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# A MULTIVARIATE ARITHMETIC FUNCTION OF COMBINATORIAL AND TOPOLOGICAL SIGNIFICANCE

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**Abstract**

We investigate properties of a multivariate function $E(m_1, m_2, \ldots, m_r)$, called *orbicyclic*, that arises in enumerative combinatorics in counting non-isomorphic maps on orientable surfaces. $E(m_1, m_2, \ldots, m_r)$ proves to be multiplicative, and a simple formula for its calculation is provided. It is shown that the necessary and sufficient conditions for this function to vanish are equivalent to familiar Harvey's conditions that characterize possible branching data of finite cyclic automorphism groups of Riemann surfaces.

## 1. Introduction

Let $(m_1, m_2, \ldots, m_r)$ be an $r$-tuple positive integers and $m = \mathrm{lcm}(m_1, m_2, \ldots, m_r)$, where $m := 1$ for $r = 0$ (an empty tuple). Introduce the following multivariate function

$$E = E(m_1, m_2, \ldots, m_r) := \frac{1}{M} \sum_{k=1}^{M} \Phi(k, m_1)\Phi(k, m_2) \cdots \Phi(k, m_r) \tag{1}$$

($E(\emptyset) = 1$), where $m|M,\ M > 0$, and $\Phi(k, n)$ stands for the von Sterneck function:

$$\Phi(k, n) := \frac{\phi(n)}{\phi\left(\frac{n}{(k,n)}\right)}\, \mu\Big(\frac{n}{(k, n)}\Big). \tag{2}$$

Here $(k, n)$ denotes the greatest common divisor of $k$ and $n$ and $\mu(n)$ and $\phi(n)$ are the Möbius and Euler functions respectively. According to O. Hölder (see, e.g., [1, Chapter 8], [39, Chapter IX]), $\Phi(k, n)$ coincides with the Ramanujan trigonometric sum:

$$\Phi(k,n) = C_n(k) \tag{3}$$

where

$$C_n(k) := \sum_{\substack{d \pmod n \\ (d,n)=1}} \exp\Bigl(\frac{2\,i\,k\,d}{n}\Bigr)$$

with the summation over a reduced residue system modulo $n$. The function $C_n(k)$ (in the literature it has diverse designations such as $C(k,n)$) satisfies the familiar Ramanujan's identity:

$$C_n(k) = \sum_{d|(k,n)} d\,\mu\Bigl(\frac{k}{d}\Bigr). \tag{4}$$

Note that $E(m_1, m_2, \ldots, m_r)$ does not depend on $M$. Indeed, by (2), $\Phi(k, m_j)$ is a periodic function of $k$ modulo the second variable and, a fortiori, modulo $m$. Thus $\prod_j \Phi(k, m_j)$ as a function of $k$ is periodic modulo $m$ as well. So, in (1) we may put $M = m$. Now, $E$ is a symmetric function of its arguments, and we might speak about the (multi-)set of arguments instead of a tuple of them. Since $\Phi(k, 1) = 1$ we may restrict $m_j$ to values greater 1, that is,

$$E(m_1, m_2, \ldots, m_{r-1}, 1) = E(m_1, m_2, \ldots, m_{r-1}). \tag{5}$$

This plays an important rôle in computational formulae for $E$ and applications. Tuples of arguments not containing 1 are called *reduced*.

$E(m_1, m_2, \ldots, m_r)$ is an essentially multivariate function in the sense that it is trivial for $r = 0, 1$. Moreover, for $r = 2$ it vanishes for unequal arguments and coincides with the Euler function otherwise. Later we will see that $E(m_1, m_2, \ldots, m_r)$ is always non-negative and integer-valued.

The function $E$ has been introduced by A. Mednykh and R. Nedela (see [29]) in the context of enumerative combinatorics: it plays a crucial rôle in counting maps on orientable surfaces up to orientation-preserving isomorphism, via the calculation of certain epimorphisms from the fundamental group of orbifolds onto cyclic groups (see Section 3). Therefore this "orbicyclic" arithmetic function (as we called it in [23]) deserves a detailed investigation by its own right.

Here we study some basic properties of $E(m_1, m_2, \ldots, m_r)$. First of all we analyze the prime-power case and establish a simple sum-free formula for $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$, $p$ prime. It is determined by three parameters (apart from $p$), denoted $r$ (reduced), $s$ and $v$, only one of which ($s$, the multiplicity of the highest power) is responsible for its vanishing. In the general case we show that $E(m_1, m_2, \ldots, m_r)$ is a multiplicative function of all its arguments. Both results provide a simple explicit formula for its calculation. The most valuable property of $E(m_1, m_2, \ldots, m_r)$ established here is the necessary and sufficient conditions of its non-vanishing. We show that they are equivalent (due to the above-mentioned connection with the enumeration of epimorphisms) to the well-known conditions discovered by W. Harvey [14] (cf. also [6]) that specify the possible actions of finite cyclic automorphism groups on Riemann surfaces. Therefore the function $E(m_1, m_2, \ldots, m_r)$ may be considered as a fruitful *enumerative refinement* of Harvey's theorem, which brings a new insight into this theory.

The familiar Jordan arithmetic function also participates in the mentioned enumeration; we discuss briefly some other links between this function and enumerative combinatorial group theory.

For the reader's convenience, the paper contains rather numerous (although restricted) references to relevant publications in the three main topics we deal with: number theory (arithmetic functions), algebraic topology (automorphisms of Riemann surfaces), and algebraic and enumerative combinatorics (map theory and map enumeration). For general material concerning these topics and the notions used in the present paper, the reader is referred to, resp., [1], [7] and [11].

## 2. Function $E(m_1, m_2, \ldots, m_r)$ and its Properties

### 2.1. Primary Case

$\Phi(k, n)$ is a multiplicative function of $n$ which is determined by the following well-known (and easily provable) formula; see, e.g., [25]:

**Lemma 1** *For $p$ prime and $a \geq 1$,*

$$\Phi(k, p^a) = \begin{cases} (p-1)p^{a-1} & if \quad p^a | k \\ -p^{a-1} & if \quad p^a \nmid k, \ p^{a-1} | k \\ 0 & otherwise. \end{cases} \tag{6}$$

This is an important result for calculating $E(m_1, m_2, \ldots, m_r)$ explicitly. As we will see below, the function $E(m_1, m_2, \ldots, m_r)$ is always non-negative, unlike $\Phi(k, n)$ (and it often vanishes like $\Phi(k, n)$).

At first, we consider the prime power case: $m = p^a$. Let $(m_1, m_2, \ldots, m_r)$ be a reduced tuple. Then $m_j = p^{a_j}, \ j = 1, 2, \ldots, r$, where without loss of generality we assume that

$$a_1 = a_2 = \ldots = a_s = a > a_{s+1} \geq a_{s+2} \cdots \geq a_r > 0, \tag{7}$$

where $r \geq s \geq 1$. Denote

$$v := \sum_{j=2}^{r}(a_j - 1) = \sum_{j=1}^{r} a_j - r - a + 1, \qquad v \geq 0. \tag{8}$$

The parameters $s(p) = s$ and $v(p) = v$, where $p|m$, can be defined for an arbitrary tuple of variables $(m_1, m_2, \dots, m_r)$ as well (see Sect. 2.3 below). As we will show, they (together with the corresponding $r(p)$ with respect to *reduced* tuples) determine the value of $E(m_1, m_2, \dots, m_r)$.

**Lemma 2** *We have*

$$E(p^{a_1}, p^{a_2}, \dots, p^{a_r}) = (p-1)^{r-s+1} p^v h_s(p), \tag{9}$$

*where the multiplicity $s$ and the exponents $a_j$ are subject to* (7)*, $v$ is determined by* (8) *and $h_s(x)$, $s \geq 1$, is the following polynomial of $x$ of degree $s-2$ (for $s > 1$):*

$$h_s(x) = \frac{(x-1)^{s-1} + (-1)^s}{x}. \tag{10}$$

*Proof.* We have

$$E(p^{a_1}, \dots, p^{a_r}) = \frac{1}{p^a} \sum_{k=1}^{p^a} \Phi(k, p^{a_1}) \cdots \Phi(k, p^{a_r}).$$

By (6), the first factor in these terms vanishes unless $k = dp^{a-1}$, so that we may restrict ourselves to such $k$ only, where $d = 1, 2, \dots, p$. Again by (6), we get (for all $s$ including $s = 1$)

$$\begin{aligned}
&E(p^{a_1}, \dots, p^{a_r}) \\
&\quad = \frac{1}{p^a}\Big(\sum_{d=1}^{p-1} \Phi(dp^{a-1}, p^{a_1}) \cdots \Phi(dp^{a-1}, p^{a_r}) + \Phi(p^a, p^{a_1}) \cdots \Phi(p^a, p^{a_r})\Big) \\
&\quad = \frac{1}{p^a}\Big((p-1)\Phi(p^{a-1}, p^{a_1}) \cdots \Phi(p^{a-1}, p^{a_r}) + \Phi(p^a, p^{a_1}) \cdots \Phi(p^a, p^{a_r})\Big) \\
&\quad = \frac{1}{p^a}\Big((p-1)(-p^{a-1})^s (p-1)^{r-s} p^{a_{s+1}-1} \cdots p^{a_r - 1} + (p-1)^r p^{a_1 - 1} \cdots p^{a_r - 1}\Big) \\
&\quad = \frac{1}{p^a}\Big((p-1)(-1)^s p^{(a-1)s} p^{\sum_{j=s+1}^{r}(a_j-1)} (p-1)^{r-s} + (p-1)^r p^{\sum_{j=1}^{r}(a_j - 1)}\Big)
\end{aligned}$$

$$= \frac{1}{p^a}\Big((p-1)(-1)^s p^{\sum_{j=1}^r (a_j-1)}(p-1)^{r-s} + (p-1)^r p^{\sum_{j=1}^r (a_j-1)}\Big)$$

$$= \frac{1}{p^a} p^{\sum_{j=1}^r (a_j-1)}(p-1)^{r-s+1}\Big((-1)^s + (p-1)^{s-1}\Big)$$

$$= p^{-1+\sum_{j=2}^r (a_j-1)}(p-1)^{r-s+1}\Big((-1)^s + (p-1)^{s-1}\Big)$$

$$= (p-1)^{r-s+1} p^v \left(\frac{(-1)^s + (p-1)^{s-1}}{p}\right).$$

□

In particular,

$$h_1(x) = 0,$$

$$h_2(x) = 1,$$

$$h_3(x) = x - 2,$$

$$h_4(x) = x^2 - 3x + 3,$$

$$h_5(x) = (x-2)(x^2 - 2x + 2).$$

Note that by (10), $(x-2)|h_s(x)$ for odd $s$ and $h_s(2) = 1$ for even $s$. Clearly $h_s(p) \geq 0$ for any $s$ and $p \geq 2$. Moreover, $h_s(p) = 0$ if and only if $s = 1$ or $p = 2$ and $s$ is odd.

**Corollary 3** (1) *$E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ is non-negative and integer.*

(2) *$E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ vanishes if and only if $s = 1$ or $p = 2$ and $s$ is odd.*

(3) *Let $p$ be odd. $p \nmid E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ if and only if $a = 1$ and $r > 1$; in this case, $s = r$.*

(4) *$E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ is odd if and only if $p = 2, a = 1$ and $s = r$ is even.*

*Proof.* Claims (1) and (2) are immediate from formulae (9) and (10).

Claims (3). Suppose that $p$ does not divide $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$. Then $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r}) \neq 0$ and by formula (9), $v = 0$. It follows from (8) that $a_2 = \cdots = a_r = 1$. But $a_1 = 1$ as well and $r > 1$, otherwise $s = 1$ and then by Claim (2), $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r}) = 0$. Thus, $a = a_1 = 1$ and $s = r$.

On the contrary, if $a = 1$ then $a_1 = a_2 = \cdots = a_r = 1$ as well. Thus, $s = r$ and by (8), $v = 0$. But in the right-hand side of formula (9), neither $p - 1$ nor $h_s(p)$ is divisible by $p$ for $p > 2$ and $s > 1$. Hence if $r > 1$ then $p \nmid E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$.

Claim (4). $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ is even for odd $p$ since always $r - s + 1 > 0$ and, thus, the factor $(p-1)^{r-s+1}$ in the right-hand side of formula (9) is even. Now suppose $p = 2$. We have the same situation as in the proof of Claim (3) with the only distinction that $2|h_s(2)$ iff $s$ is odd. □

**2.2. Multiplicativity**

Now we turn to the general case. Take a tuple $(m_1, m_2, \ldots, m_r)$ and let $a_j(p) \geq 0$ denote the exponent with which the prime $p$ divides $m_j$. For a prime $p|m$, denote

$$\{m_1, m_2, \ldots, m_r\}_p = (p^{a_1(p)}, p^{a_2(p)}, \ldots, p^{a_r(p)}). \tag{11}$$

Now denote

$$\begin{aligned} E_p(m_1, m_2, \ldots, m_r) &:= E(\{m_1, m_2, \ldots, m_r\}_p) \\ &= E(p^{a_1(p)}, p^{a_2(p)}, \ldots, p^{a_r(p)}). \end{aligned} \tag{12}$$

Then

$$E_p(m_1, m_2, \ldots, m_r) = E(\langle p^{a_1(p)}, p^{a_2(p)}, \ldots, p^{a_r(p)} \rangle), \tag{13}$$

where the angled brackets mean the removal of all the arguments equal to 1, that is, the right-hand function contains a reduced set of arguments.

**Proposition 4** $E(\emptyset) = E(1, 1, \ldots, 1) = 1$, *and for* $m > 1$,

$$E(m_1, m_2, \ldots, m_r) = \prod_{p|m \text{ prime}} E_p(m_1, m_2, \ldots, m_r). \tag{14}$$

*In other words,* $E(m_1, m_2, \ldots, m_r)$ *is a multiplicative function.*

An arithmetic function $g = g(m_1, m_2, \ldots, m_r)$ of $r \geq 1$ arguments is called *semi-multiplicative* (see, e.g., [15, 35]) if

$$g(m_1, m_2, \ldots, m_r) = g(m_1', m_2', \ldots, m_r') \cdot g(m_1'', m_2'', \ldots, m_r'')$$

whenever $m_j = m_j' m_j''$, $j = 1, 2, \ldots, r$, and $(M', M'') = 1$ where $M' = \prod_j m_j'$ and $M'' = \prod_j m_j''$. This function is called *multiplicative* if, moreover, $g(\underbrace{1, 1, \ldots, 1}_{r}) = 1$. Two most known examples of symmetric multivariate multiplicative functions are lcm() and gcd().

$\Phi(k, n)$ is multiplicative as a function of $n$ and is periodic in $k$ modulo $n$ (it possesses, in fact, stronger properties but we do not need to use them here). Therefore Proposition 4 is straightforward from the following more general assertion.

**Lemma 5** *Let a bivariate arithmetic function $f(k,n)$ be semi-multiplicative in the argument $n$ and periodic in $k$ modulo $n$. Given natural numbers $M, m_1, m_2, \ldots, m_r$, where $m_j|M$ for all $j$, the function*

$$F(m_1, m_2, \ldots, m_r) = \frac{1}{M}\sum_{k=1}^{M} f(k, m_1)f(k, m_2)\cdots f(k, m_r) \tag{15}$$

*is semi-multiplicative (with respect to all its arguments).*

*Proof.* By the same reasons as in the Introduction, $F(m_1, m_2, \ldots, m_r)$ does not depend on $M$ (provided $m_j|M,\ j = 1, 2, \ldots, r$). So that, without loss of generality, we set $M = \prod_j^r m_j$. Suppose $m_j = m'_j m''_j,\ j = 1, 2, \ldots, r$, where $(m'_i, m''_j) = 1$ for all $i, j$. Letting $M' = \prod_j^r m'_j$ and $M'' = \prod_j^r m''_j$, consider the product

$$F(m'_1, \ldots, m'_r) \cdot F(m''_1, \ldots, m''_r)$$

$$= \frac{1}{M'}\sum_{k'=1}^{M'} f(k', m'_1)\cdots f(k', m'_r) \cdot \frac{1}{M''}\sum_{k''=1}^{M''} f(k'', m''_1)\cdots f(k'', m''_r).$$

We claim that, regardless of the choice of all $m_j = m'_j m''_j$, the equality $F(m_1, \ldots, m_r) = F(m'_1, \ldots, m'_r) \cdot F(m''_1, \ldots, m''_r)$ holds. Notice first that like $F(m_1, \ldots, m_r)$, the product $F(m'_1, \ldots, m'_r) \cdot F(m''_1, \ldots, m''_r)$ contains $M = M'M''$ terms. Thus, all we need is to establish a bijection between the terms of the two functions and to show the equality of the corresponding terms.

Since the function $f(k, m)$ is periodic in $k$ modulo $m$, it is periodic modulo $M$, as well, if $m|M$, i.e., $f(k_1, m) = f(k_2, m)$ whenever $k_1 \equiv k_2 \pmod{M}$.

Consider a "generic" term in $F(m'_1, \ldots, m'_r) \cdot F(m''_1, \ldots, m''_r)$ :

$$t(k', k'') := f(k', m'_1)\cdots f(k', m'_r) \cdot f(k'', m''_1)\cdots f(k'', m''_r).$$

We look for $k$ such that $t(k', k'') = t(k, k)$. Take $k$ satisfying

$$k \equiv k' \pmod{M'}$$
$$k \equiv k'' \pmod{M''}.$$

Since $(M', M'') = 1$, by the Chinese remainder theorem, such a $k$ does exist and is unique modulo $M$. Now, due to the periodic property, $f(k', m'_j) = f(k, m'_j)$ and $f(k'', m''_j) = f(k, m''_j)$ for $j = 1, 2, \ldots, r$. Hence $t(k', k'') = t(k, k)$ as required. Therefore by the semi-multiplicativity of $f(k, m)$ in $m$ we have

$$\begin{aligned} t(k', k'') &= f(k, m'_1)\cdots f(k, m'_r) \cdot f(k, m''_1)\cdots f(k, m''_r) \\ &= f(k, m_1)\cdots f(k, m_r). \end{aligned}$$

It is clear that the established correspondence between $k$ and pairs $k', k''$ is a bijection between the sets $[1, M]$ and $[1, M'] \times [1, M'']$, which gives rise to the required bijection between the terms of $F(m_1, m_2, \dots, m_r)$ and those of $F(m'_1, \dots, m'_r) \cdot F(m''_1, \dots, m''_r)$. $\square$

**Corollary 6** *If $m_1 = m'_1 m''_1$, where $(m'_1, m''_1) = 1$, then*

$$E(m_1, m_2, \dots, m_r) = E(m'_1, m''_1, m_2, \dots, m_r).$$

Indeed, $E_p(m_1, m_2, \dots, m_r) = E_p(\langle m'_1, m''_1, m_2, \dots, m_r \rangle)$ for all prime $p$. $\square$

Therefore, increasing $r$, one can split the arguments of $E(m_1, m_2, \dots, m_r)$ into their primary factors:

$$E(m_1, m_2, \dots, m_r) = E\big(p^{a_j(p)} :\ p|m \text{ prime},\ j = 1, 2, \dots, r,\ a_j(p) \geq 1\big).$$

**Corollary 7** *The values of $E(m_1, m_2, \dots, m_r)$ are non-negative integers.* $\square$

### 2.3. Main Formulae

Given a tuple $(m_1, m_2, \dots, m_r)$ with

$$\operatorname{lcm}(m_1, m_2, \dots, m_r) = m = \prod_{p \text{ prime}} p^{a(p)}, \tag{16}$$

define for $p|m$, the parameters $s(p)$ and $v(p)$ that generalize the ones introduced in formulae (7) and (8):

$$s(p) := |\{j :\ a_j(p) = a(p),\ j = 1, 2, \dots, r\}| \tag{17}$$

and

$$v(p) := \sum_{\substack{j=1,2,\dots,r \\ a_j(p) \geq 1}} (a_j(p) - 1) - a + 1. \tag{18}$$

Moreover, we count the arguments $m_j$ divisible by $p$, that is, the ones with $a_j(p) > 0$ :

$$r(p) := |\{m_j :\ p|m_j,\ j = 1, 2, \dots, r\}|. \tag{19}$$

The next theorem follows directly from (9) and (14) and gives rise to an explicit alternating-free formula for calculating $E(m_1, m_2, \dots, m_r)$.

**Theorem 8** *We have*

$$E_p(m_1, m_2, \ldots, m_r) = (p-1)^{r(p)-s(p)+1} p^{v(p)} h_{s(p)}(p) \tag{20}$$

*and*

$$E(m_1, m_2, \ldots, m_r) = \prod_{p|m \text{ prime}} (p-1)^{r(p)-s(p)+1} p^{v(p)} h_{s(p)}(p), \tag{21}$$

*where the parameters $s(p), v(p)$ and $r(p)$ are determined, respectively, by formulae* (17), (18) *and* (19) *and the polynomial $h_s(x)$ is determined by* (10).

### 2.4. Further Properties

According to (21), the value of the function $E(m_1, m_2, \ldots, m_r)$ is determined by the set of prime divisors $p|m$ and the parameters $s(p), v(p)$ and $r(p)$, where $r \geq r(p) \geq s(p) \geq 1$ and $v(p) \geq 0$. In particular, $E(m_1, m_2, \ldots, m_r)$ does not depend directly on $a(p)$ and $m$ (indirectly, however, $a(p)$ contributes into $v(p)$). Note also that $E(m_1, m_2, \ldots, m_r)$ does not depend on $r(2)$ as formulae (20) and (18) show; so that if $4|m$, we may ignore the contributors $2^1$, that is, $a_j(2) = 1$.

**Corollary 9** (1) *$\phi(m)$ divides $E(m_1, m_2, \ldots, m_r)$.*

(2) *$E(m_1, m_2, \ldots, m_r) = \phi(m)$ if and only if for every prime $p|m$, one of the following conditions holds:*

$$\langle \{m_1, m_2, \ldots, m_r\}_p \rangle = (p^{a(p)}, p^{a(p)}),$$

*or $p = 3$ and*

$$\langle \{m_1, m_2, \ldots, m_r\}_3 \rangle = (3, 3, 3),$$

*or $p = 2$ and (up to reordering)*

$$\langle \{m_1, m_2, \ldots, m_r\}_2 \rangle = (2^{a(2)}, 2^{a(2)}, \underbrace{2, 2, \ldots, 2}_{r(2)-2}),$$

*where $a(2) \geq 1$, $r(2) \geq 3$ and $r(2)$ is even if $a(2) = 1$.*

*Proof.* Claim (1). Recall that $\phi(p^{a(p)}) = (p-1)p^{a(p)-1}$. Now, for any $p \mid m$, it follows directly from the definitions that $r(p) - s(p) + 1 \geq 1$. Besides, $v(p) \geq a(p) - 1$ if $E_p(m_1, m_2, \ldots, m_r) \neq 0$ since in this case, $s(p) > 1$ by Corollary 3 (2). Thus, in (18), the term corresponding to $j = 2$ is equal to $a(p) - 1$.

Therefore, by (20), $\phi(p^{a(p)})|E_p(m_1, m_2, \ldots, m_r)$, and we are done by (14) and the multiplicativity of $\phi$.

Claim (2). It is clear from (10) that $p$ does not divide $h_s(p)$ for $s > 1$, nor does $p-1$ for $p > 2$. Therefore by (21), $E(m_1, m_2, \ldots, m_r) = \phi(m)$ if and only if

$$r(p) - s(p) + 1 = 1,$$

$$v(p) = a(p) - 1$$

and

$$h_{s(p)}(p) = 1$$

for every prime $p|m$. Suppose $p \geq 3$. The first equality implies that $\langle \{m_1, m_2, \ldots, m_r\}_p \rangle = (p^{a(p)}, p^{a(p)}, \ldots, p^{a(p)})$ whence $v(p) = (r(p)-1)(a(p)-1)$. Then the second equality implies that $r(p) = 2$ or $a(p) = 1$. If $r(p) = s(p) = 2$ then we get $h_{s(p)}(x) = 1$. If $r(p) \neq 2$ but $a(p) = 1$, then $h_s(p) = 1$ is possible only in one exceptional case: $h_3(3) = 1$. Indeed, $h_s(x) = 1$ can be represented as the equation $y^{s-1} - y - 1 + (-1)^s = 0$, where $y = x - 1$. It has only one integer solution greater 1: $y = 2$ and $s = 3$. This solution corresponds to the tuple $(3, 3, 3)$ as claimed.

Finally, consider $p = 2$ if $2|m$. $h_s(2) = 1$ if and only if $s$ is even. Now $v(2) = a(2) - 1$, which means that $\langle \{m_1, m_2, \ldots, m_r\}_2 \rangle = (2^{a(2)}, 2^{a(2)}, 2, 2, \ldots, 2)$ with $a(2) \geq 1$. If $a(2) > 1$, then $s(2) = 2$ is even; if $a(2) = 1$, then $s(2) = r(2)$ has to be even by Corollary 3 (4). $\square$

Let us consider in more detail the behavior of the function $E$ for $r \leq 3$ arguments.

**Corollary 10** (1) $E(m) \neq 0$ *if and only if* $m = 1$.

(2) $E(m_1, m_2) \neq 0$ *if and only if* $m_1 = m_2$.

(3) $E(m_1, m_2, m_3) \neq 0$ *if and only if for any prime* $p|m$, *the numbers* $m_1, m_2$ *and* $m_3$ *(in some order) are divided by* $p$ *with the exponents* $a(p) \geq b(p) \geq c(p)$ *that meet one of the following conditions:*

- $a(p) = b(p) > c(p) > 0$*;*
- $a(p) = b(p) > c(p) = 0$*;*
- $a(p) = b(p) = c(p) > 0$ *and* $p > 2$.

*Proof.* The first claim is obvious. As to the second one, if $m_1 \neq m_2$ then there is a prime $p \mid m$ such that $s(p) = 1$. Therefore the last factor in formula (21) vanishes.

In Claim (3), if $a(p) > b(p) \geq c(p)$ for some $p \mid m$, then $s(p) = 1$ and the last factor $h_s$ in formula (20) vanishes. It also vanishes if $2 \mid m, p = 2$ and $s(2) = 3$, that is, $a(2) = b(2) = c(2)$. $\square$

Now, $E_p$ is evaluated by formula (9), and we obtain for the last case of Corollary 10,

$$E_p(m_1, m_2, m_3) = \begin{cases} (p-1)^2 p^{a(p)+c(p)-2} & \text{if } a(p) = b(p) > c(p) > 0 \\ (p-1)p^{a(p)-1} & \text{if } a(p) = b(p) > c(p) = 0 \\ (p-1)(p-2)p^{2a(p)-2} & \text{if } a(p) = b(p) = c(p) > 0. \end{cases} \tag{22}$$

Given $m = \prod_{p|m} p^{a(p)}$, Corollary 10 (3) and Proposition 4 make it possible to easily observe all triples $m_1, m_2, m_3$ for which $E(m_1, m_2, m_3) \neq 0$. Namely, for any prime $p|m$ we first form a triple $T_p = \{p^{a(p)}, p^{a(p)}, p^{c(p)}\}$ with an arbitrary integer $c(p)$, $0 \leq c(p) \leq a(p)$. Then we form three sets $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ by distributing the elements of each $T_p$ arbitrarily by them (so that, there are three different ways if $c(p) < a(p)$, and only one way otherwise). Now we obtain the desired numbers $m_j$ as the products of the elements in the corresponding $\mathcal{M}_j$: $m_j := \prod_{p|m} p^b$, $j = 1, 2, 3$, where $p^b \in \mathcal{M}_j$. It follows that there are totally $\prod_{p|m}(3a(p) + 1)$ such ordered triples.

**Corollary 11** $f_r(m) := E(\underbrace{m, m, \ldots, m}_{r})$ *is the multiplicative function of $m$ determined by the formula*

$$f_r(p^a) = (p-1)p^{(r-1)(a-1)} h_r(p), \quad p \ \text{ prime}, \ a > 0. \tag{23}$$

$\square$

According to (23), $f_r(m)$, $m > 1$, can be represented as follows:

$$f_r(m) = m^{r-1} \prod_{p|m \ \text{prime}} \frac{(p-1)h_r(p)}{p^{r-1}}.$$

It follows that $E(\underbrace{m, m, \ldots, m}_{r}) = 0$ if and only if $m$ is even and $r$ is odd or $r = 1$ and $m > 1$. Besides [33],

$$E(m, m) = \phi(m) \tag{24}$$

(in other words, $f_2(m) = \phi(m)$). This bivariate instance of $E$ is the only non-trivial particular specimen of the function defined by formula (1) that this author managed to find in the number-theoretic literature.

Likewise, relying upon formula (21), we could investigate other general properties of the function $E(m_1, m_2, \ldots, m_r)$. One such property significant in topological applications, namely non-vanishing, will be considered below in Theorem 20 (in the particular case $m = p^a$ we have already encountered it in Corollary 3 (2) and later).

## 3. Combinatorial and Topological Motivations and Applications

### 3.1. Linear Congruences (Restricted Partitions)

The following statement provides a simple combinatorial interpretation for the function $E$ :

**Lemma 12** ([29]) *Let $M$ be a natural number and $m_1, m_2, \ldots, m_r$ divisors of $M$. Let $d_1 = \frac{M}{m_1}, d_2 = \frac{M}{m_2}, \ldots, d_r = \frac{M}{m_r}$. Then the number of solutions of the system of equations*

$$\left.\begin{aligned} x_1 + x_2 + \cdots + x_r \equiv & \quad 0 \pmod{M} \\ (x_1,\, M) = & \quad d_1 \\ & \vdots \\ (x_r,\, M) = & \quad d_r \end{aligned}\right\} \tag{25}$$

*in integers modulo $M$ does not depend on $M$ and is equal to $E = E(m_1, m_2, \ldots, m_r)$.*

One could use Lemma 12 to obtain another proof of Proposition 4.

### 3.2. Jordan's Function

A more profound combinatorial interpretation for the function $E(m_1, m_2, \ldots, m_r)$ is presented below in Theorem 14 and combines $E$ with the classical Jordan function. Recall [1, 10, 36] that the Jordan function of order $k$ is defined by

$$\phi_k(n) = \sum_{d|n} d^k \mu\Big(\frac{n}{d}\Big) \tag{26}$$

or, equivalently, as a multiplicative function, by

$$\phi_k(n) = n^k \prod_{p|n \text{ prime}} (1 - p^{-k}). \tag{27}$$

In particular, $\phi_1 = \phi$ (Euler's totient). Moreover, $\phi_0(1) = 1$ and $\phi_0(n) = 0$ for $n > 1$. It follows that

$$\phi(n) |\ \phi_k(n)$$

for $k \geq 1$.

As can be noticed in the literature on arithmetic functions, it is rather typical to research the Ramanujan sums and their generalizations jointly with the Jordan functions (cf., e.g., [10, 25]).

**Remark 13** It is worth observing that the Jordan function participates (expressly or implicitly) in reductive enumeration formulae for conjugacy classes of subgroups of some finitely generated groups. The oldest formula of this kind was derived by the author [20] and can be represented as follows:

$$N_{F_r}(n) = \frac{1}{n} \sum_{d|n} \phi_{(r-1)d+1}\Big(\frac{n}{d}\Big) M_{F_r}(d), \tag{28}$$

where $F_r$ is a free group of rank $r$, $M_G(n)$ denotes the number of subgroups of index $n$ in a group $G$ and $N_G(n)$ denotes the number of conjugacy classes of such subgroups. Subsequent similar formulae obtained by A. Mednykh in the 1980s and later for $n$-index subgroups of the fundamental groups of closed surfaces (see [26, 27, 19] and [22, Sect. 2 and 5]) also contain $\phi_k(n/d)$ as a factor, where $k$ is a linear function of $d$. A considerable rôle of the Jordan function in this context has been realized only recently.

### 3.3. Orbifolds and Cyclic Automorphism Groups of Riemann Surfaces

In order to describe more sophisticated applications of $E$ we need to recall some notions of algebraic topology. Generally for automorphisms of Riemann surfaces and orbifolds we refer to [37, Sect. 2] and [7, Ch. 3]. By an *orbifold* we mean here the quotient space of a closed orientable surface with respect to an action of a finite group of orientation-preserving automorphisms. Denote by $\mathrm{Orb}(\mathcal{S}_\gamma/G)$ the set of orbifolds arising as the quotient spaces by the actions of the group $G$ on a Riemann surface $\mathcal{S}_\gamma$ of genus $\gamma$. Any orbifold $\Omega \in \mathrm{Orb}(\mathcal{S}_\gamma/G)$ is a closed surface $\mathcal{S}_g$ with a finite set of distinguished branch points; a surface is a particular case of an orbifold, with the empty set of branch points. $\Omega$ is characterized by its *signature* $(g; m_1, m_2, \ldots, m_r)$, $r \geq 0$, where $2 \leq m_j \leq \ell$, $j = 1, 2, \ldots, r$, are the orders of its branch points and $|G| = \ell$. We denote $\Omega = \Omega(g; m_1, m_2, \ldots, m_r)$. Following Mednykh and Nedela [29], by *cyclic orbifolds* we mean orbifolds corresponding to the cyclic groups $G = \mathbb{Z}_\ell$.

Given an orbifold $\Omega = \Omega(g; m_1, m_2, \ldots, m_r)$, define the orbifold fundamental group $\pi_1(\Omega)$ to be the (Fuchsian) group generated by $2g + r$ generators $x_1, y_1, x_2, y_2, \ldots, x_g, y_g$ and $z_1, z_2, \ldots, z_r$ and satisfying the relations

$$\prod_{i=1}^{g}[x_i, y_i] \prod_{j=1}^{r} z_j = 1 \quad \text{and} \quad z_j^{m_j} = 1, \quad j = 1, \ldots, r, \tag{29}$$

where $[x, y] = xyx^{-1}y^{-1}$.

Actions of a group $G$ on a surface naturally correspond to epimorphisms from the fundamental group of the corresponding orbifold onto $G$.

### 3.4. Epimorphisms $\pi_1(\Omega) \to \mathbb{Z}_\ell$

An epimorphism from $\pi_1(\Omega)$ onto a cyclic group of order $\ell$ is called *order-preserving* if it preserves the orders of the periodical generators $z_j$, $j = 1, \ldots, r$. Equivalently, order-preserving epimorphisms have torsion-free kernels; in the literature, such epimorphisms are often called *smooth*, see, e.g., [24]. We denote by $\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)$ the set of order-preserving epimorphisms $\pi_1(\Omega) \to \mathbb{Z}_\ell$. Lemma 12 makes it possible to find their number $|\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)|$ :

**Theorem 14** [29] *The number of order-preserving epimorphisms from the fundamental group $\pi_1(\Omega)$ of the cyclic orbifold $\Omega = \Omega(g; m_1, m_2, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$ onto the cyclic group $\mathbb{Z}_\ell$ is expressed by the following formula:*

$$|\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)| = m^{2g}\phi_{2g}(\ell/m) \cdot E(m_1, m_2, \ldots, m_r), \tag{30}$$

*where $m = \mathrm{lcm}(m_1, m_2, \ldots, m_r)$ and $\phi_{2g}(m)$ is the Jordan function of order $2g$.*

In particular, for $r = 0$,

$$|\mathrm{Epi_o}(\pi_1(\Omega(g; \emptyset)), \mathbb{Z}_\ell)| = \phi_{2g}(\ell)$$

and for $g = 0$,

$$|\mathrm{Epi_o}(\pi_1(\Omega(0; m_1, m_2, \ldots, m_r)), \mathbb{Z}_\ell)| = E(m_1, m_2, \ldots, m_r)$$

if, and only if, $m = \ell$.

Here and subsequently we follow the convention that an arithmetic function vanishes for non-integer arguments (besides, we direct the reader's attention to a minor change in designations with respect to $E(m_1, \ldots, m_r)$: for various reasons, from now on, the letter $\ell$ is used instead of $M$).

Due to formula (30), we call $E(m_1, m_2, \ldots, m_r)$ the *orbicyclic* (multivariate arithmetic) function[1].

According to Corollary 9 (1) and formula (30), $|\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)|$ is divisible by $\phi(\ell)$, which is obvious combinatorially since $\phi(\ell)$ is the number of primitive elements (units) in the group $\mathbb{Z}_\ell$. Moreover, by (30),

$$|\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)| = \phi(\ell)$$

if and only if $g = 0$, $m = \ell$, and $m_1, m_2, \ldots, m_r$ satisfy the conditions of Corollary 9 (2). Of course in some cases, this can be established directly from (30). For example, if $g = 0$ and $r = 2$, then it is clear from (30) that $\Omega = \Omega(0, m_1, m_2)$ exists if and only if $m_1 = m_2 = m$, and in this case $\pi_1(\Omega) = \mathbb{Z}_m$. Now, order-preserving epimorphisms from $\mathbb{Z}_m$ onto $\mathbb{Z}_\ell$ exist if and only if $m = \ell$. Finally it is obvious that $|\mathrm{Epi_o}(\mathbb{Z}_m, \mathbb{Z}_m)| = \phi(m)$.

### 3.5. Riemann–Hurwitz Equation

The signature of any orbifold $\Omega(g; m_1, m_2, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/G)$ satisfies the famous Riemann–Hurwitz equation:

$$2 - 2\gamma = \ell\Big(2 - 2g - \sum_{j=1}^{r}\Big(1 - \frac{1}{m_j}\Big)\Big), \tag{RH}$$

where $\ell = |G|$. It is clear that in (RH),

$$g \leq \gamma. \tag{31}$$

The following well-known inequalities can be easily deduced *directly* from the Riemann–Hurwitz equation (cf. [44, 4.14.27]):

**Proposition 15** **(1)** *If* $\Omega(g; m_1, m_2, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/G)$, $|G| = \ell$, *then*

$$\ell \leq 4\gamma + 2 \quad \text{for} \quad g \geq 2. \tag{32}$$

*Thus (see* [14, 5]*),*

$$\ell \leq 4\gamma + 2 \quad \text{for} \quad \gamma \geq 2 \quad \text{(Wiman, 1895)}. \tag{33}$$

**(2)** *For all* $\gamma \geq 0$,

$$r \leq 2\gamma + 2. \tag{34}$$

[1]Curiously, a similar but different term (*orbicycle index polynomial*) has been introduced in [4], again in connection with orbit enumeration.

*Proof.* Denoting $f = \sum_{j=1}^{r}(1 - 1/m_j)$, (RH) can be rewritten as follows: $4\gamma + 2 = 2\ell(2g + f - 2) + 6$. If $2\ell(2g + f - 2) + 6 < \ell$, then $\ell(4g + 2f - 5) + 6 < 0$ and $4g + 2f - 5 < 0$, which is possible only for $g = 0, 1$ since $f \geq 0$.

Wiman's inequality (33) is immediate from (31) and (32).

Given $\gamma$, it is easy to see by (RH) that $r$ is maximal whenever $g = 0$ and $m_j = m = \ell = 2$ for all $j$, in which case $r = 2\gamma + 2$. □

**Remark 16** (1) There is a subtler restriction on $\ell$ [5]:

$$\phi(\ell) \leq 2\gamma \quad \text{for} \quad \gamma \geq 2. \tag{35}$$

For example, it excludes the order $\ell = 9$ for $\gamma = 2$, which satisfy (33). The bound (35) excludes also primes $\ell > 2\gamma + 1$.

(2) Wiman's bound (33) is also valid for $\gamma = 1$ provided that $r \geq 1$.

### 3.6. Harvey's Conditions for Automorphisms of Riemann Surfaces

**Theorem 17** [14] *There exists an orbifold $\Omega(g; m_1, m_2, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$, where $\ell \geq 2$ and $\gamma \geq 2$, if and only if its parameters satisfy the Riemann–Hurwitz equation* (RH) *and the following conditions:*

(H1) $\mathrm{lcm}(m_1, m_2, \ldots, m_{j-1}, m_{j+1}, \ldots, m_r) = m$ *for every $j = 1, 2, \ldots, r$, where $m = \mathrm{lcm}(m_1, m_2, \ldots, m_r)$ (the* lcm*-condition);*

(H2) *$m$ divides $\ell$, and $m = \ell$ if $g = 0$;*

(H3) *$r \neq 1$, and $r \geq 3$ if $g = 0$;*

(H4) *if $m$ is even, then the number of $m_j$ divisible by the maximal power of 2 dividing $m$ is even.*

**Remark 18** Theorem 17 is valid for $\gamma = 0, 1$ as well with the following condition that supplements (H3) [29]:

(H3a) $r = 2$ if $\gamma = 0$, and $r \in \{0, 3, 4\}$ if $\gamma = 1$.

### 3.7. Further Applications of the Function $E$

Note that $f_r(m)$ (formula (23)) gives rise to the number of solutions of the congruences (25) when $d_1 = d_2 = \ldots = d_r = 1$. This is a particular case of the system considered and solved in [34] and later in [9] (see also [10]), where an arbitrary $n$ stands in place of 0 in the right hand side of the congruence. This was done in terms of the Ramanujan sums as well. A related enumeration problem was considered in [41].

A simple enumerative proof can be obtained for another familiar result of Harvey that supplements the bound (33):

**Corollary 19** [14] $\mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_{4\gamma+2}) \neq \emptyset$ *for every* $\gamma$. *In other words, an orientable surface of genus* $\gamma$ *possesses an (orientation-preserving) automorphism of order* $\ell = 4\gamma + 2$.

*Proof.* Indeed, $\Omega = \Omega(0; 4\gamma+2, 2\gamma+1, 2) \in \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_{4\gamma+2})$ exists since the triple $m_1 = 4\gamma+2, m_2 = 2\gamma+1, m_3 = 2$ satisfies the Riemann–Hurwitz equation (RH) with $\ell = m = 4\gamma+2$ and $g = 0$. Now by (30),

$$\begin{aligned} |\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_{4\gamma+2})| &= E(4\gamma+2, 2\gamma+1, 2) \\ &= E(4\gamma+2, 4\gamma+2) = \phi(4\gamma+2) > 0. \end{aligned}$$

□

Note that $\Omega = \Omega(g; m_1, m_2, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$ exists if and only if $|\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)| \neq 0$ (see [14, 24]). Now we can derive an enumerative counterpart of Theorem 17.

**Theorem 20** *There exists an orbifold* $\Omega(g; m_1, m_2, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$, *where* $\ell \geq 2$ *and* $m_j \geq 2$ *for all* $j = 1, \ldots, r$, *if and only if its parameters satisfy the Riemann–Hurwitz equation* (RH) *and none of the following conditions is valid:*

(E1) $m \nmid \ell$;

(E2) $g = 0$ *and* $\ell > m$;

(E3) $s(p) = 1$ *for some odd prime* $p|m$;

(E4) $2|m$ *and* $s(2)$ *is odd.*

*Proof.* According to (30), the inequality $|\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)| \neq 0$ is equivalent to the condition

$$m^{2g}\phi_{2g}(\ell/m) \cdot E(m_1, m_2, \ldots, m_r) \neq 0.$$

Now, $m^{2g} \neq 0$ and $\phi_{2g}(d) = 0$ if and only if $m \nmid \ell$ or $g = 0$ and $d > 1$ (see (27)). It is clear from (21) and Corollary 3 (2), that $E_p(m_1, m_2, \ldots, m_r) = 0$ if and only if one of conditions (E3) and (E4) of the theorem holds. □

We can compare Theorem 20 with Harvey's Theorem 17. (E1) and (E2) are equivalent to (H2). It is obvious that (E3) is equivalent to (H1) and (E4) is equivalent to (H4). As to condition (H3), by Corollary 10 (1), $r \neq 1$. Now for $r = 2$, by Corollary 10 (2), we would have $m_1 = m_2 = m$ but this contradicts (RH) for $g = 0$ and $\gamma \neq 1$. Finally, $r = 0$ (in which case $m = 1$) is also impossible by (RH) for $g = 0$ and $\gamma \neq 0$.

**Remark 21** (Additional references.) A related but different enumeration problem with respect to automorphisms groups (not necessarily cyclic) of orientable surfaces has been considered and solved by C. Maclachlan and A. Miller [24]. A simple formula (with an implicit participation of the function $\phi_k(n)/\phi(n)$) that connects $|\mathrm{Epi}(\pi_1(\Omega), \mathbb{Z}_\ell)|$ with the number of homomorphisms $|\mathrm{Hom}(\pi_1(\Omega), \mathbb{Z}_\ell)|$ has been obtained by G. Jones [16]. Different number-theoretic aspects of automorphisms of surfaces partially related to Harvey's theorem were investigated by W. Chrisman [8]; cf. also the paper [38] by M. Sierakowski. In general, the enumerative approach developed here may have certain useful interactions with combinatorics of automorphism groups of Riemann surfaces investigated in numerous recent and older publications (cf., e.g., [12, 13, 43]).

### 3.8. Map Enumeration

A (topological) *map* is a proper cell embedding of a finite connected planar graph (generally with loops and multiple edges) in an orientable surface. A map is called *rooted* if an edge-end (a vertex-edge incidence pair), is distinguished in it as its root. Unlike maps without a distinguished root (called *unrooted* maps), rooted maps do not have non-trivial automorphisms. Unrooted maps are considered here up to orientation-preserving isomorphism; i.e., as maps on a surface with a distinguished orientation. For a general combinatorial and algebraic theory of maps see [11, 17]. Most of numerous results obtained so far for counting maps are concerned with rooted *planar* maps, that is, rooted maps on the sphere. Typically maps are counted with respect to the number of edges.

Let $(g; m_1, m_2, \ldots, m_r)$ be a signature of an orbifold and $b_i \geq 0,\ i \geq 2$, denote the number of branch points with $m_j = i$. That is, up to reordering, $(g; m_1, m_2, \ldots, m_r) = (g; \underbrace{2, 2, \ldots, 2}_{b_2}, \underbrace{3, \ldots, 3}_{b_3}, \ldots) = [g; 2^{b_2} 3^{b_3} \ldots \ell^{b_\ell}]$, where brackets indicate to the use of the parameters $b_i$ rather than $m_j$. In these terms, the following theorem is valid:

**Theorem 22** [29] *The number of unrooted maps with $n$ edges on a closed orientable surface $\mathcal{S}_\gamma$ of genus $\gamma$ is*

$$\begin{aligned} \Theta_\gamma(n) \;&=\; \frac{1}{2n} \sum_{\ell | 2n} \sum_{\Omega} |\mathrm{Epi_o}(\pi_1(\Omega), \mathbb{Z}_\ell)| \\ &\quad \times \sum_{s=0}^{b_2} \binom{2n/\ell}{s} \binom{\frac{n}{\ell} - \frac{s}{2} + 2 - 2g}{b_2 - s, b_3, \ldots, b_\ell} \mathcal{N}_g\left(\frac{n}{\ell} - \frac{s}{2}\right), \end{aligned} \tag{36}$$

*where $\ell$ runs over the divisors of $2n$, $\Omega = \Omega[g; 2^{b_2} \dots \ell^{b_\ell}]$ runs over the orbifolds in* $\mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$, $\mathcal{N}_g(n)$ *denotes the number of rooted maps with $n$ edges on an orientable surface of genus $g$ (with $\mathcal{N}_g(n) := 0$ if $n$ is not integer) and* $\binom{n}{b_2, b_3, \dots, b_\ell}$ *is the multinomial coefficient.*

A similar formula holds for non-isomorphic hypermaps [30, 31].

The orbicyclic function $E(m_1, m_2, \dots, m_r)$ participates in (36) through formula (30). As for counting arbitrary rooted maps on orientable surfaces we refer to the papers [3, 2] (for any genus $g$ there exists a closed formula, which is remarkably simple, sum-free, in the planar case and becomes more and more cumbersome as $g$ grows); see also [42].

Theorem 22 is a far-reaching generalization of the formula derived by the author for arbitrary unrooted planar maps [21] (see also [22]). Combinatorially, the proof of (36) follows the general reductive scheme elaborated for planar maps: relying upon Burnside's (orbit counting) lemma, the enumeration of unrooted maps (of a certain class) is reduced to the enumeration of their corresponding rooted *quotient maps* (i.e., orbifold maps) with respect to all possible orientation-preserving automorphisms of the underlying surface. In the case of arbitrary planar maps, quotient maps are merely almost arbitrary planar maps themselves, and we had nothing to do with the functions $|\mathrm{Epi_o}|$ and $E(m_1, m_2, \dots, m_r)$ as such (naturally, we made use of Euler's $\phi(m)$ in place of them). Burnside's lemma explains the role of cyclic groups in Theorem 22.

Given $\gamma$, the above formulae make it possible to list all admissible orbifolds $\Omega = \Omega(g; m_1, m_2, \dots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$. In view of (33) and (34), for $\gamma \geq 2$, the set $\bigcup_{\ell \geq 1} \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell)$ is finite. On the contrary, for $\gamma = 0, 1$, there are two infinite families of orbifolds, namely, $\mathrm{Orb}(\mathcal{S}_0/\mathbb{Z}_\ell) = \{\Omega(0; \ell, \ell)\}$ and $\mathrm{Orb}(\mathcal{S}_1/\mathbb{Z}_\ell) = \{\Omega(1; \emptyset)\}$, $\ell = 1, 2, \dots$ (of course, in (36) they are restricted to the finite set of divisors of $2n$). Moreover, there are four other orbifolds for $\gamma = 1$ with $g = 0$ and $\ell = 2, 3, 4, 6$. Denote

$$A(\gamma) := \Big| \bigcup_{\ell \geq 1} \mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_\ell) \Big|,$$

where $\mathrm{Orb}(\mathcal{S}_\gamma/\mathbb{Z}_1) = \{\Omega(\gamma; \emptyset)\}$, and let $A_g(\gamma)$ denote the cardinality of the subset of all orbifolds $\Omega(g; m_1, m_2, \dots, m_r)$ of genus $g$ in this set. In particular [5, 29, 18],

$$A(2) = 10, A(3) = 17, A(4) = 25, A_0(2) = 8, A_0(3) = 12, A_0(4) = 18.$$

Apparently these functions could be investigated in general based upon the restrictions considered above.

## 4. Concluding Remarks

(1) No publication on multivariate arithmetic functions constructed similarly to (1) (see formula (15)) was known to this author until recently when the preprint [32] appeared, where its author, N. Minami, considered (in quite a different context) the rational-valued function $\frac{1}{m}\sum_{k=1}^{m}(k,m_1)(k,m_2)\cdots(k,m_r)$ and derived a simple formula for its calculation.

(2) Addressing the polynomial $h_s(x)$ and formula (10) it is interesting to note that $h_s(x) = \frac{\chi(\mathbf{C}_s,x)}{x(x-1)}$, where $\chi(\mathbf{C}_s,x)$ denotes the *chromatic polynomial* of a cycle of length $s$ (concerning the chromatic polynomials of graphs we refer the reader, e.g., to Chapter IX of the monograph [40]) This observation, which belongs to Alexander Mednykh [28], can easily be explained due to the interpretation of $f_s(p) = (p-1)h_s(p)$ as the number of solutions of the equation $x_1 + x_2 + \cdots + x_s = 0$ in $\mathbb{Z}_p$ where all $x_i$ are nonzero (see the first paragraph of Subsection 3.7). Indeed, given an $s$-cycle $(v_1, v_2, \ldots, v_s)$, with any proper $p$-coloring $c : \{v_1, v_2, \ldots, v_s\} \to \mathbb{Z}_p$ subject to the restriction $c(v_1) = 0$ one can associate the solution $x_i = c(v_{i+1}) - c(v_i), i = 1, 2, \ldots, s$ $(v_{s+1} := v_1)$; and vice versa, $c(v_i) = \sum_{j=1}^{i-1} x_j$, $i = 1, 2, \ldots, s$, gives rise to the corresponding proper coloring of the cycle.

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