ALTERNATIVE PROOFS ON THE 2-ADIC ORDER OF STIRLING NUMBERS OF THE SECOND KIND

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Abstract

An interesting 2-adic property of the Stirling numbers of the second kind $S(n,k)$ was conjectured by the author in 1994 and proved by De Wannemacker in 2005: $\nu_2(S(2^n,k)) = d_2(k)-1, 1 \le k \le 2^n$. It was later generalized to $\nu_2(S(c2^n,k)) = d_2(k)-1, 1 \le k \le 2^n, c \ge 1$ by the author in 2009. Here we provide full and two partial alternative proofs of the generalized version. The proofs are based on non-standard recurrence relations for $S(n,k)$ in the second parameter and congruential identities.

1 Introduction

The study of p-adic properties of Stirling numbers of the second kind offers many challenging problems. Let k and n be positive integers, and let $d_2(k)$ and $\nu_2(k)$ denote the number of ones in the binary representation of k and the highest power of two dividing k, respectively. Lengyel [5] proved that

$$\nu_2(S(2^n,k)) = d_2(k) - 1 \tag{1}$$

for all sufficiently large n (e.g., $k-2 \le n$), and conjectured that $\nu_2(S(2^n,k)) = d_2(k)-1$, for all $k : 1 \le k \le 2^n$ which was proved in

Theorem 1. ([3], Theorem 1) *Let $k, n \in \mathbb{N}$ and $1 \le k \le 2^n$. Then we have*

$$\nu_2(S(2^n,k)) = d_2(k) - 1. \tag{2}$$

At the very heart of the proof, there is an appealing recurrence for the Stirling numbers of the second kind involving a double summation

$$S(n+m,k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(n,k-i) S(m,j). \tag{3}$$

The generalization of Theorem 1 and De Wannemacker's proof can be found in [7].

Theorem 2. ([7]) *Let* $c, k, n \in \mathbb{N}$ *and* $1 \le k \le 2^n$. *Then*

$$\nu_2(S(c2^n, k)) = d_2(k) - 1. \tag{4}$$

In this paper we use Kummer's theorem on the p-adic order of binomial coefficients.

Theorem 3. (Kummer (1852)) *The power of a prime* p *that divides the binomial coefficient* $\binom{n}{k}$ *is given by the number of carries when we add* k *and* $n-k$ *in base* p. *In another form,* $\nu_p\left(\binom{n}{k}\right) = \frac{n-d_p(n)}{p-1} - \frac{k-d_p(k)}{p-1} - \frac{n-k-d_p(n-k)}{p-1} = \frac{d_p(k)+d_p(n-k)-d_p(n)}{p-1}$ *with* $d_p(n)$ *being the sum of the digits of* n *in its base* p *representation. In particular,* $\nu_2\left(\binom{n}{k}\right) = d_2(k) + d_2(n-k) - d_2(n)$ *represents the carry count in the addition of* k *and* $n-k$ *in base 2.*

We will also need

Theorem 4. ([3], Theorem 3) *Let* $k, n \in \mathbb{N}$ *and* $1 \le k \le n$. *Then*

$$\nu_2(S(n, k)) \ge d_2(k) - d_2(n). \tag{5}$$

This can be proven by an easy induction proof. Note that in general,

Theorem 5. ([6]) *For every prime* $p \ge 3$ *and integer* $k\colon 1 \le k \le n-1$,

$$\nu_p(S(n, k)) \ge \frac{d_p(k) - d_p(n) - (n-k)(p-2)}{p-1} + 1.$$

The main goal of this paper is to suggest alternative methods for proving 2-adic properties of the Stirling numbers of the second kind. In Section 2 we discuss some partial proofs of Theorem 2 while full proofs of Theorems 1 and 2 are presented in Section 3. It is remarkable that both known proofs of Theorems 1 and 2 are based on recurrence relations on $S(n, k)$ in the second parameter such as (3) and (12) or its generalization (13).

2 Preliminaries and Partial Answers

In this section we provide alternative partial proofs of Theorem 2 for two sets of values of k that are smaller than the full range $\{1, 2, \ldots, 2^n\}$. The proofs and how the tools, identity (6) and Theorem 8, are used seem to be new.

The two sets are defined by $k \leq n$ and $d_2(k) \leq \nu_2(k)$. Their respective cardinalities are n and the $(n+1)$st Fibonacci number F_{n+1}. In fact, by counting all values k with a fixed number $s = d_2(k)$ of ones in their binary representations (so that $s \leq \nu_2(k)$), we find that there are $\binom{n-s}{s}$ such ks if $s \geq 2$ and $\binom{n}{1}$ powers of two otherwise. We get that

$$|\{k \mid 1 \leq k \leq 2^n \quad \text{and } d_2(k) \leq \nu_2(k)\}| \\ = \binom{n}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \binom{n-4}{4} + \cdots = F_{n+1}, \text{if } n \geq 1.$$

Let $\pi(k; p^N)$ denote the minimum period of the sequence of Stirling numbers $\{S(n,k)\}_{n \geq k} \bmod p^N$. Kwong [4] proved the following.

Theorem 6. ([4]) *For $k > max\{4,p\}, \pi(k;p^N) = (p-1)p^{N+l_p(k)-2}$, where $p^{l_p(k)-1} < k \leq p^{l_p(k)}$, i.e., $l_p(k) = \lceil \log_p k \rceil$.*

Based on the periodicity property and Euler's theorem we can obtain:

Theorem 7. ([5], Theorem 2) *Let c and n be non-negative integers, with c odd. If $1 \leq k \leq n+2$ then $\nu_2(k!S(c2^n,k)) = k-1$, i.e., $\nu_2(S(c2^n,k)) = d_2(k) - 1$.*

The latter theorem can be proven in a slightly weakened form by replacing $k \leq n+2$ with $k \leq n$ as is shown in the following proof.

Proof. We use the identity (cf. [8, identity (188) on p. 496])

$$\sum_{d|N} \mu(d) k! S\left(\frac{N}{d}, k\right) \equiv 0 \bmod N, \tag{6}$$

for any positive integers k and N, and μ denoting the Moebius μ-function. Indeed, we set $N = 2^n, n \geq k$, and get that

$$k!S(2^n,k) - k!S(2^{n-1},k) \equiv 0 \bmod 2^n. \tag{7}$$

As above, by periodicity and Euler's theorem, we know that $\nu_2(k!S(2^n,k)) = k-1$ for any sufficiently large n, and thus, by (7), we immediately have that it holds for any $n \geq k$. This argument easily generalizes to $S(c2^n,k)$ with any $c \geq 1$ odd; however, there will be $2^{\omega(c)+1}$ terms of the form $\pm k!S(c'2^n,k)$ or $\pm k!S(c'2^{n-1},k)$ in (7) where $c' \geq 1$ is a divisor of c and $\omega(c)$ denotes the number of different prime factors of c. The proof can be completed by an induction on $\omega(c)$. ☐

Another special case can be treated by the following theorem proved by Chan and Manna [2] in a recent paper.

Theorem 8. ([2], Theorem 4.2) *Let $a, m,$ and n be positive integers with $m \geq 3$ and $n \geq a2^m + 1$. Then*

$$S(n, a2^m) \equiv a2^{m-1}\binom{\lfloor \frac{n-1}{2} \rfloor - a2^{m-2} - 1}{\lfloor \frac{n-1}{2} \rfloor - a2^{m-1}} + \frac{1 + (-1)^n}{2}\binom{\frac{n}{2} - a2^{m-2} - 1}{\frac{n}{2} - a2^{m-1}} \bmod 2^m. \tag{8}$$

This guarantees that we can determine $\nu_2(S(2^n, k))$ for any k with at least as many zeros at the end of its binary representation as the number of ones in it.

Theorem 9. *Let $k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$ with $\max\{3, d_2(k)\} \leq \nu_2(k)$. Then $\nu_2(S(2^n, k)) = d_2(k) - 1$.*

Proof. We replace n by 2^n in Theorem 8 and write k as $k = a2^m$ with some integer $a > 0$. We assume that $m \geq 3$ and $m \geq d_2(a)$, and $k = a2^m \leq 2^n$, i.e., $n \geq n_0 = \lceil \log_2(a2^m) \rceil$. Without loss of generality, we can assume that a is odd and $m = \nu_2(k)$; otherwise, we rewrite $a2^m$ as $a'2^{m'}$ with a' odd and $m' > m \geq d_2(a)$. Both (9) and (10) hold with a' and m' while n and n_0 are kept unchanged.

Now we prove that

$$S(2^n, a2^m) \equiv \binom{2^{n-1} - a2^{m-2} - 1}{2^{n-1} - a2^{m-1}} \bmod 2^m \tag{9}$$

and

$$\nu_2(S(2^n, a2^m)) = d_2(a) - 1 \tag{10}$$

by applying Theorem 8. Note that $\lfloor \frac{2^n-1}{2} \rfloor - a2^{m-2} - 1$ is even while $\lfloor \frac{2^n-1}{2} \rfloor - a2^{m-1}$ is odd; thus, there is guaranteed at least one carry in the application of Theorem 3 to the binomial coefficient of the first term in (8). This proves (9) which can be further evaluated by the last part of Theorem 3. In fact, we get that

$$\begin{aligned}
\nu_2(S(2^n, a2^m)) &= d_2(2^{n-1} - a2^{m-1}) + d_2(a2^{m-2} - 1) - d_2(2^{n-1} - a2^{m-2} - 1) \\
&= (n - n_0 + (l_2(a) - d_2(a) - \nu_2(a) + 1)) \\
&\quad + (d_2(a) + \nu_2(a) - 1 + m - 2) \\
&\quad - (n - n_0 - 1 + (m - 2) + 1 + (l_2(a) - d_2(a) + 1)) \\
&= d_2(a) - 1 < m
\end{aligned} \tag{11}$$

with $l_2(a) = \lceil \log_2(a) \rceil$. □

Note that the above proof does not require any induction (although the proof of Theorem 8 uses induction). In addition, we can generalize the proof to obtain

Theorem 10. *Let* $c, k, n \in \mathbb{N}$ *and* $1 \le k \le 2^n$ *with* $\max\{3, d_2(k)\} \le \nu_2(k)$. *Then* $\nu_2(S(c2^n, k)) = d_2(k) - 1$.

Proof. In fact, $k = a2^m \le 2^n$ implies that the nonzero binary digits of $c2^n$ and $a2^m$ avoid each other (perhaps with the exception of the rightmost one in $c2^n$ when $a = 1$ and c is odd) and thus, (11) can be easily revised:

$$\begin{aligned}
\nu_2(S(c2^n, a2^m)) &= d_2(c2^{n-1} - a2^{m-1}) + d_2(a2^{m-2} - 1) - d_2(c2^{n-1} - a2^{m-2} - 1) \\
&= (n - n_0 + (l_2(a) - d_2(a) - \nu_2(a) + 1) + d_2(c) + \nu_2(c) - 1) \\
&\quad + (d_2(a) + \nu_2(a) - 1 + m - 2) \\
&\quad - \big(n - n_0 - 1 + (m - 2) + 1 + (l_2(a) - d_2(a) + 1) \\
&\qquad\qquad + d_2(c) + \nu_2(c) - 1\big) \\
&= d_2(a) - 1 < m
\end{aligned}$$

□

3 Main Result: Alternative Proofs of Theorems 1 and 2

We now turn to another approach due to Agoh and Dilcher [1]. They developed an alternative recurrence relation for $S(n + m, k)$ which relates this quantity to terms involving $S(n, k')S(m, k - k')$ by means of a single summation rather than a double summation as in (3).

Theorem 11. ([1]) *For* $r \ge \max\{k_1, k_2\} + 2$, *we have that*

$$\begin{aligned}
&\frac{k_1!k_2!(r-1)!}{(k_1 + k_2 + 1)!} S(k_1 + k_2 + 2, r) \\
&\qquad = \sum_{i=1}^{r-1} (i-1)!(r-i-1)! S(k_1 + 1, i) S(k_2 + 1, r - i). \qquad (12)
\end{aligned}$$

The paper [1] also contains a generalization of this theorem to $s \ge 2$ factors involving Stirling numbers on the right-hand side in a summation with $s - 1$ summation indices. Theorem 11 is a special case with $s = 2$.

We will use the generalization of (12) to $r \geq 1$, cf. [1, identity (6)]. It includes a correction term involving Bernoulli numbers

$$\begin{aligned}\frac{(k-1)!(m-1)!(r-1)!}{(k+m-1)!}&S(k+m,r)\\ &= \sum_{i=1}^{r-1}(i-1)!(r-i-1)!S(k,i)S(m,r-i)\\ &\quad + (r-1)!\sum_{j=r}^{k+m-1}\left((-1)^m\binom{k-1}{j-1}+(-1)^k\binom{m-1}{j-1}\right)\frac{B_{k+m-j}}{k+m-j}S(j,r)\end{aligned} \tag{13}$$

with B_n being the nth Bernoulli number.

Now we present an alternative proof of Theorem 1.

Proof of Theorem 1. We prove by induction on n. The base case with $n = 0$ is trivial. We consider the equivalent form $\nu_2(k!S(2^n, k)) = k - 1$ of identity (1). Let us assume that $\nu_2(k!S(2^t, k)) = k - 1$ for any integers t and k such that $1 \leq t \leq n$ and $1 \leq k \leq 2^t$. We prove the statement for $t = n + 1$. We write k in its binary representation $k = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_{d_2(k)}}$ with $0 \leq b_1 < b_2 < \cdots < b_{d_2(k)}$. We have two cases according whether $k \geq 2^n + 1$ or not.

`Case 1.` First let us assume that

$$2^n < k \leq 2^{n+1}. \tag{14}$$

The assumption yields that $b_{d_2(k)} = n$ except for $k = 2^{n+1}$.

We use Theorem 11 with $k_1 = k_2 = 2^n - 1$, $r \geq 2^n + 1$, and switching from the notation r to k. After slightly rewriting (12), we obtain

$$(k-1)!S(2^{n+1},k) = \frac{(2^{n+1}-1)!}{(2^n-1)!^2}\sum_{i=1}^{k-1}\frac{1}{i(k-i)}i!S(2^n,i)\ (k-i)!S(2^n,k-i). \tag{15}$$

With $N = 2^{n+1}$, the first factor on the right-hand side of (15) is

$$\frac{(N-1)!}{(\frac{N}{2}-1)!^2} = \binom{N-1}{\frac{N}{2}}\frac{N}{2}$$

and there is no carry in the addition of $N/2$ and $N/2 - 1$. This yields an overall 2-adic order of n for the whole expression.

We have two subcases. If k is odd then we note that $i(k-i)$ in the denominator of (15) can decrease the 2-adic order, and the unique largest decrement results from setting i or $k-i$ to $2^{b_{d_2(k)}}$. By the inductive hypothesis, the last four factors at the end of (15) contribute $(i-1)+(k-i-1)=k-2$ to the 2-adic order. Hence, we get that

$$\begin{aligned}\nu_2(k(k-1)!S(2^{n+1},k)) &= \nu_2(k)+n-b_{d_2(k)}+1+(k-2)\\ &= n+k-1-b_{d_2(k)} = k-1. \qquad (16)\end{aligned}$$

If k is even and $k \neq 2^{n+1}$ then the factor $i(k-i)$ in the denominator of (15) decreases the 2-adic order the most if we set i or $k-i$ to $2^{b_{d_2(k)}}$ which yields that the other factor is an odd multiple of $2^{\nu_2(k)}$. No other pair $(i, k-i)$ can reach this decrement. If $i = k/2$ then the corresponding term occurs only once, and the decrement is $2(\nu_2(k)-1) \leq b_{d_2(k)} + \nu_2(k) - 2$. Thus, the right-hand side of (16) changes, and we obtain

$$\begin{aligned}\nu_2(k!S(2^{n+1},k)) &= \nu_2(k)+n-\big(b_{d_2(k)}+\nu_2(k)\big)+1+(k-2)\\ &= n+k-1-b_{d_2(k)} = k-1. \qquad (17)\end{aligned}$$

For $k = 2^{n+1}$, since the factor $i(k-i)$ decreases the 2-adic order the most if we set both i and $k-i$ to $2^{b_{d_2(k)}-1} = 2^n$, we get

$$\begin{aligned}\nu_2(k!S(2^{n+1},k)) &= \nu_2(k)+n-\big(b_{d_2(k)}-1+\nu_2(k)-1\big)+(k-2)\\ &= n+k-b_{d_2(k)} = k-1.\end{aligned}$$

`Case 2.` Now we assume that $k \leq 2^n$ and have two subcases. First we discuss the case with $k < 2^n$ provided that k is not a power of two then we consider the case in which $k = 2^m, m \leq n$.

Since now $k \leq 2^n$, we need the correction term in (13) which leads to the revised version of (15)

$$\begin{aligned}k(k-1)!S(2^{n+1},k) &= k\frac{(2^{n+1}-1)!}{(2^n-1)!^2}\sum_{i=1}^{k-1}\frac{1}{i(k-i)}i!S(2^n,i)\ (k-i)!S(2^n,k-i)\\ &\quad + k(k-1)!\frac{(2^{n+1}-1)!}{(2^n-1)!^2}\sum_{j=k}^{2^n}2\binom{2^n-1}{j-1}\frac{B_{2^{n+1}-j}}{2^{n+1}-j}S(j,k) \qquad (18)\end{aligned}$$

by setting k and m to 2^n and switching from r to k in (13). We proceed similarly to (16) and (17), but this time the correction term in (18) will determine the exact

2-adic order. Clearly, the factor $\binom{2^n-1}{j-1}$ in the correction term is odd for any $j, k \leq j \leq 2^n$, by Theorem 3.

If $k < 2^n$ then $b_{d_2(k)} \leq n-1$. If k is not a power of two then the right-hand sides of (16) and (17) become $n+k-1-b_{d_2(k)} \geq k$. Therefore, the first term on the right-hand side of (18) contributes an integer multiple of 2^k to (18). On the other hand, the correction term of (18) will guarantee that $\nu_2(k!S(2^{n+1},k))$ stays at $k-1$. Indeed, the 2-adic order of the jth term of the correcting sum is at least $(k-d_2(k))+n+(1+\nu_2(B_{2^{n+1}-j})-\nu_2(j))+(d_2(k)-d_2(j)) \geq n+(k-1)+(1-\nu_2(j)-d_2(j)) = n+(k-1)-d_2(j-1)$ by Theorem 4 and the fact that $\nu_2(B_n) \geq -1$. For the smallest possible value we have that

$$\min_{k \leq j \leq 2^n} n+(k-1)-d_2(j-1) = k-1 \tag{19}$$

taken uniquely at $j = 2^n$. In this case the two inequalities above become equalities since $\nu_2(S(2^n,k)) = d_2(k)-1$ and $\nu_2(B_{2^n}) = -1$. Thus, $\nu_2(k!S(2^{n+1},k)) = k-1$.

We are left with the subcases in which k is a power of two. The statement is trivially true for $k=1$. If $k = 2^m$ with $1 \leq m \leq n$ then $b_{d_2(k)} = \nu_2(k) = m$ and the right-hand side of (17) changes to

$$\begin{aligned}&\nu_2(k)+n-\big(b_{d_2(k)}-1+\nu_2(k)-1\big)+(k-2)\\&= n-m+k \geq k\end{aligned}$$

with $\max_{1\leq i\leq k-1} \nu_2(i(k-i)) = b_{d_2(k)}-1+\nu_2(k)-1$ and the unique optimum is taken at $i = k-i = 2^{m-1}$. For the correction term, (19) applies again with the same reasoning as above. □

We can generalize the above proof to obtain an alternative proof of Theorem 2 although it requires a modified version of inequality (5) of Theorem 4, cf. [7, Remark 2 and Theorem 6] in a somewhat relaxed form:

Theorem 12. *For $c \geq 3$ odd, we have*

$$\nu_2(S(c2^n,k)) \geq d_2(k)-1,\ 1 \leq k \leq 2^{n+1}. \tag{20}$$

Below, for any integer $a \geq 1$, we use the following simple fact that

$$d_2(a-1) = d_2(a)-1+\nu_2(a). \tag{21}$$

This implies $d_2(c2^n-1) = d_2(c-1)+n$ and thus,

$$d_2(c2^{n+1}-1) = d_2(c2^n-1)+1 = d_2(c)+\nu_2(c)+n. \tag{22}$$

Proof of Theorem 2. We may assume that c is an odd integer, otherwise we can factor c into a power of two and an odd integer, and k still satisfies $1 \le k \le 2^n$. We use induction on c and n. Assume that $\nu_2(k!S(s2^t,k)) = k-1, 1 \le k \le 2^t$, for all $1 \le s \le c$ and $0 \le t \le n$, and prove that it also holds for $t = n+1$. Then we prove that it also holds for the odd number $s = c+2$.

The base case with $c = 1$ is covered by the above proof of Theorem 1. Let us assume that $c \ge 3$. Clearly, $d_2(c) \ge 2$. The case with $n = 0$ is trivial since $\nu_2(S(c,1)) = 0$. Similarly to (18), we get

$$\begin{aligned} k(k-1)!S(c2^{n+1},k) & \\ &= k\frac{(c2^{n+1}-1)!}{(c2^n-1)!^2}\sum_{i=1}^{k-1}\frac{1}{i(k-i)}i!S(c2^n,i)\ (k-i)!S(c2^n,k-i) \\ &\quad + k(k-1)!\frac{(c2^{n+1}-1)!}{(c2^n-1)!^2}\sum_{j=k}^{c2^n}2\binom{c2^n-1}{j-1}\frac{B_{c2^{n+1}-j}}{c2^{n+1}-j}S(j,k) \qquad (23)\end{aligned}$$

by setting $k = m = c2^n$ and switching from r to k in (13). We will see that the correction term in (23) determines the exact 2-adic order. In fact, the first term's 2-adic order is at least

$$\begin{aligned}&\nu_2(k) + (n-1+d_2(c)) + k - 2 \\ &\quad - \begin{cases} \lfloor \log_2 k \rfloor + \nu_2(k) - 1, & \text{if } k \ge 2 \text{ is odd or even but not a power of two} \\ 2\nu_2(k) - 2, & \text{if } k \ge 2 \text{ is a power of two,} \end{cases}\end{aligned}$$

by (22) and Theorem 12, thus it is at least k. Note that the first term disappears if $k = 1$, and the statement $\nu_2(S(c2^{n+1},1)) = 0$ is trivial.

If j is odd then the corresponding Bernoulli number $B_{c2^{n+1}-j}$ in the correction term (23) is 0. If j is even then we define A as the 2-adic order of the jth term, and we have that

$$\begin{aligned} A =\ & \nu_2(k!) + \nu_2((c2^{n+1}-1)!) - 2\nu_2((c2^n-1)!) \\ & + \big(1 + d_2(j-1) + d_2(c2^n-j) - d_2(c2^n-1) - 1 - \nu_2(c2^{n+1}-j)\big) \\ & + \nu_2(S(j,k)) \\ =\ & (k - d_2(k)) + c2^{n+1} - 1 - d_2(c2^{n+1}-1) - 2\left(c2^n - 1 - d_2(c2^n-1)\right) \\ & + \big(d_2(j-1) + d_2(c2^n-j) - d_2(c2^n-1) - \nu_2(c2^{n+1}-j)\big) \\ & + \nu_2(S(j,k))\end{aligned}$$

$$= k + d_2(j-1) + d_2(c2^n - j) - \nu_2(c2^{n+1} - j) + \nu_2(S(j,k)) - d_2(k)$$
$$= k - 1 + \nu_2(j) + d_2(c2^n - j) - \nu_2(c2^{n+1} - j) + (\nu_2(S(j,k)) - d_2(k) + d_2(j))$$

by $\nu_2(B_{c2^{n+1}-j}) = -1$, (21), and (22).

Now we prove that the last quantity is at least $k-1$, and the unique value of j that achieves this lower bound is $j = c \bmod 2^{\lfloor \log_2 c \rfloor}$, i.e., when we remove the most significant binary digit of c. We set $j = c'2^{n+q}$ with c' odd and $k \le j \le c2^n$ and identify four cases according to the value of q.

If $-n \le q < 0$ then

$$A \ge k - 1 + n + q + d_2(c2^{-q} - c') - (n+q) \ge k$$

by (5) and since $c' \ne c2^{-q}$, i.e., $j \ne c2^n$. If $q = 0$, i.e., $j = c'2^n$, then

$$\begin{aligned} A &\ge k - 1 + n + d_2(c - c') - n + (d_2(k) - 1 - d_2(k) + d_2(c')) \\ &\ge k - 1 + d_2(c) - 1 \ge k \end{aligned}$$

by Theorem 12. If $q = 1$ then $2c' < c$ and

$$\begin{aligned} A &= k - 1 + n + 1 + d_2(c - 2c') - \nu_2(c - c') - (n+1) + (-1 + d_2(c')) \\ &= k - 1 + d_2(c) - 1 + \nu_2\left(\binom{c}{2c'}\right) - \nu_2(c - c') \ge k - 1 \end{aligned}$$

by the induction hypothesis as $c' < c$ and $1 \le k \le 2^{n+1}$ imply that $\nu_2(S(c'2^{n+1}, k)) = d_2(k) - 1$. It is easy to prove, e.g., by induction on the number of blocks of zeros in the binary representation of c, that A can reach the lower bound $k-1$ exactly if c' is derived from c by removing its most significant binary digit. By the way, if $c" = c2^{\lfloor \log_2 c \rfloor - i}$ with $0 \le i \le \lfloor \log_2 c \rfloor - 1$, then $d_2(c) - 1 + \nu_2\left(\binom{c}{2c"}\right) - \nu_2(c - c")$ is equal to the number of ones in $c2^{\lfloor \log_2 c \rfloor} - c"$.

If $q \ge 2$ then by (5) we get that

$$A \ge k - 1 + n + q + d_2(c - c'2^q) - (n+1) \ge k - 1 + q - 1 \ge k.$$

The proof of $\nu_2(k!S(c2^{n+1}, k)) = k - 1$ for $1 \le k \le 2^{n+1}$ and $n \ge 0$ is complete for c, and now we can proceed with the next odd c. □

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