TWO EXTENSIONS OF HILBERT’S CUBE LEMMA

Tom C. Brown
Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada
tbrown@sfu.ca

Shahram Mohsenipour
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
sh.mohsenipour@gmail.com

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Abstract
“Hilbert’s Cube Lemma” states that for every finite coloring of \( \mathbb{N} \) (the set of positive integers) and every \( n \in \mathbb{N} \), there exist \( d_1, d_2, \ldots, d_n \in \mathbb{N} \) such that infinitely many translates of

\[
\left\{ \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq 1 \right\}
\]

are monochromatic. (Given the coloring, \( d_1, d_2, \ldots, d_n \) depend on \( n \).) We show that for every finite coloring of \( \mathbb{N} \) and all \( k \geq 2 \) there exist \( d_1 < d_2 < \cdots \in \mathbb{N} \) such that for each \( n \geq 1 \), infinitely many translates of

\[
P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k - 1 \right\}
\]

are monochromatic, and \( |P_n| = k^n \). (Given the coloring, the sequence \( d_1, d_2, \ldots \) depends only on \( k \). That is, \( P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \).) We also show that for every finite coloring of \( \mathbb{N} \) and all \( n, k \in \mathbb{N} \), there exist \( a, d_1, d_2, \ldots, d_n \) such that \( d_1 < d_2 < \cdots < d_n \) and

\[
\left\{ a + \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k - 1 \right\} \cup \{d_1, \ldots, d_n\}
\]

is monochromatic. (Given the coloring, \( a, d_1, d_2, \ldots, d_n \) depend on \( n, k \).)

–Dedicated to the memory of Ron Graham
1. Introduction

Hilbert’s Cube Lemma appeared in 1892 [8] and is sometimes viewed as the first theorem in Ramsey Theory. See [1], and especially [11], for some background. Both [6] and [11] contain statements of the “density version” of this result, and proofs of the density version can be found in [5, 6].

The results described in this note are presented as extensions of Hilbert’s Cube Lemma. They may also be viewed as generalizations of van der Waerden’s theorem on arithmetic progressions ([12, 13]), which says that for every finite coloring of \( \mathbb{N} \) and all \( k \in \mathbb{N} \) there exist \( a, d \) such that \( \{a + \epsilon d : 0 \leq \epsilon < k\} \) is monochromatic. Brauer strengthened this ([6], p. 70) to the van der Waerden - Brauer theorem, which says that \( \{a + \epsilon d : 0 \leq \epsilon < k\} \cup \{d\} \) is monochromatic.

For a given fixed value of \( k \) and a given finite coloring of \( \mathbb{N} \), one may note that (for a single value of \( n \)) the existence of \( d_1, \ldots, d_n \) and a monochromatic translate of

\[
P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : \epsilon_i \in \{0, 1, \ldots, k-1\} \right\}
\]

does follow directly from the extended Hales-Jewett theorem([6, 7]). Indeed, given \( n, r \) (\( k \) is fixed at the beginning of this paragraph), the extended Hales-Jewett theorem says that if \( m \) is sufficiently large and the set \( A_k^m \) of all words of length \( m \) on the alphabet \( \{0, 1, \ldots, k-1\} \) is \( r \)-colored, then there is a monochromatic combinatorial \( n \)-space. If now the elements of \( A_k^m \) are viewed as the base \( k \) representations of the elements of \( [0, k^m - 1] \), a combinatorial \( n \)-space is precisely a translate of \( P_n \) (where each \( d_i \) is a sum of distinct powers of \( k \)).

However, in order to obtain \( P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \), as in the first extension described in the Abstract (Theorem 2 below), we need to use a different approach.

We use Theorem 1 below, which involves “uniform recurrence” of factors in certain infinite words on a finite alphabet ([9, 10]), together with van der Waerden’s theorem on arithmetic progressions. We also indicate (Theorem 3) that the fixed “\( k \)” in Theorem 2 can be replaced by any sequence \( k_i, i \geq 1 \), of positive integers.

The second extension described in the Abstract (Theorem 4 below) is proved using only the van der Waerden theorem.

2. Uniformly Recurrent Infinite Words All of Whose Factors Are Factors of a Given Infinite Word

The crucial result we need concerning infinite words is Theorem 1 below. For completeness, we include a proof, based on one due to J. Justin and G. Pirillo [9]. (A more labor-intensive proof can be obtained using the elaborate methods of symbolic dynamics. See, for example, pp. 213-215 of [4].)
We begin with some terminology.

Let \( A \) be a finite set. We denote by \( A^\omega \) the set of all infinite sequences of elements of \( A \), or infinite words on the “alphabet” \( A \). If \( c \in A^\omega \), we write \( c = c(1)c(2)c(3)\cdots \), and we regard \( c : \mathbb{N} \to A \) as a coloring of \( \mathbb{N} \), where \( A \) is the set of colors and for \( n \in \mathbb{N} \), \( c(n) \) is the color assigned to \( n \).

We denote by \( A^* \) the set of all finite sequences of elements of \( A \), or words on the alphabet \( A \), including the empty word. If \( u, v \in A^* \), say \( u = a_1a_2\cdots a_n \) and \( v = b_1b_2\cdots b_m \), with \( a_i, b_j \in A, 1 \leq i \leq n, 1 \leq j \leq m \), then their product \( uv \in A^* \) is the word \( uv = a_1a_2\cdots a_nb_1b_2\cdots b_m \). A word \( v \in A^* \) is a factor of the word \( w \in A^* \) if there exist (possibly empty) words \( p, q \in A^* \) such that \( w = pqq \). A word \( v \in A^* \) is a factor of an infinite word \( s \in A^\omega \) if there are \( p \in A^* \) and \( s' \in A^\omega \) such that \( s = pvs' \).

If \( w \in A^* \) then \( F(w) \) denotes the set of all factors of \( w \). If \( c \in A^\omega \) then \( F(c) \) denotes the set of all factors of \( c \).

If \( u = a_1a_2\cdots a_n \in A^* \), where \( a_i \in A, 1 \leq i \leq n \), then we say \( w \) has length \( n \) and write \( |w| = n \). (The empty word has length 0.)

**Definition 1.** Let \( c \in A^\omega \) and let \( u \) be a factor of \( c \). We define

\[
k(c, u) = \sup \{|v| : v \in F(c) \text{ and } u \notin F(v)\}.
\]

Thus if (and only if) \( u \) is “missing” from arbitrarily long factors of \( c \), we have \( k(c, u) = \infty \).

**Definition 2.** If \( c \in A^\omega \), \( u \in F(c) \), and \( k(c, u) < \infty \), that is, if every sufficiently long factor \( w \) of \( c \) contains \( u \) as a factor, we say that the factor \( u \) of \( c \) is uniformly recurrent (in \( c \)). If every factor \( u \) of \( c \) is uniformly recurrent in \( c \) then we say that \( c \) itself is uniformly recurrent.

First we need what is essentially König’s Lemma:

**Lemma 1.** Let \( L \) be any infinite subset of \( A^* \), where \( A \) is a finite set. Then there is an infinite word \( t \in A^\omega \) such that each factor of \( t \) is a factor of infinitely many words of \( L \).

**Proof.** Since \( A \) is finite, there is a letter in \( A \), call it \( t(1) \), which is the first letter in each word of an infinite subset \( L_1 \) of \( L \). Similarly, there is a letter in \( A \), call it \( t(2) \), such that \( t(1)t(2) \) are the first two letters of each word of an infinite subset \( L_2 \) of \( L_1 \). Continuing in this way, we produce an infinite word \( t = t(1)t(2)t(3)\cdots \in A^\omega \) such that each “prefix” \( t(1)t(2)t(3)\cdots t(n) \) of \( t \) is a prefix of an infinite subset \( L_n \) of \( L_{n-1} \subseteq \cdots \subseteq L_2 \subseteq L_1 \subseteq L \). Since each factor of \( t \) is a factor of a prefix of \( t \), every factor of \( t \) is a factor of infinitely many words of \( L \).

**Definition 3.** Let \( A \) be a finite set, and let \( c \in A^\omega \). We define a sequence of infinite words \( t_0, t_1, t_2, \ldots \) inductively as follows. We set \( t_0 = c \). Let the factors of \( c \)
be $F(c) = \{w_1, w_2, w_3, \ldots \}$. (This is an arbitrary enumeration.) For $r > 0$ assume $t_0, t_1, t_2, \ldots, t_{r-1}$ have been defined. Let $E_r$ be the set of all those factors of $t_{r-1}$ which do not contain $w_r$ as a factor. Thus  

$$E_r = \{v \in F(t_{r-1}) : w_r \notin F(v)\}.$$  

If $E_r$ is finite, we set $t_r = t_{r-1}$.

If $E_r$ is infinite, we obtain $t_r$ by using Lemma 1 in the following way. We set $L = E_r$ in the hypothesis of Lemma 1 and conclude (by Lemma 1) there is an infinite word $t_r \in A^\omega$ such that each factor of $t_r$ is a factor of (infinitely many) words of $E_r$. (Thus $F(t_r) \subseteq E_r$.)

This concludes our definition of $\{t_r\}_{r=0}^\infty$.

**Lemma 2.** For all $r > 0$, if $E_r$ is finite, then $k(t_{r-1}, w_r) < \infty$.

**Proof.** By Definition 1, $k(t_{r-1}, w_r) = \sup\{|v| : v \in F(t_{r-1}), w_r \notin F(v)\} = \sup\{|v| : v \in E_r\}$. \hfill \Box

**Lemma 3.** We have  

$$\cdots \subseteq F(t_r) \subseteq F(t_{r-1}) \subseteq \cdots \subseteq F(t_2) \subseteq F(t_1) \subseteq F(t_0) = F(c).$$

**Proof.** If $E_r$ is finite, then $t_r = t_{r-1}$. If $E_r$ is infinite, then $F(T_r) \subseteq E_r \subseteq F(t_{r-1})$. \hfill \Box

**Lemma 4.** If $w_r \in F(t_r)$ then $k(t_r, w_r) < \infty$.

**Proof.** By the definition of $E_r$, $w_r \notin E_r$. If $E_r$ is finite, then by Definition 3, $F(t_r) \subseteq E_r$. Thus, if $E_r$ is infinite, then $w_r \notin F(t_r)$. Therefore, $w_r \in F(t_r)$ implies $E_r$ is finite. Since $E_r$ is finite, Lemma 2 gives $k(t_{r-1}, w_r) < \infty$ and Definition 3 gives $t_r = t_{r-1}$, hence $k(t_r, w_r) < \infty$. \hfill \Box

We are now ready to prove the crucial Theorem 1. Recall that the term “uniformly recurrent” is defined in Definition 2.

**Theorem 1.** Given an arbitrary infinite word $c \in A^\omega$, there exists an infinite word $s \in A^\omega$ such that $s$ is uniformly recurrent and $F(s) \subseteq F(c)$.

**Proof.** We make use of the sequence of infinite words $t_0, t_1, t_2, \ldots$ defined in Definition 3. For each $i \geq 1$, let $u_i$ be any factor of $t_i$ with length $i$:  

$$u_i \in F(t_i), \quad |u_i| = i.$$  

Using Lemma 1, we let $s \in A^\omega$ be any infinite word such that every factor of $s$ is a factor of infinitely many $u_i$. From Lemma 3, it’s clear that $F(s) \subseteq F(c)$. In fact, something stronger is true, namely  

$$F(s) \subseteq \bigcap\{F(t_j) : j \geq 0\}.$$
To see this, let $w \in F(s)$, and let $j$ be arbitrary. Since $w$ is a factor of infinitely many $u_i$, choose $j_0 \geq j$ such that $w$ is a factor of $u_{j_0}$. Since $u_{j_0} \in F(t_{j_0})$, we have $w \in F(t_{j_0}) \subseteq F(t_j)$.

Now we can show that every $w \in F(s)$ is uniformly recurrent (in $s$). Since $F(s) \subseteq F(c)$, let $w = w_r$. We have just seen that $w_r \in F(t_r)$, and by Lemma 4, this gives $k(t_r, w_r) < \infty$. Finally, since $F(s) \subseteq F(t_r)$, we have $k(s, w_r) \leq k(t_r, w_r) < \infty$, which means (Definition 2) that $w_r$ is uniformly recurrent.

3. The First Extension

**Theorem 2.** Let $c$ be an arbitrary finite coloring of $\mathbb{N}$. Then for every $k \geq 2$ there exists a sequence of positive integers $d_1 < d_2 < d_3 < \cdots$ such that for all $n \geq 1$, infinitely many translates of the set

$$P_n = \left\{ \sum_{i=1}^{n} \epsilon_id_i : 0 \leq \epsilon_1, ..., \epsilon_n \leq k - 1 \right\}$$

are monochromatic. The $\{d_i\}$ depend only on $k$ (and the coloring), and can be chosen so that $|P_n| = k^n$.

**Proof.** As mentioned in the Introduction, we regard the coloring $c$ as an infinite word $c = c(1)c(2)c(3)\cdots \in A^\omega$. Here $A$ is the set of “colors,” and $c(n)$ is the color assigned to $n$, for all $n \in \mathbb{N}$. Throughout the proof, $k \geq 2$ is fixed.

Using Theorem 1, we let $s$ be any infinite word $s \in A^\omega$ such that $s$ is uniformly recurrent and $F(s) \subseteq F(c)$. To prove Theorem 2, we only need to show (for each $n$) that a single translate of $P_n$ is monochromatic under the coloring $s$. For suppose that $[p, q]$ is an interval which contains a translate $P_n^\prime$ of $P_n$, and that the coloring $s$, restricted to $[p, q]$, is constant on $P_n^\prime$; then the word

$$w = s(p)s(p+1)s(p+2)\cdots s(q)$$

occurs infinitely often in the word $s$, and hence occurs infinitely often in the word $c$ (since $F(s) \subseteq F(c)$). Thus $c$ is constant on infinitely many translates of $P_n$.

Let $k \in \mathbb{N}$ be fixed throughout the remainder of this argument.

We use induction on $n$. For $n = 1$, let $a$ be any element of $A$ which occurs in $s$.

Since $s$ is uniformly recurrent, there are $D, x_1, x_2, x_3, \cdots \in \mathbb{N}$ with $x_1 < x_2 < x_3 < \cdots$ and $x_{j+1} - x_j \leq D$ for all $j \geq 1$, such that the word $a$ (of length 1) occurs at each of the positions $x_1, x_2, x_3, \cdots$ in $s$. (That is, $s$ is constant on $\{x_1, x_2, x_3, \cdots\}$.)

Let $V = \{x_1, x_2, x_3, \cdots\}$. Then $V \cup (V+1) \cup (V+2) \cup \cdots \cup (V+D-1) = [x_1, \infty)$. Now set $V_0 = V$, $V_1 = (V+1) - V_0$, $V_2 = (V+2) - (V_0 \cup V_1)$, ..., $V_{D-1} = V + (D-1) - (V_0 \cup V_1 \cup \cdots \cup V_{D-2})$, to obtain a partition $V_0, V_1, V_2, \ldots, V_{D-1}$ (or “coloring with
D colors”) of \([x_1, \infty)\). By van der Waerden’s theorem on arithmetic progressions, some \(V_i\) contains a \(k\)-term arithmetic progression. Since \(V_i\) is a translate of \(V\), \(V\) itself contains a \(k\)-term arithmetic progression.

Thus there exists \(d_1\) such that \(s\) is constant on a translate of 

\[ P_1 = \{ \epsilon_1 d_1 : 0 \leq \epsilon_1 \leq k - 1 \}. \]

For the induction step, let \(n \geq 1\) and assume that \(d_1 < d_2 < \cdots < d_n\) exist so that \(s\) is constant on a translate of 

\[ P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k - 1 \right\}, \]

and \(|P_n| = k^n\). To be specific, assume that \(s\) is constant on the set 

\[ P_n = m + P_n. \]

The set \(P_n\) is contained in the interval \([m, m + (k - 1)(d_1 + d_2 + \cdots + d_n)]\). Let \(q = (k - 1)(d_1 + d_2 + \cdots + d_n)\), so that \(\min P_n = m, \max P_n = m + q\), and \(P_n\) is contained in the interval \([m, m + q]\).

Let \(w = s(m)s(m+1)s(m+2)\cdots s(m+q)\). Thus \(w\) is a certain factor of \(s\) with length \(|w| = q + 1\). Moreover, \(P_n\) is contained in \([m, m + q]\), and \(s\) is constant on \(P_n\). We will use the abbreviated notation 

\[ w = s[m, m + q]. \]

Since \(s\) is uniformly recurrent, there are \(D, x_1, x_2, x_3, \ldots \in \mathbb{N}\) with \(x_1 < x_2 < x_3 < \cdots\) and \(x_{j+1} - x_j \leq D\) for all \(j \geq 1\), such that the factor \(w\) begins at each of the positions \(x_1, x_2, x_3, \ldots\) in \(s\).

Again using van der Waerden’s theorem on arithmetic progressions, there exists \(d_{n+1}\) such that \(x_1, x_2, x_3, \cdots\) contains a translate of the arithmetic progression \(\{\epsilon_{n+1} d_{n+1} : \epsilon_{n+1} \in \{0, 1, \ldots, k - 1\}\}\). We can assume that \(d_{n+1} > q = (k - 1)(d_1 + \cdots + d_n)\).

Thus for some \(m' \in \mathbb{N}\), the factor \(w\) begins at each of the positions \(m', m' + d_{n+1}, m' + 2d_{n+1}, \ldots, m' + (k - 1)d_{n+1}\). We can assume that \(m' > m\).

For \(0 \leq i \leq k - 1\), let \(I_i\) denote the interval \(I_i = [m' + id_{n+1}, m' + id_{n+1} + q]\).

Since the factor \(w = s[m, m + q]\) begins at each number \(m' + id_{n+1}\), we have that 

\[ w = s[m, m + q] = s(I_i). \]

Since \(I_i = (m' - m) + id_{n+1} + [m, m + q]\) and \(s\) is constant on \(P_n \subseteq [m, m + q]\), it follows that \(s\) is constant on \((m' - m) + id_{n+1} + P_n = m' + id_{n+1} + P_n \subseteq I_i\).

Thus, \(s\) is constant on 

\[ \bigcup \{m' + id_{n+1} + P_n : 0 \leq i \leq k - 1\} = m' + P_{n+1}. \]
Since $d_{n+1} > q = (k - 1)(d_1 + \cdots + d_n)$, the intervals $I_i, 0 \leq i \leq k - 1$, are pairwise disjoint, hence $|P_{n+1}| = k|P_n| = k^{n+1}$.

\[\square\]

**Theorem 3.** Let $c$ be an arbitrary finite coloring of $\mathbb{N}$. Then for every sequence of positive integers $\{k_i\}_{i=1}^\infty$ there exists a sequence of positive integers $d_1 < d_2 < d_3 < \cdots$ such that for all $n \geq 1$, infinitely many translates of the set

\[P_n = \left\{ \sum_{i=1}^n \epsilon_i d_i : \epsilon_i \in \{0, 1, \ldots, k_i - 1\} \right\}\]

are monochromatic. Furthermore, the $d_i$ can be chosen so that $|P_n| = k_1 k_2 \cdots k_n$.

**Proof.** Trivial modifications of the proof of Theorem 2 give a proof of Theorem 3. \[\square\]

**4. The Second Extension**

Let $W(l, r)$, i.e., the van der Waerden number, be the least positive integer $m$ such that if $[1, m]$ is $r$-colored, then there exist $a, d$ such that $a, a + d, \ldots, a + (l - 1)d$ have the same color.

**Theorem 4.** Let $n, k, r$ be positive integers. There exists a least positive integer $m = WB_n(k, r)$ such that if $[1, m]$ is $r$-colored, there exist $a, d_1, d_2, \ldots, d_n$ such that $d_1 < d_2 < \cdots < d_n$, $P_n = \{a + x_1 d_1 + \cdots + x_n d_n : 0 \leq x_1, \ldots, x_n \leq k - 1\} \subseteq [1, m]$, $|P_n| = k^n$, and $P_n \cup \{d_1, \ldots, d_n\}$ is monochromatic.

**Proof.** The proof is by double induction on $n, r$. The case $r = 1$ is trivial, so let $r \geq 2$ and the non-negative integer $n$ be also fixed through the reminder of the proof.

Now fix a positive integer $k$ and assume that the following numbers exist: $WB_n(l, r)$ for all $l$ and $WB_{n+1}(k, r - 1)$ (note that for $n = 0$ we put $WB_0(l, r) = W(l, r)$).

We will show that $WB_{n+1}(k, r)$ exists.

Let

\[m = WB_n(WB_{n+1}(k, r - 1) \cdot k(k - 1) + k, r)\]

We show that $WB_{n+1}(k, r) \leq m$, which will complete the argument.

Let $[1, m]$ be $r$-colored, using the colors $\{1, 2, \ldots, r\}$. By the definition of $m$, there are $a, d_1, \ldots, d_n$ such that $d_1 < d_2 < \cdots < d_n$,

$Q_n = \{a + y_1 d_1 + \cdots + y_n d_n : 0 \leq y_1, \ldots, y_n \leq WB_{n+1}(k, r - 1) \cdot k(k - 1) + k - 1\} \subseteq [1, m]$,

and $Q_n \cup \{d_1, \ldots, d_n\}$ is monochromatic, say with color $r$. Now consider the following (monochromatic) subsets of $Q_n$:

$Q_{n+1}^1 = \{a + x_1 d_1 + \cdots + x_n d_n + x_{n+1} [k(d_1 + \cdots + d_n)] : 0 \leq x_1, \ldots, x_{n+1} \leq k - 1\}$
\(Q_{n+1}^2 = \{a + x_1d_1 + \cdots + x_nd_n + x_{n+1}[2k(d_1 + \cdots + d_n)]: 0 \leq x_1, \ldots, x_{n+1} \leq k-1\}\)

\(Q_{n+1}^3 = \{a + x_1d_1 + \cdots + x_nd_n + x_{n+1}[3k(d_1 + \cdots + d_n)]: 0 \leq x_1, \ldots, x_{n+1} \leq k-1\}\)

\[\vdots\]

\(Q_{n+1}^w = \{a + x_1d_1 + \cdots + x_nd_n + x_{n+1}[wk(d_1 + \cdots + d_n)]: 0 \leq x_1, \ldots, x_{n+1} \leq k-1\}\)

where \(w = WB_{n+1}(k, r - 1)\).

(One may note that \(\max Q_{n+1}^w = \max Q_n\).)

There are now two cases.

If at least one of \(k(d_1 + \cdots + d_n), 2k(d_1 + \cdots + d_n), \ldots, wk(d_1 + \cdots + d_n)\), say \(j_0k(d_1 + \cdots + d_n)\), has the color \(r\), we are done by setting \(d_{n+1} = j_0k(d_1 + \cdots + d_n)\), for then

\[Q_{n+1}^{j_0} \cup \{d_1, \ldots, d_{n+1}\}\]

is our desired monochromatic set.

If none of \(k(d_1 + \cdots + d_n), 2k(d_1 + \cdots + d_n), \ldots, wk(d_1 + \cdots + d_n)\) have color \(r\), then, recalling that \(w = WB_{n+1}(k, r - 1)\), all elements of

\(k(d_1 + \cdots + d_n)[1, WB_{n+1}(k, r - 1)]\)

have been colored with \(r - 1\) colors, and we are done by the induction hypothesis on \(r\). \(\Box\)

5. Remarks and Addendum

1. In proving Theorem 2, we don’t actually need the full strength of Theorem 1, which says that each factor \(w\) of the infinite word \(s\) occurs syndetically in \(s\). In order to apply van der Waerden’s theorem, it’s enough to know that each factor \(w\) of the infinite word \(s\) occurs piecewise syndetically in \(s\). This means that for each factor \(w\) of the infinite word \(s\), there exists a fixed \(d = d(w)\) such that for arbitrarily large \(m\), there is a factor \(q_1q_2 \cdots q_m\) of \(s\) with \(|q_i| = d, 1 \leq i \leq m\), such that \(w\) is a factor of each \(q_i, 1 \leq i \leq m\).

2. Theorem 2 has the same relation with van der Waerden’s theorem as Carlson-Simpson’s theorem [2] has with Hales-Jewett’s theorem. It is possible to deduce Theorem 2 from Carlson-Simpson’s theorem by generalizing the usual proof of deducing van der Waerden’s theorem from the Hales-Jewett theorem. Similarly Theorem 3 can be deduced from a generalized version of Carlson-Simpson’s theorem ([3], Section 10).

3. As the proof given for Theorem 4 uses a double induction argument, it gives no primitive recursive upper bound for \(WB_n(k, r)\). We noticed that Theorem 4 is a special case of the theorem on the existence of monochromatic \((m, p, c)\)-sets ([6], Chapter 3, Theorem 10) for \(c = 1\). Checking the standard proof given there shows
that in fact \( WB_n(k, r) \) has a primitive recursive bound belonging to the class of WOW functions ([6], 2.7). On the other hand by a straightforward modification of our proof for Theorem 4, we get a new and simpler proof of the above mentioned theorem on \((m, p, c)\)-sets.

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