GRAPHICAL ENUMERATION AND STAINED GLASS WINDOWS, 1: RECTANGULAR GRIDS

Lars Blomberg
Linghem, Sweden
lars.blomberg2@hotmail.com

S. R. Shannon
Rowville, Victoria, Australia
scott_r_shannon@hotmail.com

N. J. A. Sloane[1]
The OEIS Foundation Inc., Highland Park, New Jersey
njasloane@gmail.com

Received: 12/21/20, Revised: 5/20/21, Accepted: 7/14/21, Published: 8/23/21

Abstract

This is a survey of enumeration problems arising from the study of planar graphs formed when the edges of a polygon are marked with evenly spaced points and every pair of points is joined by a line. A few of these problems have been solved, a classical example being the the graph K_n formed when all pairs of vertices of a regular n-gon are joined by chords, which was analyzed by Poonen and Rubinstein in 1998. Most of these problems are unsolved, however, and this two-part article provides data from a number of such problems as well as colored illustrations, which are often reminiscent of stained glass windows. The polygons considered include rectangles, hollow rectangles (or frames), triangles, pentagons, pentagrams, crosses, etc., as well as figures formed by drawing semicircles joining equally-spaced points on a line. Part 1 discusses planar graphs that are based on rectangular grids.

– Dedicated to the memory of Ronald Lewis Graham (1935–2020)

1. Introduction

In 1998 Poonen and Rubinstein [18] (see also [23]) solved the problem of finding the numbers of intersection points and cells in a regular drawing of the complete graph K_n, and in 2009-2010 Legendre [11] and Griffiths [8] solved a similar problem for the complete bipartite graph $K_{n,n}$. Stated another way, [18] analyzes the planar graph

[1] To whom correspondence should be addressed.

formed by joining all pairs of vertices of a regular n-gon, while [11, 8] analyze the graph formed by taking a row of $n-1$ identical squares and drawing lines between every pair of boundary nodes.

One motivation for the present work was to see if these investigations could be extended to graphs formed from other structures, such as an $m-1 \times n-1$ array of identical squares. Take a rectangle of size $m \times n$, and place $m-1$ equally spaced points on the two vertical sides, and $n-1$ equally spaced points on the two horizontal sides. Then draw lines between every pair of the $2(m+n)$ boundary points, and place a node at each point where these lines intersect. The resulting planar graph, which we denote by $BC(m,n)$, is the main subject of Part 1 of this paper.

Although we have not been very successful in analyzing these graphs, we have collected a great deal of data, which has been entered into various sequences in the *On-Line Encyclopedia of Integer Sequences* [15].

In Part 2 [3], we continue this work by considering other structures such as hollow squares (or "frames"), triangles, pentagons, hexagons, pentagrams, etc., as well as figures formed by drawing semicircles joining equally-spaced points on an interval. The last-mentioned figures are reminiscent of juggling patterns,[2] as studied by Ron Graham in [5] and other papers, and we regret that now it is too late to ask him for help for finding a formula for those numbers.

We were also motivated by memories of stained glass windows seen in the great Gothic cathedrals of Northern Europe. In 2019 we made a colored drawing of K_{23} (Figure 1) which was reminiscent of a rose window, and we were curious to see what colored versions of other graphs would look like. Informally, our philosophy has been, if we can't solve it, make art. We make no great claims for artistic merit, but the images are certainly colorful.

Space limitations have restricted the number and quality of the images that we could include here. The corresponding entries in [15] (A007678[3] in the case of Figure 1) contain a large number of other images, with better resolution. We are especially fond of the three images of K_{41} in A007678, and in A331452 the reader should not miss the images labeled $T(10,2)$, $T(6,6)$, $T(7,7)$, which are drawings of the graphs $BC(10,2)$, $BC(6,6)$, and $BC(7,7)$ discussed below.

This paper is arranged as follows. The last section of this Introduction establishes the notation we will use, especially the terms *nodes*, *chords*, and *cells*, and provides some examples. Section 2 deals with the graphs $BC(1,n)$ (or equivalently $BC(n,1)$), where the bounding polygon is a rectangle of size $1 \times n$ (or $n \times 1$). Theorem 2.1 gives Legendre and Griffiths's enumeration of the nodes and cells in $BC(1,n)$. In 2019, Max Alekseyev (personal communication; see also A306302) pointed out that that the Legendre-Griffiths results are essentially the same as results that he and

[2]See entry A290447 in [15]

[3]Six-digit numbers prefixed by A refer to entries in the On-Line Encyclopedia of Integer Sequences [15].

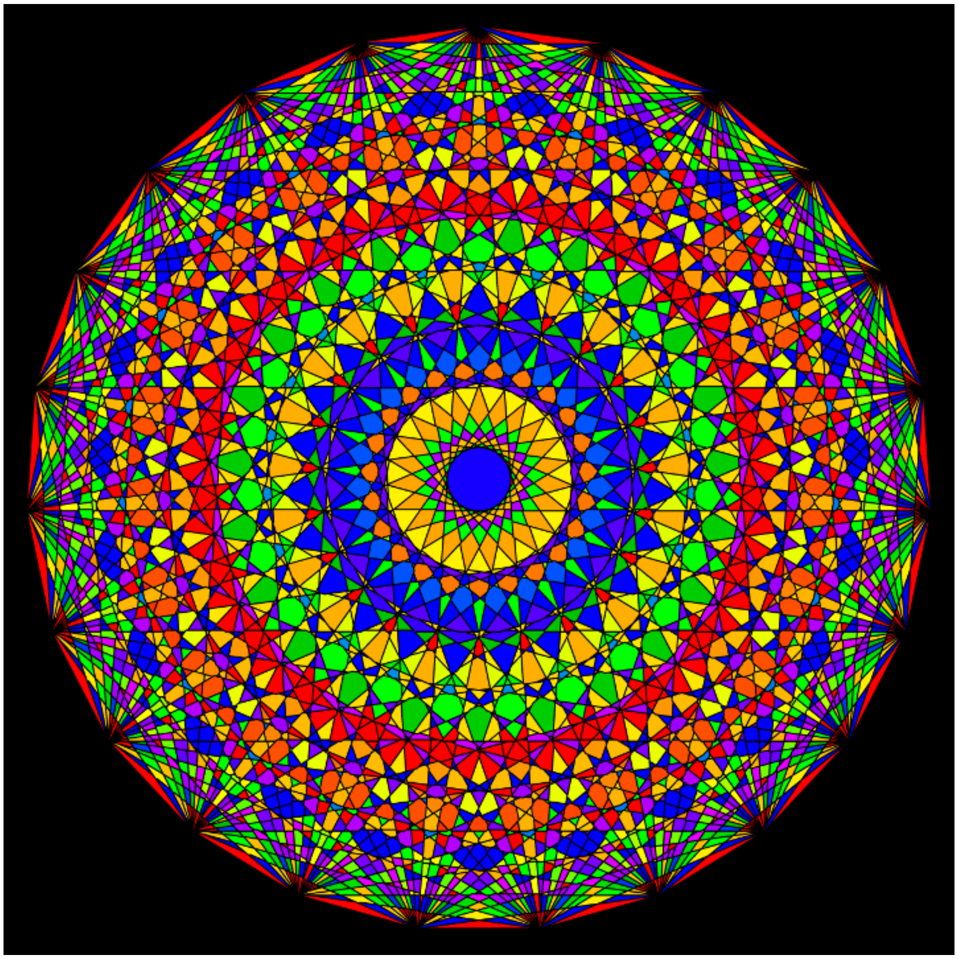


Figure 1: Colored drawing of complete graph K_{23} (see Section 10.3 for the coloring algorithm). Entry A007678 in [15] has many similar images (which are also of higher quality).

his coauthors obtained in connection with the enumeration of threshold functions [1, 2]. The family of isosceles triangle graphs $IT(n)$ (Section 3) provides a bridge between the graphs $BC(1, n)$ and two-dimensional threshold functions. Alekseyev also mentioned that their work implies a result that was apparently overlooked in the Legendre and Griffiths papers: the cells in $BC(1, n)$ are always triangles or quadrilaterals. See Theorem 3.1. The proof of this fact in [2] depends on a theorem about teaching sets for threshold functions [20, 27]. We feel that such a elementary property should have a purely geometrical proof, although no such proof is presently known. We state this question as Open Problem 3.2.

One possible attack on this problem is to study the distribution of cells in each of the n squares of $BC(1,n)$— see Section 4, and especially Tables 2, 3, 4. The *gfun* Maple program [19] suggests a form for the generating functions of the columns of these tables, but so far this is only a conjecture.

In Section 5 we consider the number of interior nodes in $BC(1,n)$ where c chords meet (Table 5). The number of simple nodes, where just two chords cross, is of the greatest interest, since these seem to dominate. But even though we have calculated 500 terms of this sequence (Table 6 and A334701) we have been unable to find a formula or recurrence (Open Problem 5.2). There have been several similar occasions during this project when we have regretted not having an oracle that would take a few hundred terms of a simple, well-defined sequence and suggest some kind of formula.[4]

The graph $BC(1,n)$ has bounding polygon which is a $1 \times n$ rectangle. If we start instead from an $m \times n$ rectangle, with m and $n > 1$, there are three natural ways to define a planar graph, which we will denote by $BC(m,n)$, $AC(m,n)$, and $LC(m,n)$. These are the subjects of Sections 6, 8, and 9, respectively. For these families we have plenty of data and pictures, but not many results. In Section 6 we conjecture that the cells in $BC(2,n)$ have at most eight sides, and for $n \geq 19$, at most six sides (Conjecture 6.2). Our main result concerning $BC(m,n)$ is an upper bound on the numbers of nodes and cells in $BC(m,n)$, presented in Section 7, which appears to be reasonably close to the true values.

The final section (Section 10) describes how we colored the graphs.

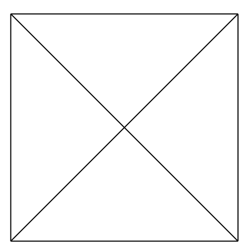


Figure 2: The planar graph $BC(1,1)$, a 1-reticulated square. There are four boundary points and two chords, and the graph has five nodes, eight edges, and four cells.

Terminology. Our subject is planar graphs, as shown in most of the figures. We start with a grid or lattice in the plane, draw a polygon on this grid, mark points

[4]The oracle might compare the sequence with shifted versions of each of the hundreds of thousands of entries in [15], and ask Bruno Salvy and Paul Zimmermann's program *gfun*, or Harm Derksen's program *guesss*, or Christian Krattenthaler's program *Rate*, or one of the other programs used by *Superseeker* [21] if there is a formula for the difference.

along the boundary of the polygon, and form the graph by joining these boundary points by lines and creating nodes where these lines cross.

Since all three concepts, grids, polygons, and graphs, involve points and lines, we will establish our terminology with some care, hoping to avoid confusion without being too pedantic.

In graph theory many different terms are used for the basic notions of node (or point, or vertex) and edge (or line). We will use *node* and *edge* for these specifically graph-theoretic terms. A planar graph divides the plane into cells (or regions, or chambers). We will use the term *cell*, with the understanding that the unbounded region exterior to the graph is not considered to be a cell.[5] Our graphs are also *maps* in the sense of Tutte [25, 26], but we will refer to them simply as planar graphs.

To construct our graph we usually start from the boundary curve of some connected polygon P in the plane, and mark various points on this boundary. We call these the *boundary points* and we call P the *defining polygon*. We assume P is connected, but not necessarily simply connected. In Part 2, for example, P may be a square annulus. Our graph is then constructed by joining pairs of boundary points by line segments, according to some specific rule. For the graphs $BC(m, n)$, every pair of distinct boundary points is joined by a line segment, which starts at one boundary point and ends at another. A line segment is called a *chord* if (apart from its end-points) it lies in the interior of P. Our graph then has *nodes* which are all the boundary points together with all the points where the chords cross. A node which is not a boundary point is called an *interior node*. When referring to the geometry of the polygon P, we will speak of its *sides* and *vertices*. Some of the line segments may coincide with the sides of P; these are not called chords.

We usually subdivide the sides of the polygon by dividing them into equal parts. To divide a side into k equal parts, we insert $k-1$ equally spaced boundary points along that side, so that the side contains a total of $k+1$ boundary points, the two vertices plus the $k-1$ additional points. We say that the side has been *k-reticulated*.

Finally, we give coordinates for our polygons P by defining them in terms of some underlying grid or lattice, which in Part 1 will be the simple square lattice $\mathbb{Z} \times \mathbb{Z}$. The *grid points* have integer coordinates (i, j).

As an example, Figure 2 shows the graph $BC(1, 1)$ (defined in Section 2), which has 5 nodes, 8 edges, and 4 cells. The polygon is a square and there are two chords which meet at the central node. Figure 3 shows the graph $BC(2, 2)$, constructed from a square in which each side has been 2-reticulated. There are 16 chords. This graph has 37 nodes, 92 edges, and 56 cells.[6] Note that for a connected planar graph,

[5] In Part 2, when we consider graphs that have a "hole" in them (such as a square annulus or frame), the hole is also not considered to be a cell.

[6] $BC(3, 3)$ is shown in Figure 14 in Section 6 and has 340 cells. There is no known formula for the sequence $4, 56, 340, 1120, 3264, \ldots$ (A255011), the number of cells in $BC(n, n)$, even though we have 52 terms.

Euler's formula states that the numbers of nodes, edges, and cells are related by

$$|\,\text{nodes}\,| \; - \; |\,\text{edges}\,| \; + \; |\,\text{cells}\,| \; = \; 1. \tag{1.1}$$

Figure 4 shows a colored version of $BC(2,2)$. The principles used to color these graphs are discussed in Section 10. For any undefined terms from graph theory see [4, 10].

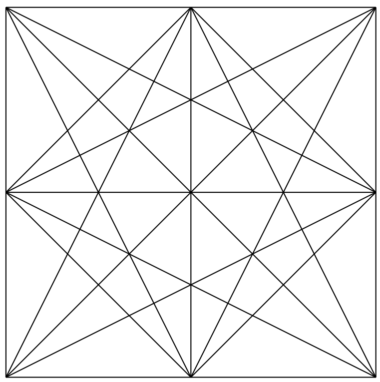


Figure 3: $BC(2,2)$: a 2-reticulated square with 56 cells.

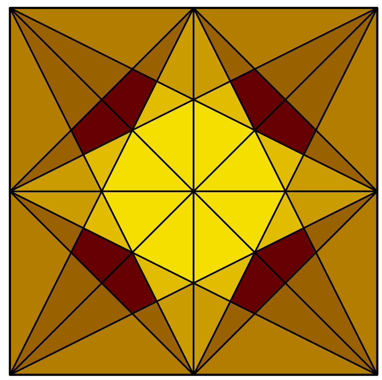


Figure 4: The same $BC(2,2)$ drawn with colored cells. See Section 10.2 for coloring scheme.

2. $BC(1,n)$: $1 \times n$ Rectangular Windows

The defining polygon for the graph $BC(1,n)$ $(n \geq 1)$ is a $1 \times n$ rectangle, which we take to have vertices $(0,0)$, $(n,0)$, $(0,1)$, and $(n,1)$. The boundary points for $BC(1,n)$ are the points $\{(i,0),(i,1) : 1 \leq i \leq n\}$, and we join every pair of distinct boundary points by a line segment. Some of these line segments lie on the sides of the rectangle, and there are in addition $n^2 + 2n - 1$ chords. Figures 2, 5, and 6 show $BC(1,n)$ for $n = 1, 2$, and 3.

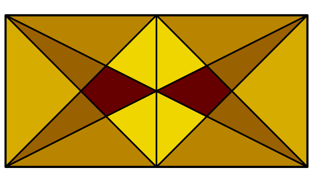


Figure 5: $BC(1,2)$.

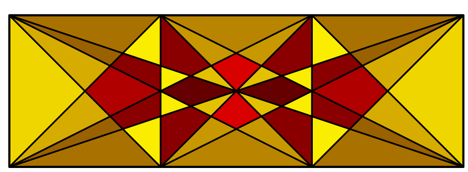


Figure 6: $BC(1,3)$.

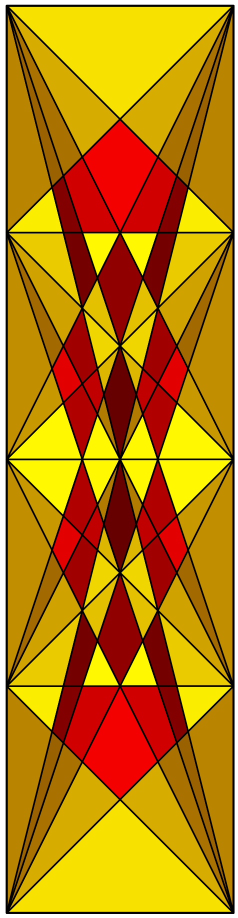


Figure 7: $BC(4,1)$, colored using the red and yellow palettes (see Section 10.2).

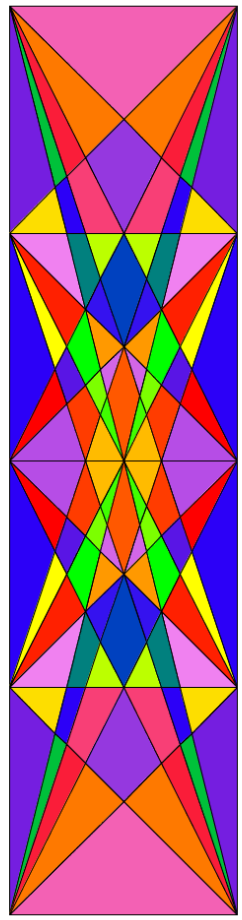


Figure 8: A version of $BC(4,1)$ colored by our 'random coloring' algorithm (see Section 10.3).

Of course we could equally well have started with a vertical rectangle of size $n \times 1$, in which case the graph would be denoted by $BC(n,1)$. Since this work was partly inspired by the windows of Gothic cathedrals, we admit to a slight preference for $BC(n,1)$ over $BC(1,n)$, although as graphs they are isomorphic. Figures 7 and 8 show our stained glass window $BC(4,1)$ using two different coloring schemes.

We will continue to discuss $BC(1,n)$, but the reader should remember that the results apply equally well to $BC(n,1)$.

Another way to construct $BC(1,n)$ is to start with the complete bipartite graph $K_{n+1,n+1}$ formed by taking $n+1$ equally spaced points in each of two horizontal rows, joining every upper point to every lower point by a line segment, placing a node at each point where these lines intersect, and then adding the line segments through the two rows of points. Thus $BC(1,2)$ in Figure 5 is the well-known nonplanar "utilities" graph $K_{3,3}$ if the two horizontal lines, the seven interior nodes, and the colors are ignored.

The graphs $BC(1,n)$ are one of the few families where explicit formulas are known for the numbers of nodes ($\mathcal{N}(1,n)$), edges ($\mathcal{E}(1,n)$), and cells ($\mathcal{C}(1,n)$). The initial values of these quantities are shown in Table 1, along with the A-numbers of the corresponding sequences.

n :	1	2	3	4	5	6	7	8	9	10	$\cdots$	[15]
$\mathcal{N}(1,n)$:	5	13	35	75	159	275	477	755	1163	1659	$\cdots$	A331755
$\mathcal{E}(1,n)$:	8	28	80	178	372	654	1124	1782	2724	3914	$\cdots$	A331757
$\mathcal{C}(1,n)$:	4	16	46	104	214	380	648	1028	1562	2256	$\cdots$	A306302

Table 1: Numbers of nodes, edges, cells in $BC(1,n)$.

Since by Euler's formula (1.1), $\mathcal{E}(1,n) = \mathcal{N}(1,n) + \mathcal{C}(1,n) - 1$, there is no need to tabulate $\mathcal{E}(1,n)$, and in the future we shall omit those numbers.

The following theorem is due to Legendre (2009) [11] and Griffiths (2010) [8], who discuss the problem from the point of view of $K_{n+1,n+1}$. First we introduce an expression that will frequently appear in these formulas. For $m,n,q \geq 1$, let

$$V(m,n,q) \;=\; \sum_{a=1..m} \; \sum_{\substack{b=1..n \\ \gcd\{a,b\}=q}} (m+1-a)(n+1-b)\,. \tag{2.1}$$

Theorem 2.1. (Legendre [11, Prop. 6], Griffiths [8, Th. 3]). *For $n \geq 1$, the number of nodes in $BC(1,n)i$ is*

$$\mathcal{N}(1,n) \;=\; 2(n+1) + V(n,n,1) - V(n,n,2)\,, \tag{2.2}$$

and the number of cells is

$$\mathcal{C}(1,n) \;=\; n^2 + 2n + V(n,n,1)\,. \tag{2.3}$$

Remarks. (i) A key step in the proof of (2.2) (see [11]) is finding a condition for three chords to meet at a point. (ii) The starting point for the proof of (2.3) (see [8]) is the observation that in the graph $BC(1,n)$ the chords contain no edges that are parallel to the two long sides of the rectangle. This means that every cell has a unique node that is closest to the upper side of the rectangle. (iii) The term $2(n+1)$ on the right-hand side of (2.2) is the number of boundary points. The difference between the other two terms is therefore the number of interior nodes in $BC(1, n)$ (A159065):

$$1, 7, 27, 65, 147, 261, 461, 737, 1143, \cdots . \tag{2.4}$$

(iv) The values of $\mathcal{N}(1, n)$ and $\mathcal{C}(1, n)$ are given in A331755 and A306302.

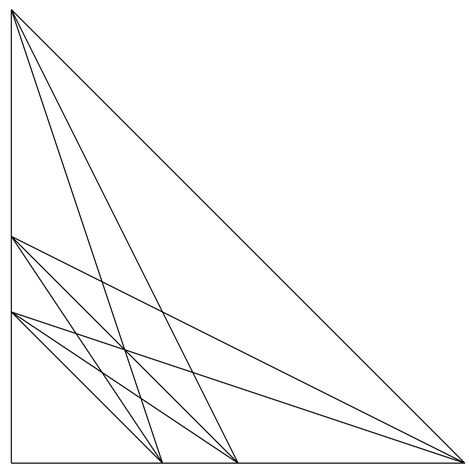


Figure 9: The isosceles triangle graph $IT(2)$. There are 14 nodes (7 on boundary, 7 in interior), 30 edges, and 17 cells (15 triangles and 2 quadrilaterals).

3. The Isosceles Triangle Graph $IT(n)$

In 2019 Max Alekseyev added a comment to A306302 pointing out that the results in Theorem 2.1 are essentially the same as the results he and his coauthors had obtained in [2] (2015) for the isosceles triangle graphs $IT(n)$.

The definition of the *isosceles triangle graph* $IT(n)$, $n \geq 1$, starts with an isosceles right triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. On the vertical side of the triangle we place n additional boundary points at the points

$$\left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \left(0, \frac{1}{4}\right), \ldots, \left(0, \frac{1}{n+1}\right),$$

and similarly on the horizontal side we place n additional boundary points at the points

$$\left(\frac{1}{2}, 0\right), \left(\frac{1}{3}, 0\right), \left(\frac{1}{4}, 0\right), \ldots, \left(\frac{1}{n+1}, 0\right).$$

Including the three vertices, there are a total of $2n+3$ boundary points. We then join every pair of distinct boundary points by a line segment. Besides the three sides of the triangle there are n^2+2n chords, and there are interior nodes at the points where the chords intersect. Figures 9, 10, 11 show $IT(2)$, $IT(3)$ and $IT(4)$. The latter two graphs have been colored using the red and yellow palettes (Section 10.2). (There are no boundary points on the hypotenuse other than its two endpoints. In Part 2 of this paper [3] we will discuss graphs formed by placing n equally-spaced boundary points on all three sides of an equilateral triangle.)

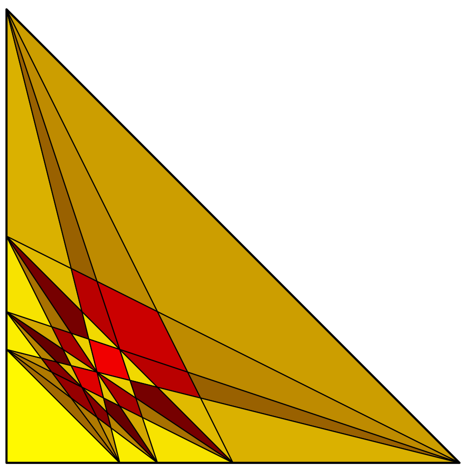


Figure 10: $IT(3)$ (33 triangles, 14 quadrilaterals)

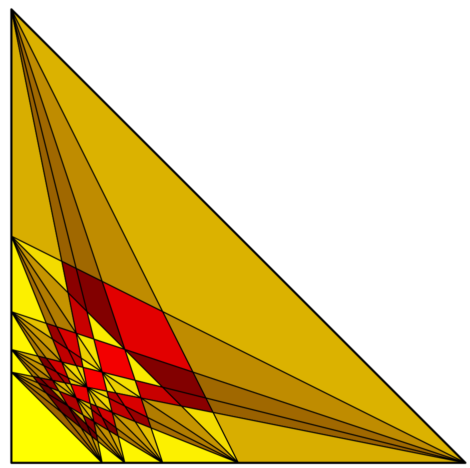


Figure 11: $IT(4)$ (71 triangles, 34 quadrilaterals)

Alekseyev pointed out that if we take the boundary points of $BC(1,n)$ to be the points $(i,0)$ and $(i,1)$ for $i=0,\ldots,n$, then the map

$$(x,y) \longmapsto \left(\frac{1-y}{x+1}, \frac{y}{x+1}\right), \tag{3.1}$$

maps $BC(1,n)$ onto $IT(n)$ minus the node and cell at the origin. Figure 12 illustrates this in the case $n=2$. The six boundary points A,B,C,E,F,G of $BC(1,2)$ are mapped to six of the seven boundary points of $IT(2)$. The point D, the point at infinity on the positive x axis (not part of $BC(1,2)$), is mapped to the origin in $IT(2)$. The region D,C,G,D to the right of $BC(1,2)$ is mapped to the triangular cell D,G,C,D at the origin in $IT(2)$.

A similar thing happens in the general case: $IT(n)$ always has one more node than $BC(1,n)$, two more edges, and one more cell. When these adjustments are made to the formulas in Theorem 2.1, we obtain the formulas in Theorem 13 of [2].

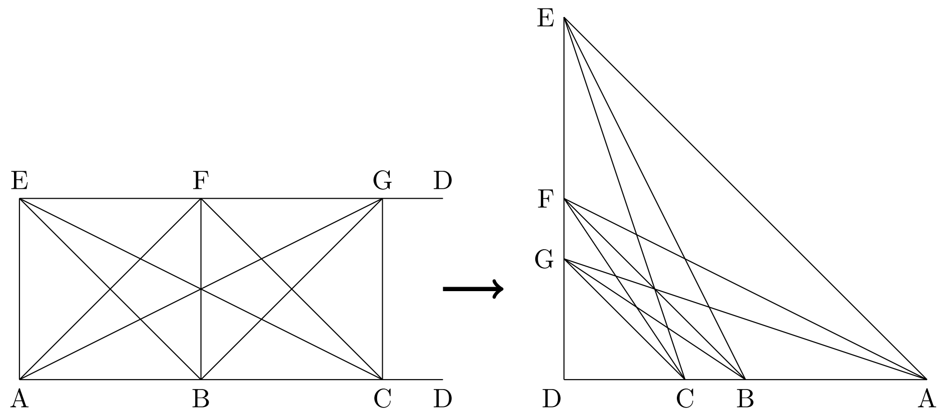


Figure 12: Illustrating the map (3.1) from $BC(1,2)$ to $IT(2)$.

The counts for nodes, edges, and cells in $IT(n)$ are given in A332362, A332360, and A332358.

However, Alekseyev (personal communication) also pointed out that Theorem 13 of [2] mentions an additional property of $IT(n)$—and hence of $BC(1,n)$—that seems to have been overlooked in [11] and [8]:

Theorem 3.1. (Alekseyev et al. [2])*: The cells in $IT(n)$ and hence $BC(1,n)$ are either triangles or quadrilaterals.*

That is, no cell in $BC(1,n)$ has five or more edges. The proof in [2] depends on a theorem about teaching sets for threshold function [20, 27]. No other proof seems to be known. :

Open Problem 3.2. Find a purely geometrical proof of Theorem 3.1.

4. The Cells in $BC(1,n)$

From Theorems 2.1 and 3.1, we can determine the numbers of triangular and quadrilateral cells in $BC(1,n)$ (sequences A324042 and A324043).

Theorem 4.1. *The $\mathcal{C}(1,n)$ cells in $BC(1,n)$ are made up of*

$$T(n) = 2V(n,n,2) + 2n(n+1) \tag{4.1}$$

triangles and

$$Q(n) = V(n,n,1) - 2V(n,n,2) - n^2 \tag{4.2}$$

quadrilaterals.

Proof. The sum $3T(n)+4Q(n)$ double-counts the edges in $BC(1,n)$ except that the $2n+2$ boundary edges are counted only once. Therefore

$$3T(n)+4Q(n)+(2n+2)=2\mathcal{E}(1,n)=2(\mathcal{N}(1,n)+\mathcal{C}(1,n)-1)\,, \tag{4.3}$$

and of course by Theorem 3.1, $T(n)+Q(n)=\mathcal{C}(1,n)$.

The proof is completed by solving these two equations for $T(n)$ and $Q(n)$ and using (2.2), (2.3). □

Figures 2, 5, 6, and 7 show the triangles and quadrilaterals for $n=1,...,4$.

One way to attack Open Problem 3.2 is to try to understand the distribution of cells in each of the n squares of $BC(1,n)$. Let $t_{n,k}$, $q_{n,k}$, and $c_{n,k}$ denote the numbers of triangles, quadrilaterals, and cells in the k-th square of $BC(1,n)$ for $1\le k\le n$ (so $t_{n,k}+q_{n,k}=c_{n,k}$ and $\sum_k c_{n,k}=\mathcal{C}(1,n)$). From Figure 5, for example, we see that $t_{1,1}=t_{1,2}=7$, $q_{1,1}=q_{1,2}=1$, and $c_{1,1}=c_{1,2}=8$.

The two end squares of $BC(1,n)$ are easily understood, and for future reference we state the result as:

Theorem 4.2. *For $n\ge 2$, the two end squares of $BC(1,n)$ both contain $2n+3$ triangles and $2n-3$ quadrilaterals.*

.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	4									
2	7	7								
3	9	14	9							
4	11	24	24	11						
5	13	30	38	30	13					
6	15	38	60	60	38	15				
7	17	44	76	86	76	44	17			
8	19	52	92	120	120	92	52	19		
9	21	58	106	146	158	146	106	58	21	
10	23	66	126	178	216	216	178	126	66	23

Table 2: Number $t_{n,k}$ of triangles in k-th square in $BC(1,n)$ (A333286).

Tables 2, 3, and 4 show the values of $t_{n,k}$, $q_{n,k}$, and $c_{n,k}$ for $n\le 10$. More extensive tables, for $n\le 80$, are given in entries A333286, A333287, A333288. However, even with 80 rows of data, we have been unable to find formulas for these numbers.

There is certainly a lot of structure in these tables. Using the Salvy-Zimmermann *gfun* Maple program [19], we attempted to find generating functions for the columns of these tables. On the basis of admittedly little evidence, we make the following conjecture.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	0									
2	1	1								
3	3	8	3							
4	5	12	12	5						
5	7	22	32	22	7					
6	9	28	40	40	28	9				
7	11	38	58	74	58	38	11			
8	13	46	74	98	98	74	46	13		
9	15	58	92	130	152	130	92	58	15	
10	17	68	104	150	180	180	150	104	68	17

Table 3: Number $q_{n,k}$ of quadrilaterals in k-th square in $BC(1, n)$ (A333287).

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	4									
2	8	8								
3	12	22	12							
4	16	36	36	16						
5	20	52	70	52	20					
6	24	66	100	100	66	24				
7	28	82	134	160	134	82	28			
8	32	98	166	218	218	166	98	32		
9	36	116	198	276	310	276	198	116	36	
10	40	134	230	328	396	396	328	230	134	40

Table 4: Total number $c_{n,k}$ of cells in k-th square in $BC(1, n)$ (A333288).

Conjecture 4.3. In all three of Tables 2, 3, and 4, the k-th column for $k \geq 3$ has a rational generating function which can be written with denominator $(1 - x^{k-2})(1 - x^{k-1})(1 - x^k)$.

For example, column 3 of Table 2, the sequence $\{t_{n,3}\}$, appears to have generating function

$$x^3 \, \frac{9 + 15x + 5x^2 - 2x^3 - 13x^4 - 11x^5 - 9x^6 + 2x^7 + 8x^8 - 4x^{10} + 4x^{12}}{(1-x)(1-x^2)(1-x^3)} \,. \tag{4.4}$$

It would be nice to know more about these quantities.

5. The Nodes in $BC(1, n)$

Besides looking at the cells of $BC(1, n)$, it is also interesting to study the nodes. For $n \geq 2$, $BC(1, n)$ has four boundary points of degree $n + 1$ and $2n - 2$ boundary points of degree $n+2$. An interior node formed when c chords (say) cross has degree $2c$. Let $v_{n,c}$ denote the number of interior nodes of degree $2c$, for $2 \leq c \leq n + 1$. Table 5 shows the values of $v_{n,c}$ for $n \leq 10$. A more extensive table, for $n \leq 100$, is given in A333275.

$n\backslash c$	2	3	4	5	6	7	8	9	10	11
1	1									
2	6	1								
3	24	2	1							
4	54	8	2	1						
5	124	18	2	2	1					
6	214	32	10	2	2	1				
7	382	50	22	2	2	2	1			
8	598	102	18	12	2	2	2	1		
9	950	126	32	26	2	2	2	2	1	
10	1334	198	62	20	14	2	2	2	2	1

Table 5: Number $v_{n,c}$ of interior nodes in $BC(1, n)$ where c chords cross (A333275).

Theorem 5.1. *For $n \geq 2$, the numbers $v_{n,c}$ satisfy:*

$$\sum_{c=2}^{n+1} v_{n,c} \;+\; 2n + 2 \;=\; \mathcal{N}(1, n)\,, \tag{5.1}$$

$$\sum_{c=2}^{n+1} c\, v_{n,c} \;+\; n^2 + 4n + 1 \;=\; \mathcal{E}(1, n)\,, \tag{5.2}$$

$$\sum_{c=2}^{n+1} \binom{c}{2} v_{n,c} \;=\; \binom{n+1}{2}^2 . \tag{5.3}$$

Proof. The first equation simply gives the total number of nodes in $BC(1, n)$. For (5.2) we count pairs (α, β), where α is a cell and β is a node, in two ways, obtaining

$$3T(n) + 4Q(n) \;=\; 4(n + 1) + (2n - 2)(n + 2) + \sum_c 2c\, v_{n,c}\,,$$

and use (4.3). To establish (5.3), we start with the observation that if all the $2n+2$ boundary points of $BC(1, n)$ are perturbed by small random amounts, there will be no triple or higher-order intersection points, all the interior nodes will be simple, and there will be $\binom{n+1}{2}^2$ of them (since any pair of nodes on the upper side of the rectangle and any pair of nodes on the lower side will determine a unique intersection point). As the boundary points are returned to their true positions, the interior nodes coalesce. If there is an interior point where c chords intersect, the $\binom{c}{2}$ interior nodes there coalesce into one, and we lose $\binom{c}{2}-1$ intersections. We are left with the $\mathcal{N}(1,n)-(2n+2)$ interior intersection points. Thus

$$\sum_{c=2}^{n+1}\left(\binom{c}{2}-1\right)v_{n,c} \;+\; \mathcal{N}(1,c)-(2n+2) \;=\; \binom{n+1}{2}^2,$$

which simplifies to give (5.3). □

However, we do not even have a formula for the number of simple interior intersection points in $BC(1,n)$ (the first column of Table 5, the sequence $\{v_{n,2}\}$, A334701), although we have computed 500 terms. The first 100 terms are shown in Table 6. We feel that a formula should exist.

Open Problem 5.2. Find a formula for the number of simple interior intersection points in $BC(1,n)$ (see Table 6 for 100 terms, or A334701 for 500 terms).

6. $BC(m,n)$: $m\times n$ Rectangular Windows

The graph $BC(1,n)$ ($n\ge 1$) is based on a $1\times n$ rectangle. In this section we consider what happens if we start more generally from an (m,n)-reticulated rectangle (where $m\ge 1$, $n\ge 1$): this is a rectangle of size $m\times n$ in which both vertical sides are divided into m equal parts, and both horizontal sides into n equal parts. There are $m-1$ nodes on each vertical side and $n-1$ nodes on each horizontal side, for a total of $4+2(m-1)+2(n-1)=2(m+n)$ boundary points.

We will discuss three families of graphs based on these rectangles, which will be denoted by by $BC(m,n)$, $AC(m,n)$, and $LC(m,n)$. The graph $BC(m,n)$ is formed by joining every pair of boundary points by a line segment and placing a node at each point where two or more line segments intersect. Figs. 3 and 4 show $BC(2,2)$, and Figure 14 shows $BC(3,3)$. ("BC" stands for "boundary chords".)

Alternatively, we could have constructed $BC(m,n)$ by starting with an $m\times n$ grid of equal squares, and then joining each pair of boundary grid points by a line segment. However, if we include the interior grid points, there are $(m+1)(n+1)$ grid points in all, and if we join *each* pair of grid points by a line segment, we obtain the graph $AC(m,n)$. ("AC" stands for "all chords".) These graphs are discussed by Huntington T. Hall [9], Marc E. Pfetsch and Günter M. Ziegler [16], and Hugo Pfoertner (entry A288187 in [15]). We shall say more about $AC(m,n)$ in Section 8.

A third family of graphs, $LC(m,n)$, arises if we extend each line segment in $AC(m,n)$ in both directions until it reaches the boundary of the grid. ("LC" stands

Terms 1–25	26–50	51–75	76–100
1	49246	679040	3264422
6	57006	732266	3438642
24	65334	790360	3616430
54	75098	849998	3805016
124	85414	914084	3998394
214	97384	980498	4202540
382	110138	1052426	4408406
598	124726	1125218	4626162
950	139642	1203980	4850198
1334	156286	1285902	5085098
1912	174018	1374300	5321854
2622	194106	1463714	5571470
3624	214570	1559064	5826806
4690	237534	1657422	6095870
6096	261666	1762004	6369534
7686	288686	1869106	6655902
9764	316770	1983922	6948566
12010	348048	2102162	7256076
14866	380798	2228512	7565826
18026	416524	2356822	7889032
21904	452794	2493834	8220566
25918	492830	2635310	8568428
30818	534962	2786090	8919298
36246	580964	2938326	9285288
42654	627822	3099230	9658638

Table 6: The first 100 terms of the number of simple interior intersection points in $BC(1,n)$.

for "long chords".) These graphs are discussed by Seppo Mustonen [12, 13, 14]. We say more about $LC(m,n)$ in Section 9.

Figure 13 shows the differences between the three definitions in the case of a $(3,2)$ reticulated rectangle, the first time the definitions differ. The black lines (both thick and thin) form the graph $BC(3,2)$. The four red lines are the additional line segments that appear when we construct $AC(3,2)$. They start at an interior grid point and so are not present in $BC(3,2)$. The four blue lines extend the red chords until they reach the boundary of the grid, and form $AC(3,2)$.

The numbers of nodes $\mathcal{N}(m,n)$ and cells $\mathcal{C}(m,n)$ in $BC(m,n)$ are shown for $m,n \le 37$ in A331453 and A331452, respectively, and the initial terms are shown in Table 7.

Regrettably, except when m or n is 1, we have been unable to find formulas for

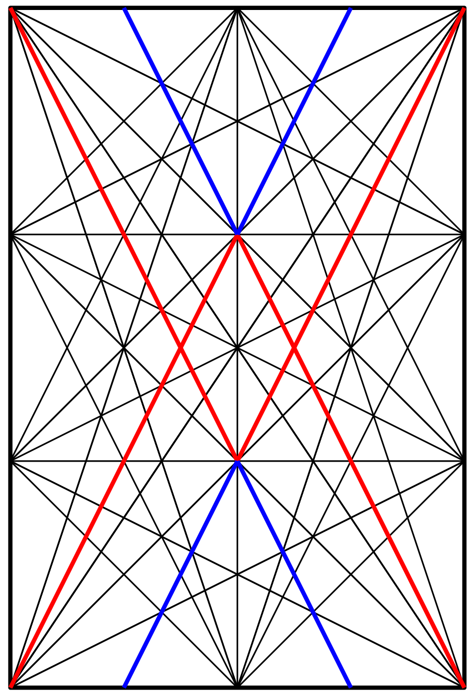


Figure 13: Comparison of the graphs $BC(3,2)$ (black lines), $AC(3,2)$ (add the red lines), and $LC(3,2)$ (also add the blue lines).

$m\backslash n$	1	2	3	4	5	6	7
1	$5, 4$	$13, 16$	$35, 46$	$75, 104$	$159, 214$	$275, 380$	$477, 648$
2	$13, 16$	$37, 56$	$99, 142$	$213, 296$	$401, 544$	$657, 892$	$1085, 1436$
3	$35, 46$	$99, 142$	$257, 340$	$421, 608$	$881, 1124$	$1305, 1714$	$2131, 2678$
4	$75, 104$	$213, 296$	$421, 608$	$817, 1120$	$1489, 1916$	$2143, 2820$	$3431, 4304$
5	$159, 214$	$401, 544$	$881, 1124$	$1489, 1916$	$2757, 3264$	$3555, 4510$	$5821, 6888$
6	$275, 380$	$657, 892$	$1305, 1714$	$2143, 2820$	$3555, 4510$	$4825, 6264$	$7663, 9360$
7	$477, 648$	$1085, 1436$	$2131, 2678$	$3431, 4304$	$5821, 6888$	$7663, 9360$	$12293, 13968$

Table 7: Numbers of nodes $\mathcal{N}(m,n)$ and cells $\mathcal{C}(m,n)$ in $BC(m,n)$ for $1 \le m, n \le 7$.

any of these quantities. The diagonal case, when $m = n$, is the most interesting (because the most symmetrical), but is also probably the hardest to solve. In accordance with our philosophy of "if you can't solve it, make art", Figure 14 shows our stained glass window $BC(3,3)$, and entry A331452 has a large number of larger and even more striking examples which space restrictions do not permit us to show here.

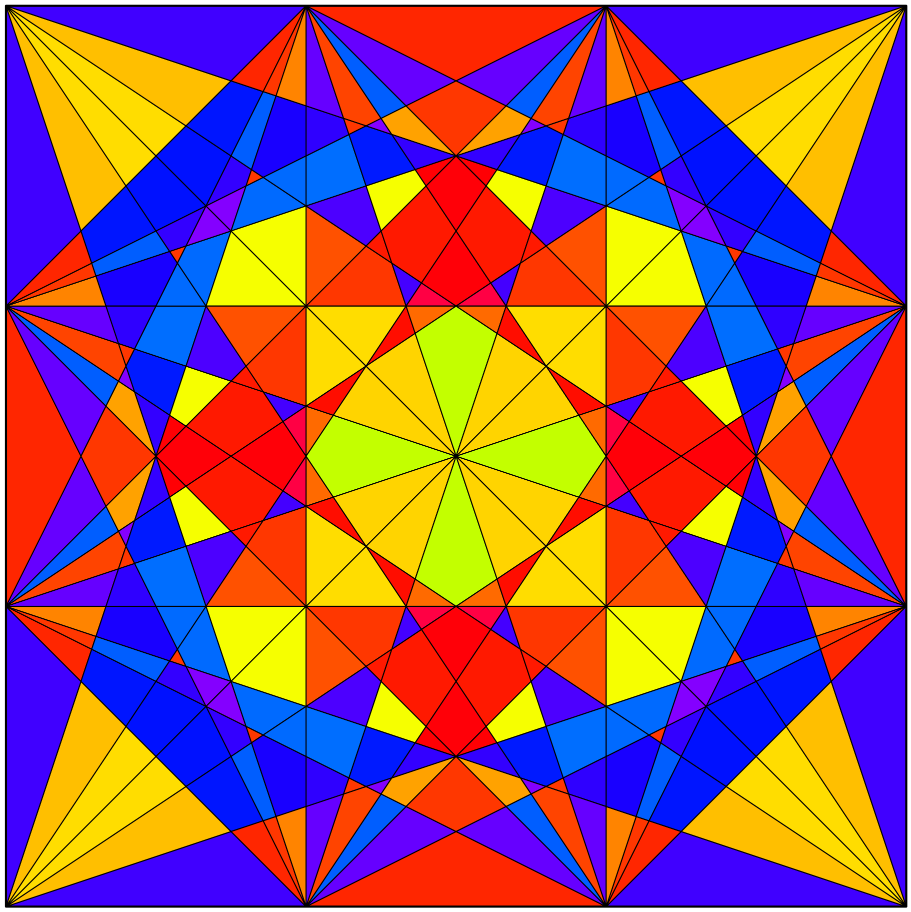


Figure 14: The graph $BC(3,3)$. There are 257 nodes and 340 cells.

Out of all these unsolved problems, the case when m (or n) is fixed at 2 would seem to be the most amenable to analysis,[7] perhaps by extending the work of Legendre [11] and Griffiths [8]. For instance, what are the conditions for three chords in $BC(2,n)$ to intersect at a common point? We emphasize this by stating:

Open Problem 6.1. Find formulas for the numbers of nodes ($\mathcal{N}(2,n)$, A331763) and cells ($\mathcal{C}(2,n)$, A331766) in $BC(2,n)$.

The first 10 terms are given in Table 8, and 100 terms are given in the entries for these two sequences in [15].

[7]Since $BC(1,n)$ is not a subgraph of $BC(2,n)$, it may be that $AC(2,n)$ (see Section 8), which *does* have $BC(1,n)$ as a subgraph, may be easier to analyze than $BC(2,n)$.

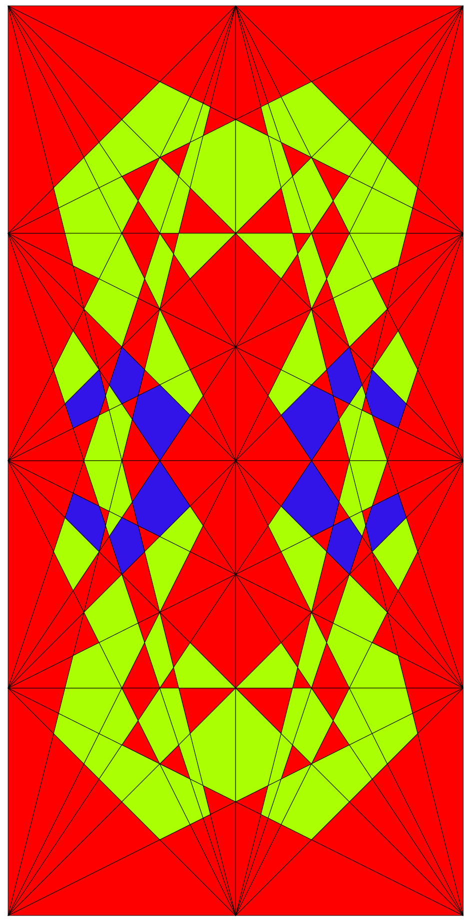


Figure 15: $BC(4, 2)$ with cells color-coded to distinguish triangles (red), quadrilaterals (yellow), and pentagons (blue).

n :	1	2	3	4	5	6	7	8	9	10	$\cdots$	[15]
$\mathcal{N}(2, n)$:	13	37	99	213	401	657	1085	1619	2327	3257	$\cdots$	A331763
$\mathcal{C}(2, n)$:	16	56	142	296	544	892	1436	2136	3066	4272	$\cdots$	A331766

Table 8: Numbers of nodes and cells in $BC(2, n)$.

In $BC(1, n)$ the cells are always triangles or quadrilaterals (Theorem 3.1). It appears that a similar phenomenon holds for $BC(2, n)$. The data strongly suggests the following conjecture.

Conjecture 6.2. The cells in $BC(2,n)$ have at most eight sides, and for $n \geq 19$, at most six sides.

We have verified the conjecture for $n \leq 106$.

Row n of Table 9 gives the number of cells in $BC(2,n)$ with k sides, for $k \geq 3$ and $n \leq 20$. For rows $n = 1, 2,$ and 3 of this table see Figs. 5, 2, and 13 (black lines only). For row 4 see Figure 15, where one can see that $BC(4,2)$ has 192 triangular cells (red), 92 quadrilaterals (yellow), and 12 pentagons (blue). Entry A335701 gives the first 106 rows of this table, and has many further illustrations. The row sums in Table 9 are the numbers $\mathcal{C}(2,n)$ given in column 2 of Table 7 and A331766.

$n\backslash k$	3	4	5	6	7	8
1	14	2				
2	48	8				
3	102	36	4			
4	192	92	12			
5	326	194	24			
6	524	336	28	4		
7	802	554	80			
8	1192	812	128	4		
9	1634	1314	112	0	4	2
10	2296	1756	200	20		
11	3074	2508	236	22		
12	4052	3252	356	28		
13	5246	4348	472	28		
14	6740	5464	652	28		
15	8398	7054	656	74		
16	10440	8760	940	52		
17	12770	11050	1040	58		
18	15512	13324	1300	60	4	
19	18782	16162	1600	70		
20	22384	19256	1948	104		

Table 9: Row n gives the number of cells in $BC(2,n)$ with k sides, for $k \geq 3$. It appears that for $n \geq 19$, no cell has more than six sides (see A335701).

Open Problem 6.3. For $BC(m,n)$, m fixed, is there an upper bound on the number of sides of a cell as n varies?

We are at least able to analyze the corner squares of $BC(2,n)$.

Theorem 6.4. *For $n = 2$ the four corner squares of $BC(2,n)$ (and $BC(n,2)$) each contain 12 triangles and 4 quadrilaterals, while for $n = 3$ they contain 15 triangles,*

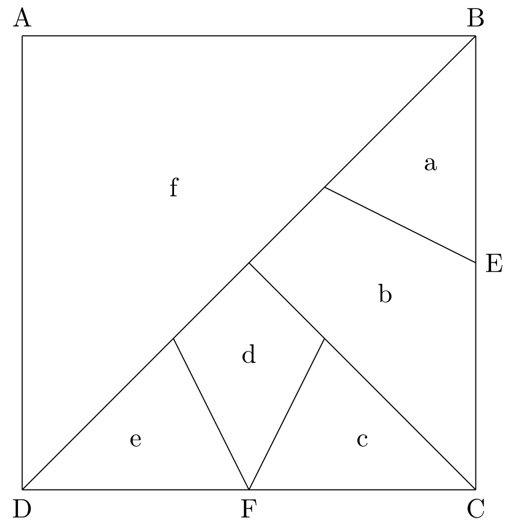


Figure 16: Dissection of corner square of $BC(n, 2)$, $n \geq 4$, used in proof of Theorem 6.4.

6 quadrilaterals, and (exceptionally) one pentagon. For $n \geq 4$, the corner squares each contain $7n + 1$ cells, consisting of $2n + 9$ triangles and $5n - 8$ quadrilaterals.

Proof. For the proof we choose a local coordinate system for $BC(2, n)$ with $(0, 0)$ at the top left, with the x-axis directed to the right, and the y-axis directed downwards. The four vertices of the rectangle defining $BC(n, 2)$ are $(0, 0)$, $(2, 0)$, $(2, n)$, and $(0, n)$. The top left corner square has vertices A, B, C, D with coordinates $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. We assume $n \geq 4$. (The case $n = 4$, where the corner squares contain 17 triangles and 12 quadrilaterals, is shown in Figure 15.)

We dissect this square into regions, in each of which the cell structure is apparent (and is such that the boundaries of the regions do not cross any cell boundaries). This is done as indicated in Figure 16. There are six regions, labeled a through f, which are defined as follows. The chord from A to the grid point $(2, 1)$ meets BC at its midpoint E, and the chord from A to $(2, 4)$ meets CD at its midpoint F. The lines AE, AC, AF, BD, and BF define the six regions.

The pencil of $n - 1$ chords from A to the grid points $(1, n)$, $(2, n)$, $(2, n - 1)$, $(2, n - 2), \ldots,$ $(2, 3)$ cuts CD at the points $(\frac{1}{n}, 1)$, $(\frac{2}{n}, 1)$, $(\frac{2}{n-1}, 1), \ldots,$ $(\frac{2}{4}, 1) = F$, $(\frac{2}{3}, 1)$. The pencil of $n - 1$ chords from B to the grid points $(0, k), 2 \leq k \leq n$ cuts CD at the points $F = (\frac{1}{2}, 1)$, $(\frac{2}{3}, 1)$, $(\frac{3}{4}, 1), \ldots,$ $(\frac{n-1}{n}, 1)$. The chord from D to E intersects both pencils of chords. The reader will now have no difficulty in verifying that the numbers of the cells in regions a, b, c, d, e, f are as shown in Table 10. $\square$

Region	triangles	quadrilaterals
a	n	0
b	2	$2n-3$
c	2	$n-2$
d	1	3
e	2	$2n-6$
f	$n+2$	0
Total	$2n+9$	$5n-8$

Table 10: Numbers of triangles and quadrilaterals in the regions shown in Figure 16.

7. $BC(m,n)$ in General Position

We can obtain reasonably good upper bounds on $\mathcal{N}(m,n)$ and $\mathcal{C}(m,n)$ by analyzing what would happen if all the intersection points in $BC(m,n)$ were simple intersections—that is, if there was no interior point where three or more chords met.

We use $BC_{GP}(m,n)$ to denote a graph obtained by perturbing the boundary points of $BC(m,n)$ (excluding the four vertices) by small random sideways displacements along the boundaries. That is, if a boundary point was a fraction $\frac{i}{j}$ say of the way along a side, we move it to a point $\frac{i}{j}+\epsilon$ of the way along the side, where ϵ is a small random real number. If the ϵ's are chosen independently, the new graph will be in "general position", and there will be no multiple intersection points in the interior.

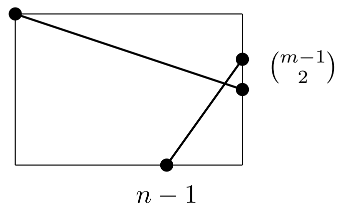
To illustrate the perturbing process, in Figure 17 below one can see (ignoring for now the supporting strut on the left) a perturbed version of $BC(1,2)$ obtained by slightly displacing just one boundary point (labeled 4) so as to avoid the triple intersection point at the center (see Figure 5).

Let $\mathcal{N}_{GP}(m,n)$ and $\mathcal{C}_{GP}(m,n)$ denote the numbers of nodes and cells, respectively, in the perturbed graph. The perturbations increase the numbers of nodes and cells, so $\mathcal{N}_{GP}(m,n) \geq \mathcal{N}(m,n)$ and $\mathcal{C}_{GP}(m,n) \geq \mathcal{C}(m,n)$

Theorem 7.1. *For $m,n \geq 1$, the number of interior nodes in $BC_{GP}(m,n)$ is*

$$\frac{1}{4}\left\{(m+n)(m+n-1)^2(m+n-4) \;+\; 2mn(2m+n-1)(m+2n-1)\right\}. \tag{7.1}$$

Proof. We start with the observation that any four boundary points of the rectangle, no three of which are on an side, determine a unique intersection point in the interior of the rectangle. There are several ways to choose these four points. They might be the four vertices of the rectangle, which can be done in just one way. They might consist of three vertices and a single node on one of the other two sides, which can be done in $4(m_1+n_1)$ ways, where $m_1 = m-1$ and $n_1 = n-1$ are the numbers of ways of choosing a single non-vertex point on a side. A more typical example consists of one vertex, and one, resp. two, points on the two opposite sides, as shown in the following drawing. This can be done in $4(m_1n_2 + m_2n_1)$ ways, where $m_2 = (m-1)(m-2)/2$, $n_2 = (n-1)(n-2)/2$ are the numbers of ways of choosing two non-vertex nodes from the sides.



There are in all seventeen different configurations for choosing four points, and when the seventeen counts are added up the result is the expression given in (7.1). We omit the details. □

Remarks. (i) Since there are $2(m+n)$ boundary points, the total number of nodes in $BC_{GP}(m,n)$ is

$$\mathcal{N}_{GP}(m,n) \;=\; \frac{1}{4}\{(m+n)(m+n-1)^2(m+n-4) + 2mn(2m+n-1)(m+2n-1)\}+2(m+n)\,. \tag{7.2}$$

This is our upper bound for $\mathcal{N}(m,n)$.

(ii) Another way to interpret $\mathcal{N}_{GP}(m,n)$ is that this is the number of nodes in $BC(m,n)$ counted with multiplicity (meaning that if there is an interior node where c chords meet, it contributes $\binom{c}{2}$ to the total).

(iii) When $m = n$, (7.2) simplifies to

$$\frac{n}{2}\,(17n^3 - 30n^2 + 19n + 4)\,, \tag{7.3}$$

which is our upper bound for $\mathcal{N}(n,n)$. For $n = 52$, $\mathcal{N}(n,n) = 52484633$ (from A331449), while (7.3) gives 60065408, too large by a factor of 1.14, which is not too bad. The moral seems to be that most interior nodes are simple.

(iv) When $m = 1$, (7.1) becomes $n^2(n+1)^2/4$, which agrees with the number mentioned in the proof of Theorem 5.1.

(v) For large m and n, the expression (7.2) is dominated by the degree 4 terms, which are

$$\frac{1}{4}\left(m^4 + n^4 + 8mn(m^2+n^2) + 16m^2n^2\right)\,. \tag{7.4}$$

Setting $m = n$, we get $\mathcal{N}_{GP}(n,n) \sim 17n^4/2$ as $n \to \infty$. We can confirm this by looking at the number of ways to choose four nodes out of the $4n$ boundary points so that no three are on a side. This is (essentially)

$$\binom{4n}{4} \;-\; 4\binom{n}{4} \;-\; 12\,n\binom{n}{3} \;\sim\; \frac{17}{2}n^4\,. \tag{7.5}$$

(vi) From (v), we have $\mathcal{N}(n,n) = O(n^4)$. In fact, we conjecture that $\mathcal{N}(n,n) \sim \mathcal{N}_{GP}(n,n) \sim 17\,n^4/2$. But to establish this we would need better information about the number of interior nodes in $BC(n,n)$ with a given multiplicity.

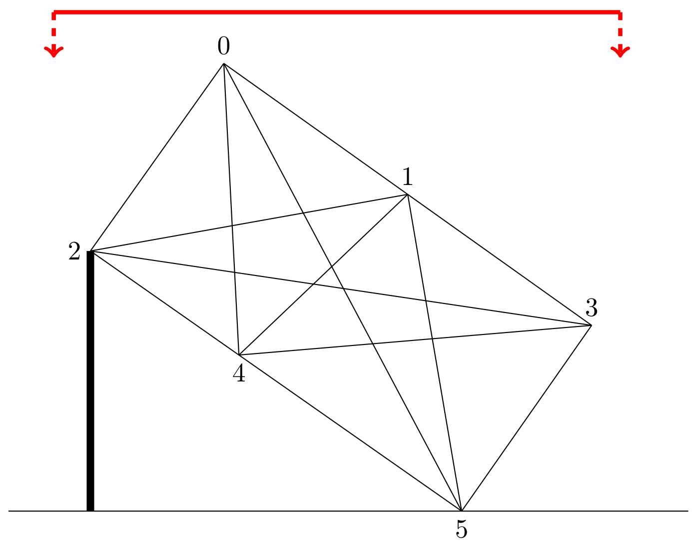


Figure 17: $BC(1,2)$ in general position: node 4 has been displaced slightly so as to avoid the triple intersection point at the center. The strut on the left tilts the figure so that the ordinates of the boundary points are in the same order as the labels. The red line is the "counting line", which descends across the picture in order to count the cells.

Now that we know the number of nodes, we can also find the number $\mathcal{C}_{GP}(m,n)$ of cells in $BC_{GP}(m,n)$. For this we use a method described by Freeman [7]. The following is a slight modification of his procedure. $BC_{GP}(m,n)$ has $2(m+n)$ boundary points. We label the top left corner vertex 0, and the bottom right corner vertex $2(m+n)-1$. The nodes along the top side we label 0, 1, 3, $5\ldots$,$2n-1$, continuing along along the right-hand side with $2n+1$, $2n+3\ldots$, $2(m+n)-1$. Along the left-hand side we place the labels $0, 2, 4, \ldots, 2m$, continuing along the bottom side with $2m+2$, $2m+4, \ldots$, $2m+2n-2$, $2(m+n)-1$.

Next, we raise the bottom left corner of the rectangle until the boundary points are at different heights, and so that the order of the heights matches the order of the labels (node 0 becomes the highest point, followed by nodes $1, 2, \ldots$ in order). Figure 17 illustrates the case $BC_{GP}(1,2)$. The black strut raises the bottom left corner so that the heights of the nodes are in the correct order.

We now take a horizontal line (Freeman calls it a "counting line"), and slide it downwards from the top of the figure to the bottom, recording each time it cuts a new cell. The counting line is shown in red in the figure.

When the counting line reaches a boundary point, with label k (say), the count is increased by the number of cells originating at k that have not yet been counted. This number is equal to the number of boundary points with label greater than k which are not on the same side as k. On the other hand, when the counting line reaches an interior node the count increases by exactly 1 (this is because there is no point where three chords meet). So the contribution to the count from the interior nodes is simply the number of interior nodes, which is known from Theorem 7.1.

In Figure 17, the count goes up by 3 at node 0, by 3 at node 1, 1 at node 2, and 1 at node 3, for a subtotal of 8. There are 9 interior nodes, so the total number of cells is 17.

From a careful study of a tilted version of the general case $BC_{GP}(m,n)$,, combined with (7.2), we obtain:

Theorem 7.2. *For $m, n \geq 1$, the number of cells in $BC_{GP}(m,n)$ is*

$$\mathcal{C}_{GP}(m,n) \;=\; \frac{1}{4}\left\{(m-1)^2(m-2)^2+(n-1)^2(n-2)^2\right\}+2mn\left(m+n-\frac{3}{2}\right)^2+\frac{9mn}{2}-1\,. \tag{7.6}$$

Remark. Asymptotically, $\mathcal{C}_{GP}(m,n)$ and $\mathcal{N}_{GP}(m,n)$ behave in the same way. In fact the difference $\mathcal{C}_{GP}(m,n) - \mathcal{N}_{GP}(m,n)$ is only $m^2+4\,mn+n^2-4\,m-4\,n+1$, a quadratic function of m and n.

$m\backslash n$	1	2	3	4	5	6	7
1	5, 4	13, 16	35, 46	75, 104	159, 214	275, 380	477, 648
2	13, 16	37, 56	121, 176	265, 388	587, 822	1019, 1452	1797, 2516
3	35, 46	121, 176	353, 520	771, 1152	1755, 2502	3075, 4392	5469, 7644
4	75, 104	265, 388	771, 1152	1761, 2584	4039, 5700	7035, 9944	12495, 17380
5	159, 214	587, 822	1755, 2502	4039, 5700	8917, 12368	15419, 21504	27229, 37572
6	275, 380	1019, 1452	3075, 4392	7035, 9944	15419, 21504	26773, 37400	47685, 65810
6	477, 648	1797, 2516	5469, 7644	12495, 17380	27229, 37572	47685, 65810	84497, 115532

Table 11: Numbers of nodes $\mathcal{N}_{AC}(m,n)$ and cells $\mathcal{C}_{AC}(m,n)$ in $AC(m,n)$ for $1 \leq m, n \leq 7$.

8. The Graphs $AC(m,n)$

The graph $AC(m,n)$ was defined in Section 6. We take an $(m+1)\times(n+1)$ square grid of nodes, and draw a line segment between *every* pair of distinct grid nodes. (If we only joined pairs of boundary points we would get $BC(m,n)$.)

Figure 13 shows $AC(3,2)$ (take the black and red lines only, not the blue lines). Hugo Pfoertner has made black and white drawings of $AC(m,n)$ for $1 \leq m, n \leq 5$

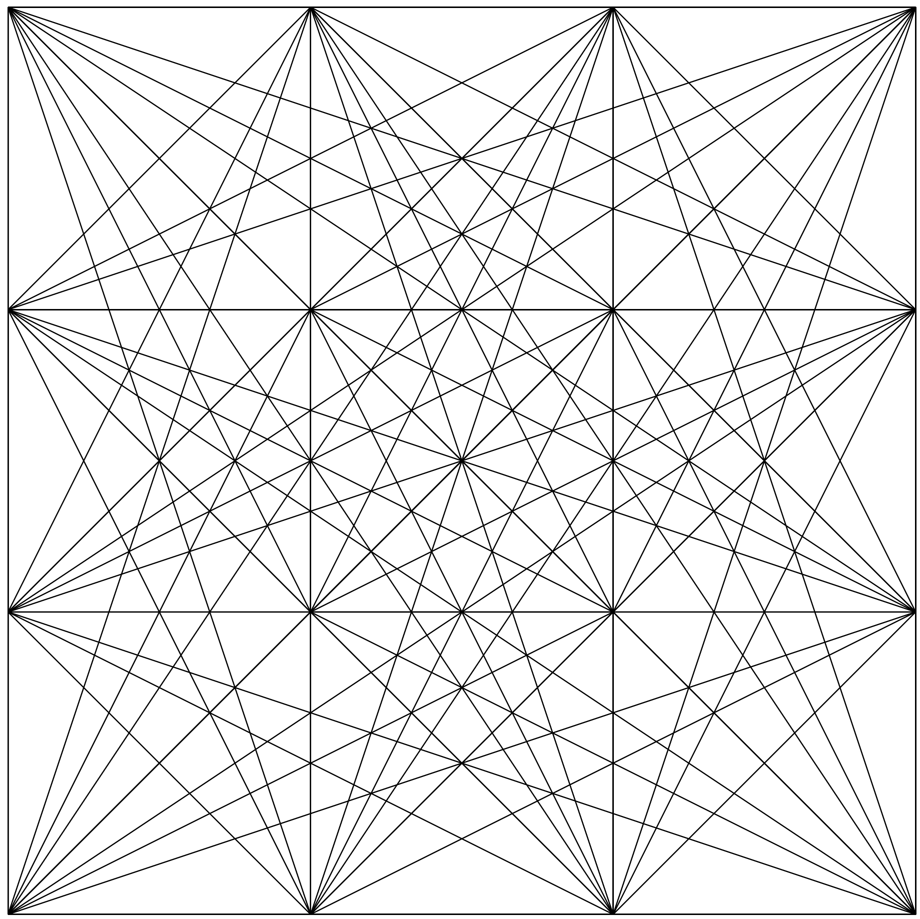


Figure 18: The graph $AC(3,3)$. There are 353 nodes and 520 cells.

in A288187. Figure 18 shows a black and white drawing of $AC(3,3)$ made using *TikZ* [6, 24].

The numbers of nodes $\mathcal{N}_{AC}(m,n)$ and cells $\mathcal{C}_{AC}(m,n)$ in $AC(m,n)$ are given for $m,n \le 9$ in A288180 and A288187, respectively, and the initial terms are shown in Table 11. The first row and column of Table 11 are the same as the first row and column of Table 7 but are included for completeness.

It is clear (compare Figs. 14 and 18) that $AC(m,n)$ contains far more nodes and cells than $BC(m,n)$. We may obtain an upper bound on $\mathcal{N}_{AC}(n,n)$ as follows. The graph $AC(n,n)$ has $(n+1)^2$ grid points. The number of ways of choosing four grid points is $\binom{(n+1)^2}{4}$, and except for a vanishingly small fraction of cases, no three points will be collinear. There are then two possibilities: the four points may form a convex quadrilateral, or a triangle with the fourth point in its interior. In the first case the intersection of the two diagonals of the quadrilaterals is a node of $AC(n, n)$ (which may or may not be a new node), but in the second case no new node is formed.

If four points in the plane are chosen at random from a square, by what is known as "Sylvester's Theorem", the probability that they form a convex quadrilateral is $25/36$ and the probability that they form a triangle with an interior point is $11/36$ (see [17, Table 4], [22, Table 3, p. 114] for the complicated history of this result). If we rescale our $(n+1) \times (n+1)$ grid into the unit square and let n go to infinity, the uniform distribution on the grid converges to Lebesgue measure. We can then conclude from Sylvester's theorem that the number of nodes in $AC(n, n)$ counted with multiplicity is asymptotically

$$\frac{25}{36}\binom{(n+1)^2}{4} \sim \frac{25}{864}\,n^8 = 0.0289\ldots\,n^8\,. \tag{8.1}$$

So $\mathcal{N}_{AC}(n, n) = O(n^8)$, compared with $\mathcal{N}(n, n) = O(n^4)$ for $BC(n, n)$.

However, in contrast to our upper bound (7.5) for $BC(n, n)$, where the constant $17/2$ seems correct, the constant $25/864$ in (8.1) seems far from the truth (at $n = 25$ it is too big by a factor of about 4). But to improve it we would need further information about the multiplicity of the chord-intersections than we have now.

Both Tom Duff (personal communication) and Keith F. Lynch (personal communication) have have carried out extensive experiments, studying what happens when four points are chosen from an $m \times n$ grid, and have confirmed that there is excellent agreement with the predictions of Sylvester's Theorem.

In a remarkable calculation, Tom Duff enumerated and classified all sets of four points chosen from an $m \times n$ grid for $m, n \le 349$. In a 349×349 grid, there are 6366733094048270910 strictly convex quadrilaterals out of 9170030499095875150 total. The fraction is 0.6942979, just a little short of Sylvester's $25/36 = 0.694444\ldots$. The deficit is explained by the not quite negligible counts of quadrilaterals with at least three collinear points. If those are included with the strictly convex quadrilaterals, the ratio is 0.6945982, slightly more than $25/36$.

$m\backslash n$	1	2	3	4	5	6	7
1	5, 4	13, 16	35, 46	75, 104	159, 214	275, 380	477, 648
2	13, 16	37, 56	129, 192	289, 428	663, 942	1163, 1672	2069, 2940
3	35, 46	129, 192	405, 624	933, 1416	2155, 3178	3793, 5612	6771, 9926
4	75, 104	289, 428	933, 1416	2225, 3288	5157, 7520	9051, 13188	16129, 23368
5	159, 214	663, 942	2155, 3178	5157, 7520	11641, 16912	20341, 29588	36173, 52368
6	275, 380	1163, 1672	3793, 5612	9051, 13188	20341, 29588	35677, 51864	63987, 92518
7	477, 648	2069, 2940	6771, 9926	16129, 23368	36173, 52368	63987, 92518	114409, 164692

Table 12: Numbers of nodes $\mathcal{N}_{LC}(m, n)$ and cells $\mathcal{C}_{LC}(m, n)$ in $LC(m, n)$ for $1 \le m, n \le 7$.

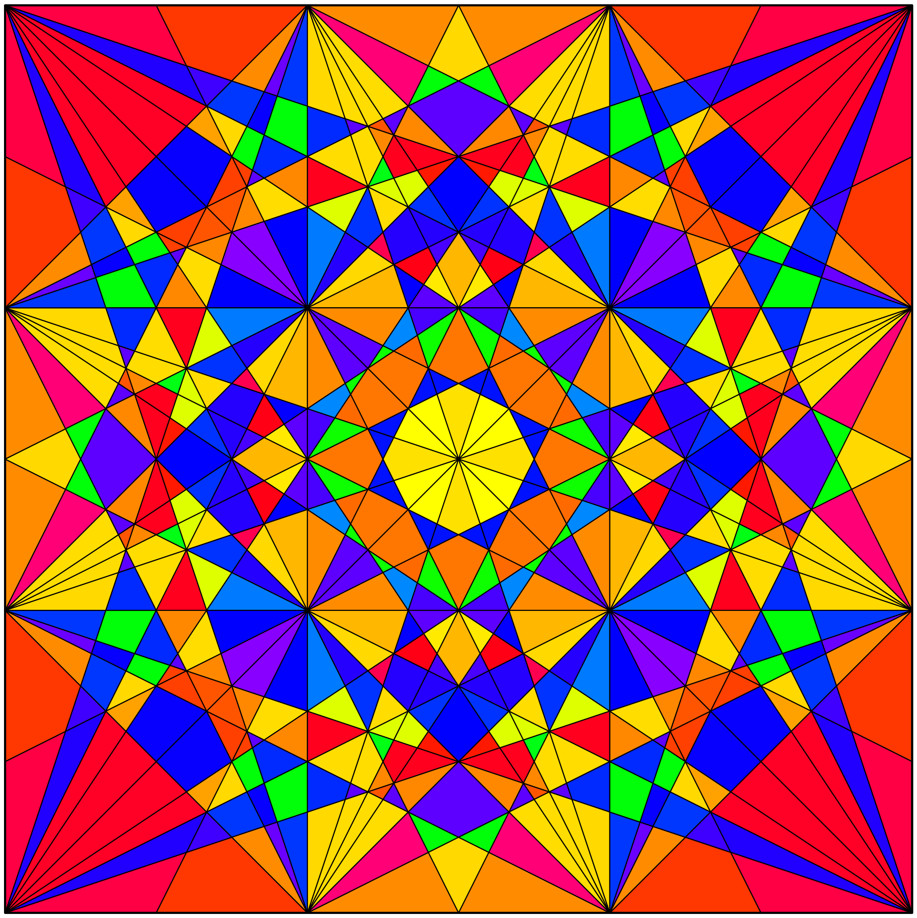


Figure 19: The graph $LC(3, 3)$. There are 405 nodes and 624 cells.

9. The Graphs $LC(m, n)$

The graph $LC(m, n)$ was defined in Section 6. We take an $(m+1) \times (n+1)$ square grid of nodes, draw a line segment between *every* pair of grid nodes, and extend these lines until they meet the boundaries of the grid. These graphs were discussed by Mustonen [12, 13, 14]. Figure 13 shows $LC(3, 2)$ (take the black, red, and blue lines), and Figure 19 shows our stained glass coloring of $LC(3, 3)$.

The numbers of nodes $\mathcal{N}_{LC}(m, n)$ and cells $\mathcal{C}_{LC}(m, n)$ in $LC(m, n)$ are given for $m, n \leq 8$ in A333284 and A333282, respectively, and the initial terms are shown in Table 12. Again the first row and column are the same as in Table 7. Mustonen [13, Table 3] gives the first 29 terms of the diagonal sequence $\mathcal{N}_{LC}(n, n)$ (A333285). For this problem we can also give an upper bound on the number of nodes counted with multiplicity, only now we have no need of Sylvester's theorem. Consider four points chosen from the $(n+1) \times (n+1)$ grid points, with no three points collinear. If the points form a triangle with a point in the interior, joining the three vertices of the triangle to the interior point and then extending these chords until they meet the sides of the triangle (something we were not allowed to do in the previous case) will produce three potentially new nodes. If the four points form a convex quadrilateral, there are also potentially three nodes that could be created: the intersection of the two diagonals, and the two points where pairs of opposite sides meet when extended. Figure 20 shows the two cases. The black nodes are the four grid points and the red nodes are the potential new nodes. Of course in the second case the two external red points may be outside the grid (or at infinity), and so would not be counted.

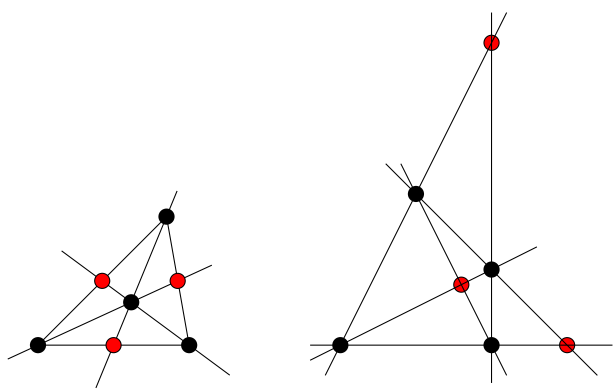


Figure 20: The two possibilities for choosing four noncollinear points from an $m \times n$ grid.

In any case, the maximum number of new nodes that are created is $3\binom{(n+1)^2}{4} + 2\binom{(n+1)^2}{2}$ (the latter term coming from the intersections of the chords with the boundaries). Asymptotically this is $n^8/8$. This is an upper bound on $\mathcal{N}_{LC}(n,n)$, because we do not always get three new nodes for each 4-tuple of grid points, and because multiple intersection points are counted multiple times. Based on his data for $n \leq 29$, Mustonen [13] makes an empirical estimate that $\mathcal{N}_{LC}(n,n) \sim Cn^8$, where C is about 0.0075. So our constant, 1/8 is, unsurprisingly, an over-estimate.

We conclude that as we progress from $BC(n,n)$ to $AC(n,n)$ to $LC(n,n)$, the graphs become progressively more dense, and so counting the nodes with multiplicity gives a steadily weaker upper bound on their number.

10. Choosing the Colors

We used three different coloring schemes.

10.1. Number-of-sides Coloring

The simplest scheme colors the cells according to the number of sides, with randomly chosen colors.. This is used in Figure 15 and in figures in [15] (entries A333282, A335701, for example) when studying the distribution of cells by number of sides.

10.2. The Yellow and Red Palettes

This is a refinement of the previous scheme, which modifies the color according to the shape of the cell. For Figs. 4, 5, 6, 7, 10, 11 the cells are either triangles or quadrilaterals, and we use colors which darken as the cell becomes more irregular. More precisely, the cells are colored according to the following rule. If the cell has n sides (where n is 3 or 4), let λ be the area of the cell divided by the area of an n-sided regular polygon with the same circumradius. Then the cell is assigned color number $\sqrt{\lambda}$ from the following palettes:

triangles

quadrilaterals

10.3. Random Colorings

For Figs. 1, 7, 14, etc. the color of a cell is assigned by first computing the average distance of the nodes of the cell from the center of the picture. These average distances are then grouped into a certain number of bins (we used 1000 bins), and the nonempty bins are assigned a random color from the standard spectrum from red to violet. This ensures a symmetrical coloring with contrasting colors for neighboring cells. In practice we do this several times and then choose the most appealing picture. We also have the option of restricting the color palette to achieve certain effects (reds, blues, and greens for a cathedral-like window, or various shades of browns for the frames that we will see in Part 2).

Acknowledgements. We thank Max Alekseyev, Gareth McCaughan, Ed Pegg, Jr., and Jinyuan Wang for their assistance during the course of this work. Tom Duff and Keith F. Lynch carried out extensive computations to check how well Sylvestor's theorem applied in practice (answer: very well, see Section 8). We made frequent use of the *gfun* Maple program [19] and the *TikZ* Latex package [6, 24]. We also thank a referee for many helpful comments, including a remark which strengthened a result in Section 8.

References

[1] M. A. Alekseyev, On the number of two-dimensional threshold functions, *SIAM J. Discr. Math.* **24:4** (2010), 1617–1631.

[2] M. A. Alekseyev, M. Basova, and N. Yu. Zolotykh, On the minimal teaching sets of two-dimensional threshold functions, *SIAM J. Discr. Math.* **29:1** (2015), 157–165.

[3] L. Blomberg, S. R. Shannon, and N. J. A. Sloane, Graphical enumeration and stained glass windows, 2: Polygons, frames, crosses, etc., in preparation..

[4] B. Bollobás, *Graph Theory: An Introductory Course*, Springer, 1979.

[5] F. Chung and R. Graham, Primitive juggling sequences, *Amer. Math. Monthly* **115:3** (2008), 185–194.

[6] J. Crémer, A very minimal introduction to *TikZ*, March 11, 2011; https://cremeronline.com/LaTeX/minimaltikz.pdf.

[7] J. W. Freeman, The number of regions determined by a convex polygon, *Math. Mag.* **49:1** (1975), 23–26.

[8] M. Griffiths, Counting the regions in a regular drawing of $K_{n,n}$, *J. Integer Sequences* **13** (2010), #10.8.5.

[9] H. T. Hall, *Counterexamples in Discrete Geometry*, PhD Dissertation, Mathematics Department, University of California Berkeley, 2004.

[10] F. Harary, *Graph Theory*, Addison-Wesley, Reading MA, 1969.

[11] S. Legendre, The number of crossings in a regular drawing of the complete bipartite graph, *J. Integer Sequences* **12** (2009), #09.5.5.

[12] S. Mustonen, Statistical accuracy of geometric constructions, September 2, 2008; http://www.survo.fi/papers/GeomAccuracy.pdf.

[13] S. Mustonen, On lines and their intersection points in a rectangular grid of points, April 16, 2009; http://www.survo.fi/papers/PointsInGrid.pdf.

[14] S. Mustonen, On lines going through a given number of points in a rectangular grid of points, May 12, 2010; http://www.survo.fi/papers/LinesInGrid2.pdf.

[15] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, 2021; https://oeis.org.

[16] M. E. Pfetsch and G. M. Ziegler, Large chambers in a lattice polygon, December 13, 2004; http://www.mathematik.tu-darmstadt.de/~pfetsch/chambers.

[17] R. E. Pfiefer, The historical development of J. J. Sylvester's Four Point Theorem, *Math. Mag.* **62:5** (1989), 309–317.

[18] B. Poonen and M. Rubinstein, The number of intersection points made by the diagonals of a regular polygon, *SIAM J. Discr. Math.* **11:1** (1998), 135–156.

[19] B. Salvy and P. Zimmermann, GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable, *ACM Trans. Math. Software* **20** (1994), 163–177.

[20] V. N. Shevchenko and N. Yu. Zolotykh, On the complexity of deciphering the threshold functions of k-valued logic, (Russian) *Dokl. Akad. Nauk* **362:5** (1998), 606–608; (English translation) *Dokl. Math.* **58** (1998), 268–270.

[21] N. J. A. Sloane, The email servers and Superseeker, 2010; https://oeis.org/ol.html.

[22] H. Solomon, *Geometric Probability*, SIAM, Philadelphia, 1978.

[23] S. E. Sommars and T. Sommars, Number of triangles formed by intersecting diagonals of a regular polygon, *J. Integer Sequences* **1** (1998), #98.1.5.

[24] T. Tantau, *The PGF/TikZ Programming Language*, Version 2.10, CTAN Org., October 25 2010.

[25] W. T. Tutte, A census of planar maps, *Canad. J. Math.* **15** (1963), 249–271.

[26] W. T. Tutte, On the enumeration of planar maps, *Bull. Amer. Math. Soc.* **74** (1968), 64–74.

[27] N. Yu. Zolotykh, On the complexity of deciphering threshold functions in two variables, (Russian), in *Proc. 11th Internat. School Seminar "Synthesis and complexity of control systems," Part I*, Center of Applied Research, Moscow State Univ. Faculty of Mechanics and Mathematics, Moscow, Russia, 2001, pp. 74–79.