



UNIVERSAL EXPANSIONS IN NEGATIVE AND COMPLEX BASES

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Expansions in noninteger positive bases have been intensively investigated since the pioneering works of Rényi (1957) and Parry (1960). The discovery of surprising unique expansions in certain noninteger bases by Erdős, Horváth and Joó (1991) was followed by many studies aiming to clarify the topological and combinatorial nature of the sets of these bases. In the present work we extend some of these studies to more general, negative or complex bases.

1. Introduction

Given a noninteger real number $q > 1$ in his seminal paper [9] Rényi initiated the study of expansions of the form

$$x = \sum_{k=1}^{\infty} \frac{c_k}{q^k}$$

on the digit set $\{0, 1, \dots, [q]\}$. Following a subsequent work of Parry [8] it turned out that the properties of these expansions are radically different in many aspects from the usual expansions in integer bases. Since most interesting phenomena appear already for $1 < q < 2$, we restrict ourselves in the sequel to expansions using only the digits zero and one.

While the expansions in base $q = 2$ are unique for all but countably many real numbers, for which there are exactly two different expansions, in bases $1 < q < 2$ the situation is quite the opposite: Sidorov [10] proved that almost all real numbers $0 < x < 1/(q - 1)$ have continuum many different expansions. The golden mean G plays a special role here: for $q < G$ every real number $0 < x < 1/(q - 1)$ has continuum many different expansions [3], while for $q = G$, for instance, the set of expansions of $x = 1$ has countably infinite [2]. More surprisingly, Erdős, Horváth and Joó [2] discovered that the expansion of $x = 1$ is unique for continuum many

bases $1 < q < 2$. Subsequently the set \mathcal{U} of such bases was characterized in [3] and their nature was clarified in [5] and [6]. For example, we have the following properties:

- the set \mathcal{U} is nowhere dense and it has zero Lebesgue measure, but its Hausdorff dimension is equal to one;
- the set \mathcal{U} is closed from above and it has a smallest element $q \approx 1.787$;
- $\overline{\mathcal{U}} \setminus \mathcal{U}$ is a countable dense set in $\overline{\mathcal{U}}$; consequently the closure of \mathcal{U} is a Cantor set (i.e., a closed set having no isolated or interior points) of zero Lebesgue measure.

Since in bases q close to 1 all real numbers satisfying $0 < x < 1/(q - 1)$ have continuum many different expansions, it was natural to ask whether there exist expansions with special properties, for instance *universal expansions* which contain all possible finite variations of the digits zero and one. A positive answer to the last question was given in [4].

The definition of expansions remains meaningful for negative or even complex bases; the set of complex numbers having at least one expansion in a given complex base was investigated by Daróczy and Kátai [1] and then in [7]. The purpose of this paper is to extend some of the above mentioned results to negative and complex bases.

In the following by a *sequence* we mean a sequence of zeroes and ones. Given a real or complex number q of modulus $|q| > 1$, by an *expansion* of a real or complex number x in base q we mean a sequence (c_k) satisfying

$$\sum_{k=1}^{\infty} \frac{c_k}{q^k} = x.$$

We denote by J_q the set of numbers having at least one expansion in base q .

2. Expansions in Positive Bases

Fix a positive real number $q > 1$. It is well-known and easy to verify that $J_q \subset [0, \frac{1}{q-1}]$ with equality if and only if $q \leq 2$; for $q > 2$ the set J_q is not connected. The endpoints of J_q always have the unique expansions 0^∞ and 1^∞ .

We recall from [2], [3] and [4] the following results:

Theorem 1. *The following hold:*

- (a) *If $q = \frac{1+\sqrt{5}}{2}$, then the set of expansions of $x = 1$ is countably infinite.*

- (b) If $q < \frac{1+\sqrt{5}}{2}$, then each interior point of J_q has continuum many expansions.
- (c) If, moreover, $q < \sqrt[4]{2}$ and q^2 is different from the second Pisot number, then each interior point of J_q has a universal expansion, i.e., an expansion containing all possible finite blocks of zeroes and ones.

We recall that the second Pisot number P_2 is the unique real root of the polynomial $x^3 - x^2 - 1$. We have $P_2 \approx 1.38$ and $\sqrt{P_2} \approx 1.175$.

3. Expansions in Negative Bases

In this section we fix a negative real number $q = -p$ with $p > 1$. The identity

$$\sum_{k=1}^{\infty} \frac{c_k}{q^k} = \frac{-p}{p^2 - 1} + \sum_{k=1}^{\infty} \frac{1 - c_{2k-1}}{p^{2k-1}} + \frac{c_{2k}}{p^{2k}}$$

establishes a bijection between the expansions in bases q and p : (c_k) is an expansion of x in base q if and only if $(1 - c_1, c_2, 1 - c_3, c_4, \dots)$ is an expansion of $x + \frac{p}{p^2 - 1}$ in base p . This identity allows us to translate many known results concerning expansions in positive bases to negative bases.

In particular, we deduce from the identity the equality

$$J_q = \frac{-p}{p^2 - 1} + J_p.$$

Using the results of the preceding section it follows that

$$J_q \subset \left[\frac{q}{q^2 - 1}, \frac{1}{q^2 - 1} \right] = \left[\frac{-p}{p^2 - 1}, \frac{1}{p^2 - 1} \right]$$

with equality if and only if $p \leq 2$; for $p > 2$ the set J_q is not connected. The endpoints of J_q always have the unique expansions $(10)^\infty$ and $(01)^\infty$.

Let us observe that the sequences (c_k) and $(1 - c_1, c_2, 1 - c_3, c_4, \dots)$ are universal at the same time. Indeed, this is the case $m' = 1$ of the following lemma:

Lemma 2. Fix a positive integer m' and for each sequence $d = (d_i)$ of zeroes and ones let us define another sequence $d' = (d'_i) = T(d)$ of zeroes and ones by the formula

$$d'_{m'i+j} := \begin{cases} 1 - d_{m'i+j} & \text{for } i = 0, 2, \dots \text{ and } j = 1, \dots, m', \\ d_{m'i+j} & \text{for } i = 1, 3, \dots \text{ and } j = 1, \dots, m'. \end{cases} \tag{1}$$

Then d and d' are universal at the same time.

Proof. First we show that if d is universal, then d' contains any fixed finite block B' . We may assume that the length of B' is a multiple of m' . It will be convenient to define the transform T by the same formula (1) for finite sequences as well.

By the universality of d it contains the finite block

$$B := T(B')0^{m-1}T(B')0^{m-2}T(B') \dots T(B')00T(B')0T(B')$$

and, depending on the position of B in d one of these m blocks $T(B')$ will be transformed into $T(T(B')) = B'$ in $d' = T(d)$, so that, indeed, d' contains the block B' . (If the index of the first element of the block B in d is $m'i + j$ for some $1 \leq j \leq m'$, then the j th subblock $T(B')$ will be transformed into a block B' .) Since $T(T(d)) = d$ for all sequences, by symmetry the universality of d' implies that of d , too. □

We deduce from Theorem 1 and Lemma 2 the following results:

Theorem 3. *The following hold:*

- (a) *If $q = -\frac{1+\sqrt{5}}{2}$, then the set of expansions of $x = 0$ is countably infinite.*
- (b) *If $-\frac{1+\sqrt{5}}{2} < q < -1$, then each interior point of J_q has continuum many expansions.*
- (c) *If, moreover, $-\sqrt[4]{2} < q < -1$ and q^2 is different from the second Pisot number, then each interior point of J_q has a universal expansion, i.e., an expansion containing all possible finite blocks of zeroes and ones.*

4. Expansions in Purely Imaginary Bases

If $q = \pm ip$ for some $p > 1$, then by distinguishing the even and odd powers of q we obtain that

$$J_q = qJ_{q^2} + J_{q^2} = \pm ipJ_{-p^2} + J_{-p^2}.$$

Identifying \mathbb{C} with \mathbb{R}^2 we deduce from the results of the preceding section that

$$J_{ip} \subset \left[\frac{-p^2}{p^4 - 1}, \frac{1}{p^4 - 1} \right] \times \left[\frac{-p^3}{p^4 - 1}, \frac{p}{p^4 - 1} \right]$$

and

$$J_{-ip} \subset \left[\frac{-p^2}{p^4 - 1}, \frac{1}{p^4 - 1} \right] \times \left[\frac{-p}{p^4 - 1}, \frac{p^3}{p^4 - 1} \right],$$

with equality if and only if $|q| \leq \sqrt{2}$. The four vertices of this closed rectangle always have the unique expansions $(1000)^\infty$, $(0100)^\infty$, $(0010)^\infty$ and $(0001)^\infty$.

Furthermore, we deduce from Theorem 3 the following result:

Theorem 4. *Let $q = \pm ip$ for some $p > 1$. The following hold:*

- (a) *If $|q| = \sqrt{\frac{1+\sqrt{5}}{2}}$, then the set of expansions of $x = 0$ is countably infinite.*
- (b) *If $|q| < \sqrt{\frac{1+\sqrt{5}}{2}}$, then each point of J_q except the four vertices has continuum many expansions.*

The existence of universal expansions follows from the case $m = 4$ of the more general Theorems 6 and 8 in Section 5 below.

5. Complex Bases

Set $\mathbb{C}_R := \{z \in \mathbb{C} : |z| \leq R\}$ for brevity.

Theorem 5. *Fix a nonreal complex number ω of modulus one and a positive real number R . If $p > 1$ is sufficiently close to 1, then every complex number $z \in \mathbb{C}_R$ has continuum many expansions in base $q := p\omega$.*

Proof. For $\omega = \pm i$ the result follows from Theorem 4 (b): the condition

$$1 < p < \min \left\{ \sqrt{\frac{1 + \sqrt{5}}{2}}, \sqrt[4]{\frac{R + 1}{R}} \right\}$$

is sufficient. We may therefore assume henceforth that $\omega^4 \neq 1$.

Applying Proposition 2.1 from [7] with ω^2 and $2R$ instead of ω and R we obtain that if $p > 1$ is sufficiently close to 1, then every complex number $w \in \mathbb{C}_{2R}$ has at least one expansion in base q^2 . Now fix a sufficiently large positive integer n such that

$$\frac{1}{p^{2n+1}} + \frac{1}{p^{2n+3}} + \dots < R.$$

Now given $z \in \mathbb{C}_R$ and an arbitrary subset $A \subset \{2n + 1, 2n + 3, \dots\}$, applying this result with

$$w := z - \sum_{k \in A} \frac{1}{q^k} \in \mathbb{C}_{2R}$$

we obtain an expansion

$$z - \sum_{k \in A} \frac{1}{q^k} = \frac{c_2}{q^2} + \frac{c_4}{q^4} + \dots$$

This yields a different expansion of z in base q for every subset A . □

Now fix a positive integer $m \geq 3$, a complex number satisfying $\omega^m = 1$ and consider $q = p\omega$ with $1 < p \leq 2^{1/m}$. It follows from the identity

$$\sum_{i=1}^{\infty} \frac{d_i}{q^i} = \sum_{j=1}^m (p\omega)^{m-j} \sum_{i=0}^{\infty} \frac{d_{mi+j}}{p^{m(i+1)}} \tag{2}$$

and from the inequalities $1 < p^m \leq 2$ that J_q coincides with the set of numbers

$$\sum_{j=1}^m (p\omega)^{m-j} \alpha_j$$

with

$$0 \leq \alpha_j \leq \frac{1}{p^m - 1} \quad \text{for all } j.$$

We will prove the following theorem:

Theorem 6. *If $1 < p < 2^{1/4m}$ and p^{2m} is different from the second Pisot number, then each interior point of $(p^m - 1)J_q$ has a universal expansion in base $q = p\omega$, i.e., an expansion containing all possible finite blocks of zeroes and ones.*

For $\omega = \pm 1$ the theorem follows from Theorem 1 (c) and Theorem 3 (c). Henceforth we assume that ω is not real. We divide the proof into three steps.

First step. Every interior point z of $(p^m - 1)J_q$ may be written in the form

$$z = \sum_{j=1}^m (p\omega)^{m-j} \alpha_j$$

with suitable real numbers satisfying $0 < \alpha_j < 1$ for all j . Indeed, since $tz \in (p^m - 1)J_q$ for a sufficiently small $t > 1$, dividing by t a representation of tz we may assume that $0 \leq \alpha_j < 1$ for all j . Adding a small positive multiple of the equality

$$0 = \sum_{j=1}^m \omega^{m-j}$$

we may also assume that $0 < \alpha_j < 1$ for all j .

Using the identity (2) it suffices to construct a universal sequence (d_i) satisfying the equalities

$$\alpha_j = \sum_{i=0}^{\infty} \frac{d_{mi+j}}{p^{m(i+1)}}, \quad j = 1, \dots, m. \tag{3}$$

Second step. The construction will be based on the following generalization of Lemma 4.1 in [4].

Lemma 7. *Let m and p be as in the statement of the theorem. Given $0 < \alpha_j \leq 1$, $j = 1, \dots, m$ and a finite sequence $c_1 \dots c_{mN}$ of integers $\in \{0, 1\}$ there exists another finite sequence $d_1 \dots d_{mn+mN}$ of integers $\in \{0, 1\}$ ending with $c_1 \dots c_{mN}$:*

$$d_{mn+i} = c_i, \quad i = 1, \dots, mN,$$

and satisfying the inequalities

$$0 < \alpha_j - \sum_{i=0}^{n+N-1} \frac{d_{mi+j}}{p^{m(i+1)}} < \frac{1}{p^{m(n+N)}}, \quad j = 1, \dots, m.$$

Proof. Let $0 = y_0 < y_1 < \dots$ be the sequence of numbers of the form $P(p^m)$ where P runs over the polynomials with coefficients belonging to the set $\{0, 1\}$. We know from [4] that $y_k \rightarrow \infty$ and $y_{k+1} - y_k \rightarrow 0$. Set

$$A_j = \sum_{i=0}^{N-1} \frac{c_{mi+j}}{p^{m(i+1)}}, \quad j = 1, \dots, m.$$

If n is sufficiently large, then

$$p^{mn} \alpha_j > A_j, \quad j = 1, \dots, m,$$

so that there exist indices k_1, \dots, k_m satisfying

$$y_{k_j} < p^{mn} \alpha_j - A_j \leq y_{k_j+1}, \quad j = 1, \dots, m.$$

Since $y_{k_j} < p^{mn}$ for all j , we have

$$y_{k_j} = a_{j,0} + a_{j,m} p^m + \dots + a_{j,(n-1)m} p^{(n-1)m}, \quad j = 1, \dots, m,$$

with suitable coefficients $a_{j,km} \in \{0, 1\}$.

Since $n \rightarrow \infty$ implies $k_j \rightarrow \infty$ for each j , choosing a sufficiently large n we also have

$$0 < y_{k_j+1} - y_{k_j} < p^{-mN}, \quad j = 1, \dots, m.$$

It follows that

$$0 < \alpha_j - p^{-mn} y_{k_j} - p^{-mn} A_j < p^{-m(n+N)}, \quad j = 1, \dots, m. \quad \square$$

Third step. We fix a sequence B_1, B_2, \dots of finite blocks, containing all possible finite variations of the digits 0 and 1. Let $c_1 \dots c_{mN} := B_1 0 \dots 0$ with additional zeroes if necessary. Applying the preceding lemma with $c_1 \dots c_{mN}$ we obtain

a finite sequence d_1, \dots, d_{mi_1} (with $i_1 := n + N$) ending with the block B_1 and satisfying the inequalities

$$0 < \alpha_j - \sum_{i=0}^{i_1-1} \frac{d_{mi+j}}{p^{m(i+1)}} < \frac{1}{p^{mi_1}}, \quad j = 1, \dots, m.$$

Repeating the construction with the numbers

$$\alpha_{j,2} := p^{mi_1} \left(\alpha_j - \sum_{i=0}^{i_1-1} \frac{d_{mi+j}}{p^{m(i+1)}} \right), \quad j = 1, \dots, m$$

instead of α_j and with the block B_2 instead of B_1 , we may extend the sequence d_1, \dots, d_{mi_1} to a sequence d_1, \dots, d_{mi_2} ending with the block B_2 and satisfying the inequalities

$$0 < \alpha_{j,2} - \sum_{i=i_1}^{i_2-1} \frac{d_{mi+j}}{p^{m(i+1-i_1)}} < \frac{1}{p^{m(i_2-i_1)}}, \quad j = 1, \dots, m.$$

This can be rewritten in the following form:

$$0 < \alpha_j - \sum_{i=0}^{i_2-1} \frac{d_{mi+j}}{p^{m(i+1)}} < \frac{1}{p^{mi_2}}, \quad j = 1, \dots, m.$$

Proceeding by induction we construct in this way a sequence (d_i) and an infinite sequence of indices $i_1 < i_2 < \dots$ such that d_1, \dots, d_{mi_k} ends with the block B_k , and

$$0 < \alpha_j - \sum_{i=0}^{i_k-1} \frac{d_{mi+j}}{p^{m(i+1)}} < \frac{1}{p^{mi_k}}, \quad j = 1, \dots, m$$

for each $k = 1, 2, \dots$, so that the equalities (3) are satisfied.

If m is even, then we may weaken the assumptions on p in order to obtain universal expansions.

Theorem 8. *Assume that $m = 2m'$ is even, $\omega^{m'} = -1$, $1 < p < 2^{1/2m}$ and p^m is different from the second Pisot number. Then every number of the form*

$$z = \sum_{j=1}^{m'} (p\omega)^{m'-j} \alpha_j \tag{4}$$

with real numbers

$$-\frac{p^{m'}}{p^{2m'} - 1} < \alpha_j < 1 - \frac{p^{m'}}{p^{2m'} - 1} = \frac{p^{2m'} - p^{m'} - 1}{p^{2m'} - 1}$$

has a universal expansion.

Proof. It follows from the identity

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{d_i}{q^i} &= \sum_{i=1}^{\infty} \frac{d_i}{(p\omega)^i} \\ &= \sum_{j=1}^{m'} (p\omega)^{m'-j} \sum_{i=0}^{\infty} \frac{d_{m'+i+j}}{(p\omega)^{m'+i+j}} \\ &= \sum_{j=1}^{m'} (p\omega)^{m'-j} \sum_{i=0}^{\infty} \frac{d_{m'+i+j}}{(-p^{m'})^{i+1}} \end{aligned}$$

and from the results of Section 3 (our assumption on p implies that $1 < p^{m'} \leq 2$) that a complex number z belongs to J_q if and only if z has the form (4) with suitable real numbers

$$\alpha_j \in \left[\frac{-p^{m'}}{p^{2m'} - 1}, \frac{1}{p^{2m'} - 1} \right].$$

Rewriting the above identity in the form

$$\begin{aligned} \left(\sum_{i=1}^{\infty} \frac{d_i}{q^i} \right) + \frac{p^{m'}}{p^{2m'} - 1} \sum_{j=1}^{m'} (p\omega)^{m'-j} \\ = \sum_{j=1}^{m'} (p\omega)^{m'-j} \left(\frac{1 - d_j}{p^{m'}} + \frac{d_{j+m'}}{(p^{m'})^2} + \frac{1 - d_{j+2m'}}{(p^{m'})^3} + \frac{d_{j+3m'}}{(p^{m'})^4} - \dots \right) \end{aligned} \quad (5)$$

we also see that $z \in J_q$ if and only if

$$z' := z + \frac{p^{m'}}{p^{2m'} - 1} \sum_{j=1}^{m'} (p\omega)^{m'-j} = \sum_{j=1}^{m'} (p\omega)^{m'-j} \alpha_j$$

with suitable real numbers

$$\alpha_j \in \left[0, \frac{1}{p^{m'} - 1} \right].$$

Furthermore, the identity (5) establishes a bijection between the expansions

$$z = \sum_{i=1}^{\infty} \frac{d_i}{q^i} = \sum_{j=1}^{m'} (p\omega)^{m'-j} \sum_{i=0}^{\infty} (-1)^{i+1} \frac{d_{m'+i+j}}{(p^{m'})^{i+1}} \tag{6}$$

of an element $z \in J_q$ and the expansion

$$z' = \sum_{i=1}^{\infty} \frac{d'_i}{q^i} = \sum_{j=1}^{m'} (p\omega)^{m'-j} \sum_{i=0}^{\infty} \frac{d'_{m'+i+j}}{(p^{m'})^{i+1}} \tag{7}$$

of its translate

$$z' := z + \frac{p^{m'}}{p^{2m'} - 1} \sum_{j=1}^{m'} (p\omega)^{m'-j}, \tag{8}$$

where the sequence (d'_i) of zeroes and ones is obtained from (d_i) by the transformation (1) of Lemma 2.

If z satisfies the hypotheses of the theorem, then by what we have shown in the first step, its translate z' defined by (8) may be written in the form

$$z' = \sum_{j=1}^{m'} (p\omega)^{m'-j} \alpha_j$$

with real numbers $0 < \alpha_j < 1$. Repeating the second and third part of the proof of Theorem 6 we conclude that z' has a universal expansion (7). By Lemma 2 its transform $d := T(d')$ provides a universal expansion (6). □

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