Claim 13,

$$D_i:=(1-z)^{\omega_i+\rho_i+1}\left(\frac{d}{dz}\right)^{\rho_i+1}(1-z)^{-\omega_i}.$$

We will show that

$$D_0 I\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right)=I\left(z \,\middle|\, \substack{\vec{\omega}^{*0}\\ \vec{\omega}^{*0}}\right).$$

Substituting yields

$$\begin{aligned}D_0 I\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right)&=(1-z)^{\omega_0+\rho_0+1}\left(\frac{d}{dz}\right)^{\rho_0+1}(1-z)^{-\omega_0}I\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right)\\&=(1-z)^{\omega_0+\rho_0+1}\left(\frac{d}{dz}\right)^{\rho_0+1}\frac{(-1)^{\sigma-1}}{2\pi i}\int_\gamma(1-z)^{\xi-\omega_0}\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi.\end{aligned}$$

As

$$\left(\frac{d}{dz}\right)^{\rho_0+1}(1-z)^{\xi-\omega_0}=(-1)^{\rho_0+1}(\xi-\omega_0)^{\underline{\rho_0+1}}(1-z)^{\xi-\omega_0-\rho_0-1},$$

differentiating under the integral eliminates the $k=0$ factor in the product, giving

$$D_0 I\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)=\frac{(-1)^{\sigma-\rho_0-2}}{2\pi i}\int_\gamma (1-z)^\xi \prod_{k=1}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi = I\big(z \,\big|\, \substack{\vec{\omega}^{*0} \\ \vec{\rho}^{*0}}\big).$$

We iterate, using $D_1,\dots,D_{M-1}$ successively to remove all but the final coordinates of $\vec{\omega},\vec{\rho}$, arriving at

$$D_{M-1}\cdots D_1 D_0 I\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)=I\big(z \,\big|\, \substack{\langle\omega_M\rangle \\ \langle\rho_M\rangle}\big)=\frac{(-1)^{\rho_M}}{2\pi i}\int_\gamma \frac{(1-z)^\xi}{(\xi-\omega_M)^{\underline{\rho_M+1}}}\,d\xi.$$

By partial fractions [5, equation (5.41) in Section 5.3]

$$\frac{(-1)^{\rho_M}}{(\xi-\omega_M)^{\underline{\rho_M+1}}}=\frac{1}{\rho_M!}\sum_{r=0}^{\rho_M}\frac{(-1)^r}{\xi-\omega_M-r}\binom{\rho_M}{r},$$

and with Cauchy's Integral Formula we conclude

$$\begin{aligned}
\frac{(-1)^{\rho_M}}{2\pi i}\int_\gamma \frac{(1-z)^\xi}{(\xi-\omega_M)^{\underline{\rho_M+1}}}\,d\xi &= \frac{1}{2\pi i}\int_\gamma \frac{(1-z)^\xi}{\rho_M!}\sum_{r=0}^{\rho_M}\frac{(-1)^r}{\xi-\omega_M-r}\binom{\rho_M}{r}\,d\xi\\
&=\frac{1}{\rho_M!}\sum_{r=0}^{\rho_M}\binom{\rho_M}{r}(-1)^r\frac{1}{2\pi i}\int_\gamma\frac{(1-z)^\xi}{\xi-\omega_M-r}\,d\xi\\
&=\frac{1}{\rho_M!}\sum_{r=0}^{\rho_M}\binom{\rho_M}{r}(-1)^r(1-z)^{\omega_M+r}\\
&=\frac{(1-z)^{\omega_M}}{\rho_M!}\sum_{r=0}^{\rho_M}\binom{\rho_M}{r}(z-1)^r\\
&=\frac{(1-z)^{\omega_M}}{\rho_M!}\,z^{\rho_M}.
\end{aligned}$$

Thus, $I\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$ satisfies the DE in Claim 13. We now show that it also satisfies the initial conditions given there, and so by Claim 13 we will have $I\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)=\textsc{Rem}\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$.

As for the initial conditions, it remains to show that $\frac{d^r}{dz^r}I\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)\big|_{z=0}=0$ for $0\le r\le\sigma-2$, and for $r=\sigma-1$ we get 1. We start with

$$\frac{d^r}{dz^r}I\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)=\frac{(-1)^{\sigma-1}}{2\pi i}\int_\gamma (-1)^r\xi^{\underline{r}}(1-z)^{\xi-r}\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi$$

and evaluating this at $z=0$ gives

$$\frac{(-1)^{\sigma-r-1}}{2\pi i}\int_\gamma \frac{\xi^{\underline{r}}}{\prod_{k=0}^{M}(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi. \tag{3}$$

We may take γ to be a circle with large radius N, where $N > \omega_k + r$ for $0 \le k \le M$ and $0 \le r \le \rho_k + 1$. We now appeal to an argument that has little to do with our particular integrand, and so we generalize. Let $P(\xi) = \prod(\xi - p_j)$ be a monic polynomial of degree r, and let $Q(\xi) = \prod(\xi - q_j)$ be a monic polynomial of degree σ with all of its roots inside $|\xi| = N$. Then, using the substitution $\xi \mapsto N^2/u$, which reverses the orientation of the contour,

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\xi|=N}\frac{P(\xi)}{Q(\xi)}\,d\xi &= \frac{1}{2\pi i}\int_{|u|=N}\frac{P(N^2/u)}{Q(N^2/u)}\,\frac{N^2}{u^2}\,du\\ &= \frac{N^2}{2\pi i}\int_{|u|=N}\frac{\prod(N^2/u - p_j)}{\prod(N^2/u - q_j)}\,\frac{du}{u^2}\\ &= \frac{N^2}{2\pi i}\int_{|u|=N}\frac{\prod(N^2 - up_j)}{\prod(N^2 - uq_j)}u^{\sigma-r-2}\,du.\end{aligned}$$

As all the roots of the denominator $\prod(N^2 - uq_j)$ are outside the contour, this integral is 0 provided that $\sigma - r - 2 \ge 0$, that is, provided $r \le \sigma - 2$. If $r = \sigma - 1$, then

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\xi|=N}\frac{P(\xi)}{Q(\xi)}\,d\xi = \frac{N^2}{2\pi i}\int_{|u|=N}&\frac{\prod(N^2 - up_j)}{\prod(N^2 - uq_j)}u^{\sigma-r-2}\,du\\ &= N^2\cdot\frac{\prod(N^2 - 0\cdot p_j)}{\prod(N^2 - 0\cdot q_j)} = N^2\cdot\frac{(N^2)^r}{(N^2)^\sigma} = 1.\end{aligned}$$

□

Claim 17. *Let γ_m be a simple positively oriented contour enclosing all $\rho_m + 1$ of the complex numbers $\omega_m + r$ $(0 \le r \le \rho_m)$ and none of $\omega_k + r$ $(0 \le k \le M, k \ne m, 0 \le r \le \rho_m)$. Then*

$$\textsc{Poly}_m\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big) = \frac{(-1)^{\sigma-1}}{2\pi i}\int_{\gamma_m}(1-z)^{\xi-\omega_m}\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi.$$

Proof. As no pair of the ω_i has a difference that is an integer, the σ numbers $\omega_m + r$, where $0 \le m \le M, 0 \le r \le \rho_m$, are distinct. Set

$$\Phi_{r,m}(\xi) \coloneqq (\xi - \omega_m - r)\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}},$$

where we understand the removable singularity to be removed. Observe that each $\Phi_{r,m}$ has $\sigma - 1$ simple poles. We will evaluate $\textsc{Rem}\big(z \,\big|\, \tfrac{\vec{\omega}}{\vec{\rho}}\big)$ using Cauchy's Integral Formula. Let $\gamma_{r,m}$ be a simple closed contour enclosing $\omega_m + r$, but none of the roots of $\Phi_{r,m}$. From Claim 16, we find that

$$\begin{aligned}
\textsc{Rem}\left(z \,\middle|\, {\textstyle{\vec{\omega} \atop \vec{\rho}}}\right) &= \frac{(-1)^{\sigma-1}}{2\pi i} \int_{\gamma} (1-z)^{\xi} \prod_{k=0}^{M} \frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\, d\xi \\
&= (-1)^{\sigma-1} \sum_{m=0}^{M} \sum_{r=0}^{\rho_m} \frac{1}{2\pi i} \int_{\gamma_{r,m}} \frac{(1-z)^{\xi}\,\Phi_{r,m}(\xi)}{\xi-\omega_m-r}\, d\xi \\
&= (-1)^{\sigma-1} \sum_{m=0}^{M} \sum_{r=0}^{\rho_m} (1-z)^{\omega_m+r}\,\Phi_{r,m}(\omega_m+r) \\
&= \sum_{m=0}^{M} \left((-1)^{\sigma-1} \sum_{r=0}^{\rho_m} (1-z)^{r}\,\Phi_{r,m}(\omega_m+r) \right) (1-z)^{\omega_m}.
\end{aligned}$$

We now notice that

$$\textsc{Poly}_m\left(z \,\middle|\, {\textstyle{\vec{\omega} \atop \vec{\rho}}}\right) = (-1)^{\sigma-1} \sum_{r=0}^{\rho_m} (1-z)^{r}\,\Phi_{r,m}(\omega_m+r), \tag{4}$$

as this is a polynomial of the required degree.

Also,

$$\begin{aligned}
&\frac{(-1)^{\sigma-1}}{2\pi i} \int_{\gamma_m} (1-z)^{\xi-\omega_m} \prod_{k=0}^{M} \frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\, d\xi \\
&\qquad = (-1)^{\sigma-1} \sum_{r=0}^{\rho_m} \frac{1}{2\pi i} \int_{\gamma_{m,r}} \frac{(1-z)^{\xi-\omega_m}\,\Phi_{r,m}(\xi)}{\xi-\omega_m-r}\, d\xi \\
&\qquad = (-1)^{\sigma-1} \sum_{r=0}^{\rho_m} (1-z)^{(\omega_m+r)-\omega_m}\,\Phi_{r,m}(\omega_m+r) \\
&\qquad = (-1)^{\sigma-1} \sum_{r=0}^{\rho_m} (1-z)^{r}\,\Phi_{r,m}(\omega_m+r) \\
&\qquad = \textsc{Poly}_m\left(z \,\middle|\, {\textstyle{\vec{\omega} \atop \vec{\rho}}}\right).
\end{aligned}$$

□

Claim 18. $\textsc{Poly}_m\left(z \,\middle|\, {\textstyle{\vec{\omega} \atop \vec{\rho}}}\right) = \dfrac{1}{\rho_m!} \displaystyle\sum_{r=0}^{\rho_m} (z-1)^{r} \binom{\rho_m}{r} \prod_{\substack{k=0 \\ k\neq m}}^{M} \frac{1}{(\omega_k-\omega_m-r)^{\overline{\rho_k+1}}}.$

Proof. We continue with the notation of the proof of Claim 17. In particular, we simplify the expression (4). Observe that

$$\Phi_{r,m}(\xi) = \left[\prod_{\substack{k=0 \\ k\neq m}}^{M} \frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}} \right] \cdot \left[\prod_{\substack{r'=0 \\ r'\neq r}}^{\rho_m} \frac{1}{\xi-\omega_m-r'} \right]$$

so that

$$\Phi_{r,m}(\omega_m + r) = \left[\prod_{\substack{k=0\\k\neq m}}^{M} \frac{1}{(\omega_m - \omega_k + r)^{\underline{\rho_k+1}}}\right] \cdot \left[\prod_{\substack{r'=0\\r'\neq r}}^{\rho_m} \frac{1}{r - r'}\right].$$

Now,

$$\prod_{\substack{r'=0\\r'\neq r}}^{\rho_m} (r - r') = (r)(r-1)\cdots(2)(1)(-1)(-2)\cdots(r-\rho_m) = (-1)^{r-\rho_m}\, r!\,(\rho_m - r)!,$$

so that

$$\frac{1}{\prod_{\substack{r'=0\\r'\neq r}}^{\rho_m}(r - r')} = \frac{(-1)^{r-\rho_m}}{\rho_m!}\binom{\rho_m}{r}.$$

Also,

$$\begin{aligned}\prod_{\substack{k=0\\k\neq m}}^{M} (\omega_m - \omega_k + r)^{\underline{\rho_k+1}} &= \prod_{\substack{k=0\\k\neq m}}^{M} (-1)^{\rho_k+1}(\omega_k - \omega_m - r)^{\overline{\rho_k+1}}\\ &= (-1)^{\sigma-\rho_m-1}\prod_{\substack{k=0\\k\neq m}}^{M} (\omega_k - \omega_m - r)^{\overline{\rho_k+1}}.\end{aligned}$$

We now have $\textsc{Poly}_m\left(z \,\middle|\, \genfrac{}{}{0pt}{}{\vec{\omega}}{\vec{\rho}}\right)$ as claimed. □

Claim 19. *Set* $W \coloneqq W(m,k) = \omega_k - \omega_m$, *and define* $C_{m,k,r}$ *by*

$$C_{m,k,r} \coloneqq \binom{\rho_k}{r},$$

if $m = k$, *by*

$$C_{m,k,r} \coloneqq (-1)^{\rho_k+1}\binom{r}{\rho_k}^{-1}\frac{\Gamma(r+1)}{\Gamma(r+1-W)}\frac{\Gamma(r-\rho_k-W)}{\Gamma(r-\rho_k+1)}$$

if $m \neq k$ *and* $\rho_k < r$, *and by*

$$C_{m,k,r} \coloneqq (-1)^{r}\binom{\rho_k}{r}\frac{\Gamma(r+1)}{\Gamma(r+1-W)}\frac{\Gamma(\rho_k-r+1)}{\Gamma(\rho_k-r+1+W)}\frac{\pi}{\sin(\pi W)}$$

if $m \neq k$ *and* $\rho_k \geq r$. *Then, we have*

$$\textsc{Poly}_m\left(z \,\middle|\, \genfrac{}{}{0pt}{}{\vec{\omega}}{\vec{\rho}}\right) = \frac{1}{\vec{\rho}!}\sum_{r=0}^{\rho_m}(z-1)^r\prod_{k=0}^{M} C_{m,k,r}.$$

Proof. We begin from Claim 18, writing W in place of $\omega_k - \omega_m$:

$$\textsc{Poly}_m\left(z \,\middle|\, \substack{\vec{\omega}\\ \vec{\rho}}\right) = \frac{1}{\rho_m!}\sum_{r=0}^{\rho_m}(z-1)^r\binom{\rho_m}{r}\prod_{\substack{k=0\\k\neq m}}^{M}\frac{1}{(W-r)^{\overline{\rho_k+1}}}$$

$$= \frac{1}{\vec{\rho}!}\sum_{r=0}^{\rho_m}(z-1)^r\binom{\rho_m}{r}\prod_{\substack{k=0\\k\neq m}}^{M}\frac{\rho_k!}{(W-r)^{\overline{\rho_k+1}}}.$$

If $k \neq m$ and $\rho_k < r$, then

$$\rho_k! = \binom{r}{\rho_k}^{-1}\frac{r!}{(r-\rho_k)!} = \binom{r}{\rho_k}^{-1}\frac{\Gamma(r+1)}{\Gamma(r-\rho_k+1)}$$

and

$$(W-r)^{\overline{\rho_k+1}} = (-1)^{\rho_k+1}(r-W)^{\underline{\rho_k+1}} = (-1)^{\rho_k+1}\frac{\Gamma(r-W+1)}{\Gamma(r-W-\rho_k)}.$$

Combining these,

$$\frac{\rho_k!}{(W-r)^{\overline{\rho_k+1}}} = (-1)^{\rho_k+1}\binom{r}{\rho_k}^{-1}\frac{\Gamma(r+1)}{\Gamma(r-\rho_k+1)}\frac{\Gamma(r-W-\rho_k)}{\Gamma(r-W+1)}$$

$$= (-1)^{\rho_k+1}\binom{r}{\rho_k}^{-1}\frac{\Gamma(r+1)}{\Gamma(r+1-W)}\frac{\Gamma(r-\rho_k-W)}{\Gamma(r-\rho_k+1)}$$

$$= C_{m,k,r}.$$

If $k \neq m$ and $\rho_k \geq r$, then

$$\rho_k! = \binom{\rho_k}{r}r!(\rho_k-r)! = \binom{\rho_k}{r}\Gamma(r+1)\Gamma(\rho_k-r+1)$$

and

$$(W-r)^{\overline{\rho_k+1}} = (W-r)^{\overline{r}}\cdot W^{\overline{\rho_k+1-r}}$$

$$= (-1)^r(r-W)^{\underline{r}}\cdot(W+\rho_k-r)^{\underline{\rho_k-r+1}}$$

$$= (-1)^r\frac{\Gamma(r-W+1)}{\Gamma(r-W-r+1)}\cdot\frac{\Gamma(\rho_k-r+W+1)}{\Gamma(\rho_k-r+W-(\rho_k-r+1)+1)}$$

$$= (-1)^r\frac{\Gamma(r+1-W)\Gamma(\rho_k-r+1+W)}{\Gamma(1-W)\Gamma(W)}.$$

By Euler's reflection formula for the Gamma fucntion, $\Gamma(1-W)\Gamma(W) = \pi/\sin(\pi W)$. Combining these,

$$\frac{\rho_k!}{(W-r)^{\overline{\rho_k+1}}} = (-1)^r\frac{\pi}{\sin(\pi W)}\frac{\Gamma(r+1)}{\Gamma(r+1-W)}\frac{\Gamma(\rho_k-r+1)}{\Gamma(\rho_k-r+1+W)}\binom{\rho_k}{r}$$

$$= C_{m,k,r}.$$

This concludes the proof. □

Claim 20. *For $M \geq 1$, we can represent $\textsc{Poly}_m\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$ as an M-fold iterated (principal value) contour integral as*

$$\textsc{Poly}_m\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big) = \frac{Q_m}{\vec{\rho}!} \int_{(G)} T_m^{-\omega_m - 1} \Bigg(\prod_{\substack{k=0 \\ k \neq m}}^{M} t_k^{\omega_k} (1 + t_k)^{\rho_k} \Bigg) \left(1 - (-1)^M \, \frac{1-z}{T_m}\right)^{\rho_m} \, d\vec{t},$$

where

$$Q_m := \prod_{\substack{k=0 \\ k \neq m}}^{M} \frac{1}{2i \sin(\pi(\omega_k - \omega_m))}$$

$$T_m := \prod_{\substack{k=0 \\ k \neq m}}^{M} t_k$$

$$\int_{(G)} := \int_{|t_0|=1} \cdots \int_{|t_{m-1}|=1} \int_{|t_{m+1}|=1} \cdot \int_{|t_M|=1}$$

$$d\vec{t} := dt_M \cdots dt_{m+1} \, dt_{m-1} \cdots t_1.$$

Proof. By induction and integration-by-parts, we notice that

$$\text{P. V.} \int_{|t|=1} t^{x-1} (1+t)^{\rho} \, dt = \int_{-\pi}^{\pi} e^{i(x-1)t} (1 + e^{it})^{\rho} \, ie^{it} dt = \frac{2i \sin(\pi x) \, \rho!}{x^{\overline{\rho+1}}}, \tag{5}$$

provided that ρ is a nonnegative integer and $x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. In this claim and its proof, all integrals are understood to be principal values.

Beginning with Claim 18, we may write $\textsc{Poly}_m\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big)$ as

$$\begin{aligned}
&\textsc{Poly}_m\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big) \\
&= \frac{1}{\rho_m!} \sum_{r=0}^{\rho_m} (z-1)^r \binom{\rho_m}{r} \prod_{\substack{k=0 \\ k \neq m}}^{M} \frac{1}{(\omega_k - \omega_m - r)^{\overline{\rho_k + 1}}} \\
&= \frac{1}{\vec{\rho}!} \sum_{r=0}^{\rho_m} (z-1)^r \binom{\rho_m}{r} \prod_{\substack{k=0 \\ k \neq m}}^{M} \frac{1}{2i \sin(\pi(\omega_k - \omega_m - r))} \frac{2i \sin(\pi(\omega_k - \omega_m - r)) \, \rho_k!}{(\omega_k - \omega_m - r)^{\overline{\rho_k + 1}}}.
\end{aligned}$$

We now use equation (5) to continue

$$\begin{aligned}
&\textsc{Poly}_m\big(z \,\big|\, \substack{\vec{\omega} \\ \vec{\rho}}\big) \\
&= \frac{1}{\vec{\rho}!} \sum_{r=0}^{\rho_m} (z-1)^r \binom{\rho_m}{r} \prod_{\substack{k=0 \\ k \neq m}}^{M} (-1)^r Q_m \int_{|t_k|=1} t_k^{\omega_k - \omega_m - r - 1} (1 + t_k)^{\rho_k} \, dt_k \\
&= \frac{Q_m}{\vec{\rho}!} \sum_{r=0}^{\rho_m} (-1)^{rM} (z-1)^r \binom{\rho_m}{r} \prod_{\substack{k=0 \\ k \neq m}}^{M} \int_{|t_k|=1} t_k^{\omega_k - \omega_m - r - 1} (1 + t_k)^{\rho_k} \, dt_k
\end{aligned}$$

Compressing the product of integrals using the $\int_{(G)}$ notation, we continue with

$$\begin{aligned}
&\mathrm{POLY}_m\big(z \,\big|\, \begin{smallmatrix}\vec{\omega}\\ \vec{\rho}\end{smallmatrix}\big)\\
&= \frac{Q_m}{\vec{\rho}!} \int_{(G)} \sum_{r=0}^{\rho_m} (-1)^{rM} (z-1)^r \binom{\rho_m}{r} \prod_{\substack{k=0\\ k\neq m}}^{M} t_k^{\omega_k-\omega_m-r-1} (1+t_k)^{\rho_k}\, d\vec{t}\\
&= \frac{Q_m}{\vec{\rho}!} \int_{(G)} \sum_{r=0}^{\rho_m} (-1)^{rM} (z-1)^r \binom{\rho_m}{r} T_m^{-r}\, T_m^{-\omega_m-1} \prod_{\substack{k=0\\ k\neq m}}^{M} t_k^{\omega_k} (1+t_k)^{\rho_k}\, d\vec{t}\\
&= \frac{Q_m}{\vec{\rho}!} \int_{(G)} T_m^{-\omega_m-1} \Bigg(\prod_{\substack{k=0\\ k\neq m}}^{M} t_k^{\omega_k} (1+t_k)^{\rho_k} \Bigg) \sum_{r=0}^{\rho_m} \binom{\rho_m}{r} \left((-1)^M \frac{z-1}{T_m} \right)^r d\vec{t}\\
&= \frac{Q_m}{\vec{\rho}!} \int_{(G)} T_m^{-\omega_m-1} \Bigg(\prod_{\substack{k=0\\ k\neq m}}^{M} t_k^{\omega_k} (1+t_k)^{\rho_k} \Bigg) \left(1 - (-1)^M \frac{1-z}{T_m} \right)^{\rho_m} d\vec{t},
\end{aligned}$$

as asserted in the Claim. □

4.5. Hypergeometric Functions

Claim 21 below is Theorem 4(v), and Claim 23 is Theorem 5(iv).

The Meijer G-function [8] is defined for natural numbers m, n, p, q, provided $m \leq q$ and $n \leq p$, although we only encounter it here with $n = 0, m = p = q = M + 1$. It is denoted

$$G_{p,\,q}^{m,\,n}\left(z \,\middle|\, \begin{matrix} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{matrix} \right)$$

and defined as

$$\frac{1}{2\pi i} \int_C \frac{\prod_{k=1}^{m} \Gamma(s+b_k) \prod_{k=1}^{n} \Gamma(1-a_k-s)}{\prod_{k=n+1}^{p} \Gamma(s+a_k) \prod_{k=m+1}^{q} \Gamma(1-b_k-s)} z^{-s}\, ds,$$

where C is a particular infinite contour that separates the poles of $\Gamma(1-a_k-s)$ from those of $\Gamma(b_k+s)$; the particular contour required for convergence varies depending on m, n, p, q, z.

Claim 21. *$\mathrm{REM}\big(z \,\big|\, \begin{smallmatrix}\vec{\omega}\\ \vec{\rho}\end{smallmatrix}\big)$, when $|z| < 1$ and $|1-z| < 1$, is a special value of Meijer's G-function*

$$\mathrm{REM}\big(z \,\big|\, \begin{smallmatrix}\vec{\omega}\\ \vec{\rho}\end{smallmatrix}\big) = G_{M+1,\; M+1}^{M+1,\; 0}\left(1-z \,\middle|\, \begin{matrix} \vec{\omega}+\vec{\rho}+1 \\ \vec{\omega} \end{matrix} \right).$$

Sketch of Proof. With $m = p = q = M + 1, n = 0$, we see that

$$\prod_{k=1}^{n} \Gamma(1-a_k-s) = \prod_{k=m+1}^{q} \Gamma(1-b_k-s) = 1.$$

Further, with $a_{k+1} = \omega_k + \rho_k + 1, b_{k+1} = \omega_k$,

$$\frac{\prod_{k=1}^{m} \Gamma(s+b_k)}{\prod_{k=n+1}^{p} \Gamma(s+a_k)} = \prod_{k=0}^{M} \frac{\Gamma(s+\omega_k)}{\Gamma(s+\omega_k+\rho_k+1)} = \prod_{k=0}^{M} \frac{1}{(s+\omega_k)^{\overline{\rho_k+1}}}.$$

We now have

$$\begin{aligned} G_{M+1,\ M+1}^{M+1,\ 0}\left(1-z \,\middle|\, \begin{matrix} \vec{\omega}+\vec{\rho}+1 \\ \vec{\omega} \end{matrix}\right) &= \frac{1}{2\pi i}\int_C (1-z)^{-s} \prod_{k=0}^{M} \frac{1}{(s+\omega_k)^{\overline{\rho_k+1}}}\, ds \\ &= \frac{1}{2\pi i}\int_C (1-z)^{\xi} \prod_{k=0}^{M} \frac{(-1)^{\rho_k+1}}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\, (-d\xi) \\ &= \frac{(-1)^{\sigma-1}}{2\pi i}\int_C (1-z)^{\xi} \prod_{k=0}^{M} \frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\, d\xi \\ &= \operatorname{REM}\left(z \,\middle|\, \tfrac{\vec{\omega}}{\vec{\rho}}\right). \end{aligned}$$

Admittedly, we have played fast-and-loose with the contour, and therefore the conditions $|z| < 1$ and $|1-z| < 1$ are not explained. □

Theorem 22 (Slater's Theorem [12])**.** *Provided that $a_j - b_h$ is not a positive integer (with $j \le n, h \le m$), and $b_j - b_k$ is not an integer (with $1 \le j < k \le q$), and $0 < |z| < 1$,*

$$G_{p,\ q}^{m,\ n}\left(z \,\middle|\, \begin{matrix} a_1,\ a_2,\ \ldots,\ a_p \\ b_1,\ b_2,\ \ldots,\ b_q \end{matrix}\right) = \sum_{h=1}^{m} \frac{\prod_{\substack{k=1\\k\neq h}}^{m} \Gamma(b_k-b_h) \prod_{k=1}^{n} \Gamma(1+b_h-a_k)}{\prod_{k=m+1}^{q} \Gamma(1+b_h-b_k) \prod_{k=n+1}^{p} \Gamma(a_k-b_h)}\ {}_pF_{q-1}\left[\begin{matrix} \vec{a} \\ \vec{b} \end{matrix}\,;\,(-1)^{m+n-p} z\right] z^{b_h},$$

where $\vec{a_h} = \langle 1+b_h-a_1, \ldots, 1+b_h-a_p\rangle$ and $\vec{b_h} = \langle 1+b_h-b_1, \ldots, 1+b_h-b_q\rangle$ (with the $1+b_h-b_h$ term omitted).

Claim 23. *The Padé approximant $\operatorname{POLY}_m\left(z \,\middle|\, \tfrac{\vec{\omega}}{\vec{\rho}}\right)$ is associated with a generalized hypergeometric function by*

$$\operatorname{POLY}_m\left(z \,\middle|\, \tfrac{\vec{\omega}}{\vec{\rho}}\right) = \frac{1}{\rho_m!}\left(\prod_{\substack{k=0\\k\neq m}}^{M} \frac{1}{(\omega_k-\omega_m)^{\overline{\rho_k+1}}}\right) {}_{M+1}F_M\left[\begin{matrix} \omega_m-\vec{\omega}-\vec{\rho} \\ (1+\omega_m-\vec{\omega})^{\star m} \end{matrix}\,;\,1-z\right].$$

Proof. Using Claim 21 and Theorem 22, we may write $\operatorname{REM}\left(z \,\middle|\, \tfrac{\vec{\omega}}{\vec{\rho}}\right)$ as

$$\sum_{m=0}^{M} \frac{\prod_{\substack{k=0\\k\neq m}}^{M} \Gamma(\omega_k-\omega_m)}{\prod_{k=0}^{M} \Gamma(\omega_k+\rho_k+1-\omega_m)}\ {}_{M+1}F_M\left[\begin{matrix} 1+\omega_m-\vec{\omega}-\vec{\rho}-1 \\ (1+\omega_m-\vec{\omega})^{\star m} \end{matrix}\,;\,1-z\right] (1-z)^{\omega_m},$$

which we manipulate into the form

$$\sum_{m=0}^{M} \frac{1}{\rho_m!}\Bigg(\prod_{\substack{k=0\\k\neq m}}^{M} \frac{1}{(\omega_k-\omega_m)^{\overline{\rho_k+1}}}\Bigg)\ {}_{M+1}F_M\left[\begin{matrix}\omega_m-\vec{\omega}-\vec{\rho}\\(1+\omega_m-\vec{\omega})^{\star m}\end{matrix};1-z\right](1-z)^{\omega_m}.$$

One coordinate of $\omega_m-\vec{\omega}-\vec{\rho}$ is $-\rho_m$, a nonpositive integer. Consequently,

$$\frac{1}{\rho_m!}\Bigg(\prod_{\substack{k=0\\k\neq m}}^{M} \frac{1}{(\omega_k-\omega_m)^{\overline{\rho_k+1}}}\Bigg)\ {}_{M+1}F_M\left[\begin{matrix}\omega_m-\vec{\omega}-\vec{\rho}\\(1+\omega_m-\vec{\omega})^{\star m}\end{matrix};1-z\right]$$

is a polynomial with degree at most ρ_m. Therefore, it must be $\textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$. □

4.6. Power Series

Claim 24 is Theorem 4(iv).

Claim 24. *Let g_n be the coefficients in the power series expansion of $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ at $z=0$, i.e., $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$. Then for $n \geq 0$ we have*

$$g_n = (-1)^n \sum_{m=0}^{M} \frac{1}{\rho_m!} \sum_{r=0}^{\rho_m} \binom{\rho_m}{r} \frac{(-1)^r(\omega_m+r)^{\underline{n}}}{\prod_{\substack{k=0\\k\neq m}}^{M}(\omega_k-\omega_m-r)^{\overline{\rho_k+1}}}.$$

In particular, $g_n = 0$ for $0 \leq n \leq \sigma-2$ and $g_{\sigma-1}=1$.

Proof. We begin with the contour integral representation of $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$ given in Claim 16, replace $(1-z)^{\xi}$ with its power series, and then integrate term by term, obtaining

$$\begin{aligned}
\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big) &= \frac{(-1)^{\sigma-1}}{2\pi i}\int_{\gamma}(1-z)^{\xi}\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi \\
&= \frac{(-1)^{\sigma-1}}{2\pi i}\int_{\gamma}\Bigg(\sum_{n=0}^{\infty}(-1)^n\xi^{\underline{n}}\frac{z^n}{n!}\Bigg)\prod_{k=0}^{M}\frac{1}{(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi \\
&= \sum_{n=0}^{\infty}(-1)^n\Bigg(\frac{1}{2\pi i}\int_{\gamma}\frac{(-1)^{\sigma-1}\xi^{\underline{n}}}{\prod_{k=0}^{M}(\xi-\omega_k)^{\underline{\rho_k+1}}}\,d\xi\Bigg)\frac{z^n}{n!}.
\end{aligned}$$

Continuing as in the proof of Claims 17 and 18, we see that

$$\begin{aligned}
g_n &= (-1)^n \left(\frac{1}{2\pi i} \int_\gamma \frac{(-1)^{\sigma-1} \xi^{\underline{n}}}{\prod_{k=0}^{M} (\xi-\omega_k)^{\underline{\rho_k+1}}} \, d\xi \right) \qquad (6)\\
&= (-1)^n \sum_{m=0}^{M} \sum_{r=0}^{\rho_m} (-1)^{\sigma-1} (\omega_m+r)^{\underline{n}} \Phi_{r,m}(\omega_m+r)\\
&= (-1)^n \sum_{m=0}^{M} \sum_{r=0}^{\rho_m} (-1)^{\sigma-1} (\omega_m+r)^{\underline{n}} \frac{(-1)^{r-\rho_m}}{\rho_m!} \binom{\rho_m}{r} \prod_{\substack{k=0\\k\neq m}}^{M} \frac{1}{(\omega_m+r-\omega_k)^{\overline{\rho_k+1}}}\\
&= (-1)^n \sum_{m=0}^{M} \sum_{r=0}^{\rho_m} (-1)^{\sigma-1} (\omega_m+r)^{\underline{n}} \frac{(-1)^{r-\rho_m}}{\rho_m!} \binom{\rho_m}{r} \prod_{\substack{k=0\\k\neq m}}^{M} \frac{(-1)^{\rho_k+1}}{(\omega_k-\omega_m-r)^{\overline{\rho_k+1}}}\\
&= (-1)^n \sum_{m=0}^{M} \frac{1}{\rho_m!} \sum_{r=0}^{\rho_m} \binom{\rho_m}{r} \frac{(-1)^r (\omega_m+r)^{\underline{n}}}{\prod_{\substack{k=0\\k\neq m}}^{M} (\omega_k-\omega_m-r)^{\overline{\rho_k+1}}}.
\end{aligned}$$

That $g_n = 0$ for $0 \le n \le \sigma-2$ and $g_{\sigma-1} = 1$ follow from the definition of $\textsc{Rem}\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}}\big)$. Alternatively, the expression on line (6) is shown directly to have these values in the proof of Claim 16, beginning with equation (3). □

4.7. Perfection

We remind our reader that our vectors are indexed from 0, so that the j-th coordinate of $\langle \rho_0, \ldots, \rho_M \rangle$ is ρ_j. The coordinates of the $(M+1)\times(M+1)$ matrix $\mathbf{H}$ in the next claim is indexed in the same manner.

Claim 25 is Theorem 6.

Claim 25. *Fix $\vec{\rho} \in \mathbb{N}^{M+1}$ and $\vec{\epsilon}_0, \vec{\epsilon}_1, \ldots, \vec{\epsilon}_M \in \mathbb{Z}^{M+1}$ with each $\vec{\rho}+\vec{\epsilon}_k$ having nonnegative coordinates, and denote the j-th coordinate of $\vec{\epsilon}_i$ as $\vec{\epsilon}_{i,j}$. Let S be maximum of $\sum_{i=0}^{M} \vec{\epsilon}_{i,\beta(i)}$ taken over all permutations β of $0,1,\ldots,M$, and let T be the minimum of $\sum_{j=0}^{M} \vec{\epsilon}_{i,j}$ taken over $0 \le i \le M$. Suppose the following two conditions are satisfied:*

1. *There is a unique permutation α of $0,1,\ldots,M$ with $S = \sum_{i=0}^{M} \vec{\epsilon}_{i,\alpha(i)}$;*
2. *$T + M = S$.*

Then the $(M+1)\times(M+1)$ matrix $\mathbf{H}$, whose (k,m) coordinate is the polynomial $\textsc{Poly}_m\big(z \,\big|\, {\vec{\omega} \atop \vec{\rho}+\vec{\epsilon}_k}\big)$, has determinant

$$C z^{\sigma(\vec{\rho})+T-1},$$

where C is nonzero and does not depend on z.

Proof. The determinant of $\mathbf{H}$, by the familiar permutation expansion, is

$$\det(\mathbf{H}) = \sum_{\beta\in\Sigma_{[0,M]}} (-1)^{\operatorname{sgn}(\beta)} \prod_{k=0}^{M} \textsc{Poly}_{\beta(k)}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_k}\big),$$

which is clearly a polynomial. Notice that

$$\begin{aligned}\deg\left(\prod_{k=0}^{M} \textsc{Poly}_{\beta(k)}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_k}\big)\right) &= \sum_{k=0}^{M} \deg\left(\textsc{Poly}_{\beta(k)}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_k}\big)\right)\\ &= \sum_{k=0}^{M} (\rho_{\beta(k)} + \vec{\epsilon}_{k,\beta(k)}) \le \sigma(\vec{\rho}) - (M+1) + S,\end{aligned}$$

with equality achieved for (and only for) $\beta = \alpha$. Consequently,

$$\deg(\det(\mathbf{H})) = \sigma(\vec{\rho}) - M - 1 + S = \sigma(\vec{\rho}) + T - 1,$$

and in particular $\det(\mathbf{H})$ is not identically 0.

Let $\vec{v}$ be the column vector $\langle (1-z)^{\omega_0}, (1-z)^{\omega_1}, \ldots, (1-z)^{\omega_M}\rangle^T$. By definition $\mathbf{H}\vec{v}$ is a column of $M+1$ functions of z: in row k it is $\textsc{Rem}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_k}\big)$, which has a zero of order $\sigma(\vec{\rho}+\vec{\epsilon}_k) - 1 = \sigma(\vec{\rho}) + \left(\sum_{j=0}^{M} \vec{\epsilon}_{k,j}\right) - 1 \ge \sigma(\vec{\rho}) + T - 1$. Now multiply $\mathbf{H}\vec{v}$ by the adjoint of $\mathbf{H}$, which is also a matrix of polynomials. We have

$$\begin{aligned}\det(\mathbf{H})\vec{v} &= \operatorname{adj}(\mathbf{H})\mathbf{H}\vec{v}\\ &= \operatorname{adj}(\mathbf{H}) \begin{pmatrix} \textsc{Rem}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_0}\big)\\ \textsc{Rem}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_1}\big)\\ \vdots\\ \textsc{Rem}\big(z \,\big|\, {}^{\vec{\omega}}_{\vec{\rho}+\vec{\epsilon}_M}\big)\end{pmatrix}\\ &= \operatorname{adj}(\mathbf{H}) \left(z^{\sigma(\vec{\rho})+T-1} \sum_{n=0}^{\infty} \vec{v}_n z^n \right)\\ &= z^{\sigma(\vec{\rho})+T-1} \sum_{n=0}^{\infty} (\operatorname{adj}(\mathbf{H})\vec{v}_n) z^n\end{aligned}$$

for some column vectors $\vec{v}_0 \ne \vec{0}, \vec{v}_1, \ldots$. That $\vec{v}_0 \ne \vec{0}$ follows from the definition of T. Each coordinate of $\det(\mathbf{H})\vec{v}$ has the form $\det(\mathbf{H})(1-z)^{\omega}$, and so has a zero at $z=0$ of order at most $\deg(\det(\mathbf{H})) = \sigma(\vec{\rho}) - M - 1 + S$. By the above displayed equations, each coordinate of $\det(\mathbf{H})\vec{v}$ has a zero at $z=0$ of order at least $\sigma(\vec{\rho}) + T - 1$ with equality for some coordinate. But $T + M = S$, by hypothesis, so that $\det(\mathbf{H})$ is a polynomial whose degree coincides with the order of its zero at $z = 0$. Therefore

$$\det(\mathbf{H}) = Cz^{\sigma(\vec{\rho})+T-1} = Cz^{\sigma(\vec{\rho})+S-M-1},$$

as claimed. The constant C is nonzero as $\det(\mathbf{H})$ is not identically 0. $\square$

5. Opportunities for Further Work

1. Is there a nice iterated integral representation of $\text{POLY}_m\left(z \,\middle|\, \frac{\vec{\omega}}{\vec{\rho}}\right)$ without contours, similar to the representation in Theorem 4(i) for $\text{REM}\left(z \,\middle|\, \frac{\vec{\omega}}{\vec{\rho}}\right)$?

2. For fixed $\vec{\omega}$, which degree vectors $\vec{\rho}^{(0)}, \vec{\rho}^{(1)}, \ldots, \vec{\rho}^{(M)}$ lead to a perfect system? There seems to be some geometry involved. That is, a modest amount of computation suggests that for each M there is B such that if any coordinate of any $\vec{\epsilon}_k - \vec{\epsilon}_j$ is not between $-B$ and B, then the resulting system is not perfect for any ρ (the determinant of $\mathbf{H}$ doesn't have the form Cz^n).

3. What is the value of C in Theorem 6?

4. What is the nice power series expression for $\text{POLY}_m\left(z \,\middle|\, \frac{\vec{\omega}}{\vec{\rho}}\right)$? For $M = 1$, this is an important part of the best explicit irrationality measure for $2^{1/3}$.

References

[1] A. Baker, Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers, Quart. J. Math. Oxford Ser. (2) **15** (1964), 375–383, DOI 10.1093/qmath/15.1.375.

[2] George A. Baker Jr. and Peter Graves-Morris, *Padé Approximants*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 59, Cambridge University Press, Cambridge, 1996.

[3] Michael A. Bennett, Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$, J. Reine Angew. Math. **535** (2001), 1–49, DOI 10.1515/crll.2001.044.

[4] G. V. Chudnovsky, On the method of Thue-Siegel, Ann. of Math. (2) **117** (1983), no. 2, 325–382, DOI 10.2307/2007080.

[5] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994. A foundation for computer science.

[6] H. Jager, *A multidimensional generalization of the Padé table. I, II, III, IV, V, VI*, Nederl. Akad. Wetensch. Proc. Ser. A 67=Indag. Math. **26** (1964), 193–249.

[7] Kurt Mahler, Ein Beweis des Thue-Siegelschen Satzes über die Approximation algebraischer Zahlen für binomische Gleichungen, Math. Ann. **105** (1931), no. 1, 267–276, DOI 10.1007/BF01455819 (German). English translation by Karl Levy at `https://arxiv.org/abs/1507.01447`.

[8] *NIST Digital Library of Mathematical Functions*, `http://dlmf.nist.gov/`. Accessed April 25, Release 1.0.8 of 2014. Online companion to [9].

[9] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, 2010. Print companion to [8].

[10] B. Riemann, *Oeuvres Mathimatiques*, Albert Blanchard, Paris, 1968.

[11] C. L. Siegel, Approximation algebraische zahlen, Mat. Zeit. **10** (1921), 127–213.

[12] Lucy Joan Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[13] A. Thue, Bemerkung über gewisse Nährungsbrüche algebraischer Zahlen, Skrifter udigvne af Videnskabsselskabet i Christiania (1908).