#A13 INTEGERS 10 (2010), 179-185

RECOUNTING THE NUMBER OF RISES, LEVELS, AND DESCENTS IN FINITE SET PARTITIONS

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Received: 10/12/09, Accepted: 12/17/09, Published: 4/20/10

Abstract

A finite set partition is said to have a *descent at* i if it has a descent at i in its canonical representation as a restricted growth function (and likewise for *level* and *rise*). In this note, we provide direct combinatorial proofs as well as extensions of recent formulas for the total number of rises, levels, and descents in all the partitions of an n-set with a prescribed number of blocks. In addition, we supply direct proofs of formulas for the number of partitions having a fixed number of levels.

1. Introduction

A *partition* of $[n] = \{1, 2, \ldots, n\}$ is a decomposition of $[n]$ into non-overlapping subsets $B_1, B_2, \ldots, B_k$, called *blocks*, which are listed in increasing order of their least elements ($1 \leqslant k \leqslant n$). We will represent a partition $\pi = B_1, B_2, \ldots, B_k$ in the canonical sequential form $\pi = \pi_1\pi_2 \cdots \pi_n$ such that $j \in B_{\pi_j}$, $1 \leqslant j \leqslant n$. Therefore, a sequence $\pi = \pi_1\pi_2 \cdots \pi_n$ over the alphabet $[k]$ represents a partition of $[n]$ with k blocks if and only if it is a *restricted growth function* of $[n]$ onto $[k]$ (see, e.g., [1, 4, 6, 7] for details). For instance, 123214154 is the canonical sequential form of the partition $\{1, 5, 7\}$, $\{2, 4\}$, $\{3\}$, $\{6, 9\}$, $\{8\}$ of [9]. Partitions will be identified with their corresponding canonical sequences throughout.

Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be any given partition represented by its canonical sequence. Given an integer $t > 1$, we say that π has a *t-rise at i* if $\pi_i < \pi_{i+1} < \cdots < \pi_{i+t-1}$, a *$t$-descent* if $\pi_i > \pi_{i+1} > \cdots > \pi_{i+t-1}$, and a *$t$-level* if $\pi_i = \pi_{i+1} = \cdots = \pi_{i+t-1}$. For example, if $t = 3$, then the partition 1222345322 of [10] has two 3-rises (at $i = 4$ and $i = 5$), one 3-level (at $i = 2$), and one 3-descent (at $i = 7$). The set of partitions of $[n]$ will be denoted by $\mathfrak{B}_n$ and the subset of partitions with k blocks by $\mathfrak{B}_{n,k}$. The cardinality of $\mathfrak{B}_{n,k}$ is the Stirling number of the second kind $S(n, k)$, following the notation of [5].

Mansour and Munagi [3] found formulas for the total number of 2-rises, 2-descents, and t-levels in all the members of $\mathfrak{B}_{n,k}$. More specifically, they showed

$$\# \text{ of 2-rises in } \mathfrak{B}_{n,k} = (k-1)S(n,k) + \sum_{j=2}^{k} \binom{j}{2} \sum_{i=k}^{n-2} j^{n-2-i} S(i,k), \tag{1}$$

$$\# \text{ of 2-descents in } \mathfrak{B}_{n,k} = \binom{k}{2} S(n-1,k) + \sum_{j=2}^{k} \binom{j}{2} \sum_{i=k}^{n-2} j^{n-2-i} S(i,k), \tag{2}$$

and

$$\# \text{ of } t\text{-levels in } \mathfrak{B}_{n,k} = kS(n-t+1,k) + \sum_{j=1}^{k} j \sum_{i=k}^{n-t} j^{n-t-i} S(i,k). \tag{3}$$

(Formula (3) is a slight correction over that appearing in [3]). Formulas (1)–(3) were obtained algebraically by partially differentiating certain joint generating functions and the question of finding direct combinatorial proofs is raised. Here, we provide the requested combinatorial proofs of (1)–(3). The arguments may be extended to yield formulas for the total number of t-descents as well as for the total number of 3-rises. In addition, we supply direct proofs of formulas which count the members of $\mathfrak{B}_{n,k}$ having a fixed number of levels.

2. Proofs of (1)–(3)

Proof of (1)*:* Upon replacing i by $n-i$ in the inner sum of the right side of (1), we'll show that the total number of 2-rises in all members of $\mathfrak{B}_{n,k}$ is given by

$$(k-1)S(n,k) + \sum_{j=2}^{k} \binom{j}{2} \sum_{i=2}^{n-k} j^{i-2} S(n-i,k).$$

First consider the rises occurring each time a new block is started; there are clearly $k-1$ such rises within each member of $\mathfrak{B}_{n,k}$ and hence their total is $(k-1)S(n,k)$. So to complete the proof we must show that the number of rises caused by two members of $[n]$ belonging to different blocks where neither member is the smallest element of its block is given by the double sum above. We'll call such rises *non-trivial*.

Given i and j, where $2 \leqslant i \leqslant n-k$ and $2 \leqslant j \leqslant k$, consider all the members of $\mathfrak{B}_{n,k}$ which may be decomposed uniquely as

$$\pi = \pi' j \alpha \beta, \tag{4}$$

where π' is a partition with $j-1$ blocks, α is a word of length i in the alphabet $[j]$ whose last two letters form a rise, and β is possibly empty. For example, if $i=5$, $j=3$, and $\pi = 122323113342 \in \mathfrak{B}_{12,4}$, then $\pi' = 122$, $\alpha = 23113$, and $\beta = 342$. The total number of non-trivial rises can then be obtained by finding the number of partitions which may be expressed as in (4) for each i and j and then summing over all possible values of i and j. And there are $\binom{j}{2} j^{i-2} S(n-i,k)$ members of $\mathfrak{B}_{n,k}$ which may be expressed as in (4) since there are j^{i-2} choices for the first $i-2$ letters of α, $\binom{j}{2}$ choices for the final two letters in α (as the last letter must exceed its predecessor), and $S(n-i,k)$ choices for the remaining letters $\pi' j \beta$ which necessarily constitute a partition of an $(n-i)$-set into k blocks. $\square$

Proof of (2)*:* The sum on the right side of (2) above counts all 2-descents where neither element is the smallest (= first) member of its block, upon reasoning as above. To show that $\binom{k}{2} S(n-1,k)$ counts all descents at i for some i where i is the smallest member of its block in some member of $\mathfrak{B}_{n,k}$, first choose two numbers a and b in $[k]$, where $a<b$. Given $\lambda \in \mathfrak{B}_{n-1,k}$, let m denote the smallest member of block b. Increase all members of $[m+1,n-1] = \{m+1, m+2, \ldots, n-1\}$ in λ by one (leaving them within their blocks) and then add $m+1$ to block a. This produces a descent between the first element of block b and an element of block a within some member of $\mathfrak{B}_{n,k}$. For example, if $n=7, k=4, a=1$, and $b=3$, then $\lambda = \{1,5\},\{2\},\{3,4\},\{6\} \in \mathfrak{B}_{6,4}$ would give rise to the descent between 3 and 4 in $\{1,4,6\},\{2\},\{3,5\},\{7\} \in \mathfrak{B}_{7,4}$. $\square$

Proof of (3)*:* Reasoning as above, the sum on the right side of (3) counts all t-levels which do *not* start with a number that is the smallest element of a block, upon replacing i with $n-t+1-i$ in the inner sum. (Decompose a partition π exactly as in (4) above except now α is a j-ary word of length $i+t-1$ whose final t letters are the same.) To see that there are $kS(n-t+1,k)$ t-levels that do start with the smallest element of a block, let $\lambda \in \mathfrak{B}_{n-t+1,k}$ and suppose ℓ is the smallest element of block a, where $1 \leqslant a \leqslant k$. Increase all members of $[\ell+1, n-t+1]$ by $t-1$ within λ and add all members of $[\ell+1, \ell+t-1]$ to block a to obtain a t-level at ℓ. $\square$

Remark 1. One may also count the total number of t-levels in $\mathfrak{B}_{n,k}$ by considering those t-levels caused by each set of the form $[i, i+t-1]$, $1 \leqslant i \leqslant n-t+1$. Note that there are $S(n-t+1,k)$ t-levels in $\mathfrak{B}_{n,k}$ caused by a set $[i,i+t-1]$, upon regarding it as a single element, and hence $(n-t+1)S(n-t+1,k)$ t-levels in all. Equating this with the right-hand side of (3) above, re-indexed, and replacing

$n-t+1$ by m yields the Stirling number recurrence

$$(m-k)S(m,k)=\sum_{j=1}^{k}\sum_{i=1}^{m-k} j^i S(m-i,k), \tag{5}$$

which seems to be new. Finally, trying to count the total number of 2-rises and 2-descents in a similar manner by considering those caused by each doubleton $\{i,i+1\}$ does not seem to yield any interesting alternative formulas to (1) and (2).

Remark 2. A straightforward generalization of the proof of (2) yields

$$\#\text{ of } t\text{-descents in } \mathfrak{B}_{n,k}=\binom{k}{t}S(n-t+1,k)+\sum_{j=t}^{k}\binom{j}{t}\sum_{i=t}^{n-k} j^{i-t}S(n-i,k) \tag{6}$$

for any $t>1$.

3. An Explicit Formula for 3-Rises

In this section, we establish an explicit formula for the total number of 3-rises taken over all the members of $\mathfrak{B}_{n,k}$. If $\pi\in\mathfrak{B}_n$ and $i\in[n]$, we will say that i is *minimal* if i is the smallest element of a block within π, i.e., if the $i^{\underline{th}}$ slot of the canonical representation of π corresponds to the first occurrence of a letter. Unlike the case for t-levels and t-descents, there does not seem to be a simple explicit formula for the total number of t-rises within all the members of $\mathfrak{B}_{n,k}$ for general t.

Proposition 3. *The number of* 3*-rises in all the partitions of* $[n]$ *with* k *blocks is given by*

$$\sum_{r=1}^{n-2}\sum_{a,b\geqslant 0}\binom{n-r-2}{a,b}S(n-2-a-b,k-2)+\sum_{j=3}^{k}\binom{j}{3}\sum_{i=3}^{n-k} j^{i-3}S(n-i,k)$$
$$+\sum_{j=2}^{k-1}\binom{j}{2}\sum_{i=3}^{n-k+1} j^{i-3}\left[(k-j)S(n-i,k)+S(n-i,k-1)\right].$$

Proof. Suppose a 3-rise occurs at r in $\pi\in\mathfrak{B}_{n,k}$ for some $r\geqslant 1$. We consider three cases. First assume that both $r+1$ and $r+2$ are minimal. The first sum above then counts all such rises, upon choosing a elements of $[r+3, n]$ to go in the block with $r+1$ and b additional elements of $[r+3, n]$ to go in the block with $r+2$ and then partitioning the remaining $n-2-a-b$ members of $[n]$ into $k-2$ blocks. The second sum above counts all 3-rises where none of $\{r, r+1, r+2\}$ are minimal, by the same reasoning used to count non-trivial 2-rises in the prior section.

To complete the proof, we must show that the third sum counts all 3-rises where $r+2$ is minimal, but $r+1$ is not. Suppose i and j are given, where $3 \leqslant i \leqslant n-k+1$ and $2 \leqslant j \leqslant k-1$. Consider those members of $\mathfrak{B}_{n,k}$ which may be decomposed uniquely as

$$\pi = \pi' j \alpha j + 1 \beta, \tag{7}$$

where π' is a partition with $j-1$ blocks, α is a j-ary word of length $i-1$ whose last two letters form a 2-rise, and β is possibly empty. So counting all possible 3-rises at some r where $r+2$ is minimal but $r+1$ is not is equivalent to counting all members of $\mathfrak{B}_{n,k}$ which may be expressed as in (7) for each i and j and then summing over all possible values of i and j.

So it suffices to show that there are $\binom{j}{2} j^{i-3} \left[(k-j)S(n-i,k) + S(n-i,k-1)\right]$ members of $\mathfrak{B}_{n,k}$ which may be expressed as in (7). Suppose first that the $(j+1)^{\underline{st}}$ block of π is a singleton. Then there are j^{i-3} choices for the first $i-3$ letters of α, $\binom{j}{2}$ choices for the last two letters of α, and $S(n-i,k-1)$ choices for the remaining letters $\pi' j \beta$ of π which together necessarily constitute a partition of an $(n-i)$-set into $k-1$ blocks, which explains the second term.

Now suppose that the $(j+1)^{\underline{st}}$ block of π is not a singleton and say that π' is a partition of $[t]$ for some $t \geqslant 1$. Then the same reasoning as before applies except now the remaining letters $\pi' j \beta$ constitute a partition having k blocks with the number $t+i+1 \in [n]$ to be added to one of the final $k-j$ blocks of this partition. $\square$

4. Partitions with a Fixed Number of Levels

In [3], formulas were given which counted the members of $\mathfrak{B}_{n,k}$ having a fixed number of levels. For example, there are $\binom{n-1}{r} S(n-r-1, k-1)$ members of $\mathfrak{B}_{n,k}$ with exactly r occurrences of 2-levels. When $r = 0$, this reduces to the well-known fact that there are $S(n-1, k-1)$ members of $\mathfrak{B}_{n,k}$ where no block contains two consecutive integers (see [2] for a bijective proof). The $r = 0$ case implies the general case as follows. First select r members of $\{2, 3, \ldots, n\}$ to be the second numbers in r 2-levels. Partition the remaining $n-r$ members of $[n]$ into k blocks so that no two "consecutive" elements go in the same block, which can be done in $S(n-r-1, k-1)$ ways. Then add the chosen r members of $[n]$ to the appropriate blocks so as to create r 2-levels.

On the other hand, there do not appear to be simple formulas for the number of partitions of $[n]$ with k blocks having r 2-rises or r 2-descents for general r. The following proposition gives the number of members of $\mathfrak{B}_{n,k}$ having r 3-levels and was established in [3] algebraically using generating functions. Here, we provide a combinatorial proof.

Proposition 4. *The number of partitions of $[n]$ with k blocks without three levels is given by*

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} S(n-j-1, k-1),$$

and the number of partitions of $[n]$ with k blocks with r occurrences of 3-levels, $r \geqslant 1$, is given by

$$\sum_{v=1}^{r} \sum_{j=v}^{\lfloor \frac{n-r}{2} \rfloor} \binom{r-1}{v-1} \binom{j}{v} \binom{n-r-j}{j} S(n-r-j-1, k-1).$$

Proof. We prove only the second formula, rewritten as

$$\sum_{v=1}^{r} \binom{r-1}{v-1} \sum_{j=0}^{\lfloor \frac{n-r}{2} \rfloor - v} \binom{n-r-j-v}{v} \binom{n-r-j-2v}{j} S(n-r-j-v-1, k-1),$$

upon replacing j by $j+v$ in the inner sum and using trinomial revision. Given $\lambda \in \mathfrak{B}_{n,k}$ with exactly r 3-levels, let v denote the number of maximal sets of consecutive integers of length at least three belonging to a single block in λ and let $S_1, S_2, \ldots, S_v$ denote these sets. For example, if $\lambda = 122223331222123 \in \mathfrak{B}_{15,3}$, then $S_1 = \{2,3,4,5\}$, $S_2 = \{6,7,8\}$, $S_3 = \{10,11,12\}$, and $v = 3$. Let j be the number of 2-levels in λ involving two members of $[n] - \bigcup_{i=1}^{v} S_i$. Let $x_i = |S_i| - 3$, $1 \leqslant i \leqslant v$. Since there are to be r 3-levels and since all 3-levels must involve members of some S_i, we have $x_1 + x_2 + \cdots + x_v = r - v$, where $x_i \geqslant 0$ $\forall i$. So there are $\binom{(r-v)+v-1}{v-1} = \binom{r-1}{v-1}$ choices for the cardinalities of the S_i and $(r-v) + 3v = r + 2v$ members of $[n]$ used to form the S_i.

Within λ, we will call any member of $[n]$ not involved in a t-level for any $t > 1$ a *1-level*. Outside of the S_i, there are to be exactly j 2-levels and hence

$n-r-2v-2j$ 1-levels, which we will regard as doubleton and singleton sets. Therefore, there are $\binom{n-r-2v-j}{j}$ choices concerning the relative order of the 1- and 2-levels outside of the S_i. Once this is determined, one needs to decide how the $n-r-j-2v$ 1- and 2-levels are to be arranged relative to the v sets S_i, which is equivalent to finding the number of nonnegative integer solutions to $y_1+y_1+\cdots+y_{v+1}=n-r-j-2v$, whence there are $\binom{n-r-j-v}{v}$ ways.

Finally, the 1- and 2-levels and the sets S_i must be arranged in a partition having k blocks without any levels (so as not to create any additional 3-levels). Since there are $S(n-1,m-1)$ partitions of an n-object set having m blocks and no 2-levels, it follows that there are $S(n-r-j-v-1,k-1)$ ways to arrange these $n-r-j-v$ items in a partition having k blocks (one compares 1- and 2-levels and the sets S_i by comparing smallest elements). Summing over i and j yields all members of $\mathfrak{B}_{n,k}$ having r 3-levels. $\square$

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