TWO EXTENSIONS OF HILBERT'S CUBE LEMMA

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Abstract

"Hilbert's Cube Lemma" states that for every finite coloring of $\mathbb{N}$ (the set of positive integers) and every $n \in \mathbb{N}$, there exist $d_1, d_2, \ldots, d_n \in \mathbb{N}$ such that infinitely many translates of

$$\left\{\sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq 1\right\}$$

are monochromatic. (Given the coloring, $d_1, d_2, \ldots, d_n$ depend on n.) We show that for every finite coloring of $\mathbb{N}$ and all $k \geq 2$ there exist $d_1 < d_2 < \cdots \in \mathbb{N}$ such that for each $n \geq 1$, infinitely many translates of

$$P_n = \left\{\sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k-1\right\}$$

are monochromatic, and $|P_n| = k^n$. (Given the coloring, the sequence $d_1, d_2, \ldots$ depends only on k. That is, $P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots$.) We also show that for every finite coloring of $\mathbb{N}$ and all $n, k \in \mathbb{N}$, there exist $a, d_1, d_2, \ldots, d_n$ such that $d_1 < d_2 < \cdots < d_n$ and

$$\left\{a + \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k-1\right\} \cup \{d_1, \ldots, d_n\}$$

is monochromatic. (Given the coloring, $a, d_1, d_2, \ldots, d_n$ depend on n, k.)

–Dedicated to the memory of Ron Graham

1. Introduction

Hilbert's Cube Lemma appeared in 1892 [8] and is sometimes viewed as the first theorem in Ramsey Theory. See [1], and expecially [11], for some background. Both [6] and [11] contain statements of the "density version" of this result, and proofs of the density version can be found in [5, 6].

The results described in this note are presented as extensions of Hilbert's Cube Lemma. They may also be viewed as generalizations of van der Waerden's theorem on arithmetic progressions ([12, 13]), which says that for every finite coloring of $\mathbb{N}$ and all $k \in \mathbb{N}$ there exist a, d such that $\{a + \epsilon d : 0 \leq \epsilon < k\}$ is monochromatic. Brauer strengthened this ([6], p. 70) to the van der Waerden - Brauer theorem, which says that $\{a + \epsilon d : 0 \leq \epsilon < k\} \cup \{d\}$ is monochromatic.

For a given fixed value of k and a given finite coloring of $\mathbb{N}$, one may note that (for a single value of n) the existence of $d_1, \ldots, d_n$ and a monochromatic translate of

$$P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : \epsilon_i \in \{0, 1, \ldots, k-1\} \right\}$$

does follow directly from the extended Hales-Jewett theorem([6, 7]). Indeed, given n, r (k is fixed at the beginning of this paragraph), the extended Hales-Jewett theorem says that if m is sufficiently large and the set A_k^m of all words of length m on the alphabet $\{0, 1, \ldots, k-1\}$ is r-colored, then there is a monochromatic combinatorial n-space. If now the elements of A_k^m are viewed as the base k representions of the elements of $[0, k^m - 1]$, a combinatorial n-space is precisely a translate of P_n (where each d_i is a sum of distinct powers of k).

However, in order to obtain $P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots$, as in the first extension described in the Abstract (Theorem 2 below), we need to use a different approach.

We use Theorem 1 below, which involves "uniform reccurrence" of factors in certain infinite words on a finite alphabet ([9, 10]), together with van der Waerden's theorem on arithmetic progressions. We also indicate (Theorem 3) that the fixed "k" in Theorem 2 can be replaced by any sequence $k_i, i \geq 1$, of positive integers.

The second extension described in the Abstract (Theorem 4 below) is proved using only the van der Waerden theorem.

2. Uniformly Recurrent Infinite Words All of Whose Factors Are Factors of a Given Infinite Word

The crucial result we need concerning infinite words is Theorem 1 below. For completeness, we include a proof, based on one due to J. Justin and G. Pirillo [9]. (A more labor-intensive proof can be obtained using the elaborate methods of symbolic dynamics. See, for example, pp. 213-215 of [4].)

We begin with some terminology.

Let A be a finite set. We denote by A^ω the set of all infinite sequences of elements of A, or *infinite words* on the "alphabet" A. If $c \in A^\omega$, we write $c = c(1)c(2)c(3)\cdots$, and we regard $c : \mathbb{N} \to A$ as a *coloring* of $\mathbb{N}$, where A is the set of colors and for $n \in \mathbb{N},\ c(n)$ is the color assigned to n.

We denote by A^* the set of all *finite* sequences of elements of A, or *words* on the alphabet A, including the empty word. If $u, v \in A^*$, say $u = a_1a_2\cdots a_n$ and $v = b_1b_2\cdots b_m$, with $a_i, b_j \in A, 1 \le i \le n, 1 \le j \le m$, then their *product* $uv \in A^*$ is the word $uv = a_1a_2\cdots a_nb_1b_2\cdots b_m$. A word $v \in A^*$ is a *factor* of the word $w \in A^*$ if there exist (possibly empty) words $p, q \in A^*$ such that $w = pvq$. A word $v \in A^*$ is a factor of an infinite word $s \in A^\omega$ if there are $p \in A^*$ and $s' \in A^\omega$ such that $s = pvs'$.

If $w \in A^*$ then $F(w)$ denotes the set of all factors of w. If $c \in A^\omega$ then $F(c)$ denotes the set of all factors of c.

If $u = a_1a_2\cdots a_n \in A^*$, where $a_i \in A, 1 \le i \le n$, then we say w has *length* n and write $|w| = n$. (The empty word has length 0.)

Definition 1. Let $c \in A^\omega$ and let u be a factor of c. We define

$$k(c, u) = \sup\ \{|v| : v \in F(c) \text{ and } u \notin F(v)\}.$$

Thus if (and only if) u is "missing" from arbitrarily long factors of c, we have $k(c, u) = \infty$.

Definition 2. If $c \in A^\omega,\ u \in F(c)$, and $k(c, u) < \infty$, that is, if every sufficiently long factor w of c contains u as a factor, we say that the factor u of $c \in A^\omega$ is *uniformly recurrent* (in c). If *every* factor u of c is uniformly recurrent in c then we say that c itself is *uniformly recurrent.*

First we need what is essentially König's Lemma:

Lemma 1. *Let L be any infinite subset of A^*, where A is a finite set. Then there is an infinite word $t \in A^\omega$ such that each factor of t is a factor of infinitely many words of L.*

Proof. Since A is finite, there is a letter in A, call it $t(1)$, which is the first letter in each word of an infinite subset L_1 of L. Similarly, there is a letter in A, call it $t(2)$, such that $t(1)t(2)$ are the first two letters of each word of an infinite subset L_2 of L_1. Continuing in this way, we produce an infinite word $t = t(1)t(2)t(3)\cdots \in A^\omega$ such that each "prefix" $t(1)t(2)t(3)\cdots t(n)$ of t is a prefix of an infinite subset L_n of $L_{n-1} \subseteq \cdots \subseteq L_2 \subseteq L_1 \subseteq L$. Since each factor of t is a factor of a prefix of t, every factor of t is a factor of infinitely many words of L. ☐

Definition 3. Let A be a finite set, and let $c \in A^\omega$. We define a sequence of infinite words $t_0, t_1, t_2, \ldots$ inductively as follows. We set $t_0 = c$. Let the factors of c

be $F(c) = \{w_1, w_2, w_3, \dots\}$. (This is an arbitrary enumeration.) For $r > 0$ assume $t_0, t_1, t_2, \dots, t_{r-1}$ have been defined. Let E_r be the set of all those factors of t_{r-1} which do not contain w_r as a factor. Thus

$$E_r = \{v \in F(t_{r-1}) : w_r \notin F(v)\}.$$

If E_r is finite, we set $t_r = t_{r-1}$.

If E_r is infinite, we obtain t_r by using Lemma 1 in the following way. We set $L = E_r$ in the hypothesis of Lemma 1 and conclude (by Lemma 1) there is an infinite word $t_r \in A^\omega$ such that each factor of t_r is a factor of (infinitely many) words of E_r. (Thus $F(t_r) \subseteq E_r$.)

This concludes our definition of $\{t_r\}_{r=0}^{\infty}$.

Lemma 2. *For all $r > 0$, if E_r is finite, then $k(t_{r-1}, w_r) < \infty$.*

Proof. By Definition 1, $k(t_{r-1}, w_r) = \sup\{|v| : v \in F(t_{r-1}), w_r \notin F(v)\} = \sup\{|v| : v \in E_r\}$. □

Lemma 3. *We have*

$$\cdots \subseteq F(t_r) \subseteq F(t_{r-1}) \subseteq \cdots \subseteq F(t_2) \subseteq F(t_1) \subseteq F(t_0) = F(c).$$

Proof. If E_r is finite, then $t_r = t_{r-1}$. If E_r is infinite, then $F(T_r) \subseteq E_r \subseteq F(t_{r-1})$. □

Lemma 4. *If $w_r \in F(t_r)$ then $k(t_r, w_r) < \infty$.*

Proof. By the definition of E_r, $w_r \notin E_r$. If E_r is infinite, then by Definition 3, $F(t_r) \subseteq E_r$. Thus, if E_r is infinite, then $w_r \notin F(t_r)$. Therefore, $w_r \in F(t_r)$ implies E_r is finite. Since E_r is finite, Lemma 2 gives $k(t_{r-1}, w_r) < \infty$ and Definition 3 gives $t_r = t_{r-1}$, hence $k(t_r, w_r) < \infty$. □

We are now ready to prove the crucial Theorem 1. Recall that the term "uniformly recurrent" is defined in Definition 2.

Theorem 1. *Given an arbitrary infinite word $c \in A^\omega$, there exists an infinite word $s \in A^\omega$ such that s is uniformly recurrent and $F(s) \subseteq F(c)$.*

Proof. We make use of the sequence of infinite words $t_0, t_1, t_2, \dots$ defined in Definition 3. For each $i \geq 1$, let u_i be any factor of t_i with length i:

$$u_i \in F(t_i), \quad |u_i| = i.$$

Using Lemma 1, we let $s \in A^\omega$ be any infinite word such that every factor of s is a factor of infinitely many u_i. From Lemma 3, it's clear that $F(s) \subseteq F(c)$. In fact, something stronger is true, namely

$$F(s) \subseteq \bigcap \{F(t_j) : j \geq 0\}.$$

To see this, let $w \in F(s)$, and let j be arbitrary. Since w is a factor of infinitely many u_i, choose $j_0 \geq j$ such that w is a factor of u_{j_0}. Since $u_{j_0} \in F(t_{j_0})$, we have $w \in F(t_{j_0}) \subseteq F(t_j)$.

Now we can show that every $w \in F(s)$ is uniformly recurrent (in s). Since $F(s) \subseteq F(c)$, let $w = w_r$. We have just seen that $w_r \in F(t_r)$, and by Lemma 4, this gives $k(t_r, w_r) < \infty$. Finally, since $F(s) \subseteq F(t_r)$, we have $k(s, w_r) \leq k(t_r, w_r) < \infty$, which means (Definition 2) that w_r is uniformly recurrent. □

3. The First Extension

Theorem 2. *Let c be an arbitrary finite coloring of $\mathbb{N}$. Then for every $k \geq 2$ there exists a sequence of positive integers $d_1 < d_2 < d_3 < \cdots$ such that for all $n \geq 1$, infinitely many translates of the set*

$$P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k-1 \right\}$$

are monochromatic. The $\{d_i\}$ depend only on k (and the coloring), and can be chosen so that $|P_n| = k^n$.

Proof. As mentioned in the Introduction, we regard the coloring c as an infinite word $c = c(1)c(2)c(3)\cdots \in A^\omega$. Here A is the set of "colors," and $c(n)$ is the color assigned to n, for all $n \in \mathbb{N}$. Throughout the proof, $k \geq 2$ is fixed.

Using Theorem 1, we let s be any infinite word $s \in A^\omega$ such that s is uniformly recurrent and $F(s) \subseteq F(c)$. To prove Theorem 2, we only need to show (for each n) that a single translate of P_n is monochromatic under the coloring s. For suppose that $[p, q]$ is an interval which contains a translate $\overline{P_n}$ of P_n, and that the coloring s, restricted to $[p, q]$, is constant on $\overline{P_n}$; then the word

$$w = s(p)s(p+1)s(p+2)\cdots s(q)$$

occurs infinitely often in the word s, and hence occurs infinitely often in the word c (since $F(s) \subseteq F(c)$). Thus c is constant on infinitely many translates of P_n.

Let $k \in \mathbb{N}$ be fixed throughout the remainder of this argument.

We use induction on n. For $n = 1$, let a be any element of A which occurs in s.

Since s is uniformly recurrent, there are $D, x_1, x_2, x_3, \cdots \in \mathbb{N}$ with $x_1 < x_2 < x_3 < \cdots$ and $x_{j+1} - x_j \leq D$ for all $j \geq 1$, such that the word a (of length 1) occurs at each of the positions $x_1, x_2, x_3, \cdots$ in s. (That is, s is constant on $\{x_1, x_2, x_3, \ldots\}$.)

Let $V = \{x_1, x_2, x_3, \cdots\}$. Then $V \cup (V+1) \cup (V+2) \cup \cdots \cup (V+D-1) = [x_1, \infty)$. Now set $V_0 = V, V_1 = (V+1) - V_0, V_2 = (V+2) - (V_0 \cup V_1), \ldots, V_{D-1} = V + (D-1) - (V_0 \cup V_1 \cup \cdots \cup V_{D-2})$, to obtain a partition $V_0, V_1, V_2, \ldots, V_{D-1}$ (or "coloring with

D colors") of $[x_1, \infty)$. By van der Waerden's theorem on arithmetic progressions, some V_i contains a k-term arithmetic progression. Since V_i is a translate of V, V itself contains a k-term arithmetic progression.

Thus there exists d_1 such that s is constant on a translate of

$$P_1 = \{\epsilon_1 d_1 : 0 \leq \epsilon_1 \leq k-1\}.$$

For the induction step, let $n \geq 1$ and assume that $d_1 < d_2 < \cdots < d_n$ exist so that s is constant on a translate of

$$P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : 0 \leq \epsilon_1, \ldots, \epsilon_n \leq k-1 \right\},$$

and $|P_n| = k^n$. To be specific, assume that s is constant on the set

$$\overline{P_n} = m + P_n.$$

The set $\overline{P_n}$ is contained in the interval $[m,\ m + (k-1)(d_1 + d_2 + \cdots + d_n)]$. Let $q = (k-1)(d_1 + d_2 + \cdots + d_n)$, so that $\min \overline{P_n} = m, \max \overline{P_n} = m + q$, and $\overline{P_n}$ is contained in the interval $[m,\ m+q]$.

Let $w = s(m)s(m+1)s(m+2)\cdots s(m+q)$. Thus w is a certain factor of s with length $|w| = q+1$. Moreover, $\overline{P_n}$ is contained in $[m, m+q]$, and s is constant on $\overline{P_n}$. We will use the abbreviated notation

$$w = s[m,\ m+q].$$

Since s is uniformly recurrent, there are $D, x_1, x_2, x_3, \cdots \in \mathbb{N}$ with $x_1 < x_2 < x_3 < \cdots$ and $x_{j+1} - x_j \leq D$ for all $j \geq 1$, such that the factor w *begins* at each of the positions $x_1, x_2, x_3, \cdots$ in s.

Again using van der Waerden's theorem on arithmetic progressions, there exists d_{n+1} such that $x_1, x_2, x_3, \cdots$ contains a translate of the arithmetic progression $\{\epsilon_{n+1} d_{n+1} : \epsilon_{n+1} \in \{0, 1, \ldots, k-1\}\}$. We can assume that $d_{n+1} > q = (k-1)(d_1 + \cdots + d_n)$.

Thus for some $m' \in \mathbb{N}$, the factor w begins at each of the positions $m', m' + d_{n+1}, m' + 2d_{n+1}, \ldots, m' + (k-1)d_{n+1}$. We can assume that $m' > m$.

For $0 \leq i \leq k-1$, let I_i denote the interval $I_i = [m' + id_{n+1}, m' + id_{n+1} + q]$. Since the factor $w = s[m, m+q]$ begins at each number $m' + id_{n+1}$, we have that for $0 \leq i \leq k-1$,

$$w = s[m, m+q] = s(I_i).$$

Since $I_i = (m' - m) + id_{n+1} + [m, m+q]$ and s is constant on $\overline{P_n} \subseteq [m, m+q]$, it follows that s is constant on $(m' - m) + id_{n+1} + \overline{P_n} = m' + id_{n+1} + P_n \subseteq I_i$.

Thus, s is constant on

$$\bigcup \{m' + id_{n+1} + P_n : 0 \leq i \leq k-1\} = m' + P_{n+1}.$$

Since $d_{n+1} > q = (k-1)(d_1 + \cdots + d_n)$, the intervals $I_i, 0 \le i \le k-1$, are pairwise disjoint, hence $|P_{n+1}| = k|P_n| = k^{n+1}$.

□

Theorem 3. *Let c be an arbitrary finite coloring of $\mathbb{N}$. Then for every sequence of positive integers $\{k_i\}_{i=1}^{\infty}$ there exists a sequence of positive integers $d_1 < d_2 < d_3 < \cdots$ such that for all $n \ge 1$, infinitely many translates of the set*

$$P_n = \left\{ \sum_{i=1}^{n} \epsilon_i d_i : \epsilon_i \in \{0, 1, \ldots, k_i - 1\} \right\}$$

are monochromatic. Furthermore, the d_i can be chosen so that $|P_n| = k_1 k_2 \cdots k_n$.

Proof. Trivial modifications of the proof of Theorem 2 give a proof of Theorem 3. □

4. The Second Extension

Let $W(l, r)$, i.e., the van der Waerden number, be the least positive integer m such that if $[1, m]$ is r-colored, then there exist a, d such that $a, a + d, \ldots, a + (l-1)d$ have the same color.

Theorem 4. *Let n, k, r be positive integers. There exists a least positive integer $m = WB_n(k, r)$ such that if $[1, m]$ is r-colored, there exist $a, d_1, d_2, \ldots, d_n$ such that $d_1 < d_2 < \cdots < d_n$, $\mathcal{P}_n = \{a + x_1 d_1 + \cdots + x_n d_n : 0 \le x_1, \ldots, x_n \le k-1\} \subseteq [1, m]$, $|\mathcal{P}_n| = k^n$, and $\mathcal{P}_n \cup \{d_1, \ldots, d_n\}$ is monochromatic.*

Proof. The proof is by double induction on n, r. The case $r = 1$ is trivial, so let $r \ge 2$ and the non-negative integer n be also fixed through the reminder of the proof. Now fix a positive integer k and assume that the following numbers exist: $WB_n(l, r)$ for all l and $WB_{n+1}(k, r-1)$ (note that for $n = 0$ we put $WB_0(l, r) = W(l, r)$). We will show that $WB_{n+1}(k, r)$ exists.

Let

$$m = WB_n(WB_{n+1}(k, r-1) \cdot k(k-1) + k, r).$$

We show that $WB_{n+1}(k, r) \le m$, which will complete the argument.

Let $[1, m]$ be r-colored, using the colors $\{1, 2, \ldots, r\}$. By the definition of m, there are $a, d_1, \ldots, d_n$ such that $d_1 < d_2 < \cdots < d_n$,

$$\mathcal{Q}_n = \{a + y_1 d_1 + \cdots + y_n d_n : 0 \le y_1, \ldots, y_n \le WB_{n+1}(k, r-1) \cdot k(k-1) + k - 1\} \subseteq [1, m],$$

and $\mathcal{Q}_n \cup \{d_1, \ldots, d_n\}$ is monochromatic, say with color r. Now consider the following (monochromatic) subsets of $\mathcal{Q}_n$:

$$\mathcal{Q}_{n+1}^1 = \{a + x_1 d_1 + \cdots + x_n d_n + x_{n+1}[k(d_1 + \cdots + d_n)] : 0 \le x_1, \ldots, x_{n+1} \le k-1\}$$

$$\mathcal{Q}^2_{n+1} = \{a + x_1d_1 + \cdots + x_nd_n + x_{n+1}[2k(d_1 + \cdots + d_n)] : 0 \le x_1, \ldots, x_{n+1} \le k-1\}$$
$$\mathcal{Q}^3_{n+1} = \{a + x_1d_1 + \cdots + x_nd_n + x_{n+1}[3k(d_1 + \cdots + d_n)] : 0 \le x_1, \ldots, x_{n+1} \le k-1\}$$
$$\cdots$$
$$\mathcal{Q}^w_{n+1} = \{a + x_1d_1 + \cdots + x_nd_n + x_{n+1}[wk(d_1 + \cdots + d_n)] : 0 \le x_1, \ldots, x_{n+1} \le k-1\}$$

where $w = WB_{n+1}(k, r-1)$.

(One may note that $\max \mathcal{Q}^w_{n+1} = \max \mathcal{Q}_n$.)

There are now two cases.

If at least one of $\{k(d_1 + \cdots + d_n), 2k(d_1 + \cdots + d_n), \ldots, wk(d_1 + \cdots + d_n)\}$, say $j_0k(d_1 + \cdots + d_n)$, has the color r, we are done by setting $d_{n+1} = j_0k(d_1 + \cdots + d_n)$, for then

$$\mathcal{Q}^{j_0}_{n+1} \cup \{d_1, \ldots, d_{n+1}\}$$

is our desired monochromatic set.

If none of $\{k(d_1 + \cdots + d_n), 2k(d_1 + \cdots + d_n), \ldots, wk(d_1 + \cdots + d_n)\}$ have color r, then, recalling that $w = WB_{n+1}(k, r-1)$, all elements of

$$k(d_1 + \cdots + d_n)[1, WB_{n+1}(k, r-1)]$$

have been colored with $r-1$ colors, and we are done by the induction hypothesis on r. □

5. Remarks and Addendum

1. In proving Theorem 2, we don't actually need the full strength of Theorem 1, which says that each factor w of the infinite word s occurs *syndetically* in s. In order to apply van der Waerden's theorem, it's enough to know that each factor w of the infinite word s occurs *piecewise syndetically* in s. This means that for each factor w of the infinite word s, there exists a fixed $d = d(w)$ such that for arbitrarily large m, there is a factor $q_1q_2\cdots q_m$ of s with $|q_i| = d, 1 \le i \le m$, such that w is a factor of each $q_i, 1 \le i \le m$.

2. Theorem 2 has the same relation with van der Waerden's theorem as Carlson-Simpson's theorem [2] has with Hales-Jewett's theorem. It is possible to deduce Theorem 2 from Carlson-Simpson's theorem by generalizing the usual proof of deducing van der Waerden's theorem from the Hales-Jewett theorem. Similarly Theorem 3 can be deduced from a generalized version of Carlson-Simpson's theorem ([3], Section 10).

3. As the proof given for Theorem 4 uses a double induction argument, it gives no primitive recursive upper bound for $WB_n(k, r)$. We noticed that Theorem 4 is a special case of the theorem on the existence of monochromatic (m, p, c)-sets ([6], Chapter 3, Theorem 10) for $c = 1$. Checking the standard proof given there shows

that in fact $WB_n(k, r)$ has a primitive recursive bound belonging to the class of WOW functions ([6], 2.7). On the other hand by a straightforward modification of our proof for Theorem 4, we get a new and simpler proof of the above mentioned theorem on (m, p, c)-sets.

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