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UPPER AND LOWER BOUNDS FOR A FUNCTION RELATED TO BROWN'S LEMMA

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Abstract

The well-known Brown's lemma says that for every finite coloring of the positive integers, there exist a fixed positive integer d and arbitrarily large monochromatic sets $A = \{a_1 < a_2 < \cdots < a_n\}$ such that $\max_{1 \le i \le n-1}(a_{i+1} - a_i) \le d$. We provide upper and lower bounds for some of the functions associated with the "finite form" of this result.

1. Introduction

The following two facts are equivalent.

Fact A. *For any finite coloring of the positive integers, there exist a fixed positive integer d and arbitrarily large monochromatic sets $A = \{a_1 < a_2 < \cdots < a_n\}$ such that $\max_{1 \le i \le n-1}(a_{i+1} - a_i) = d$.*

Fact B. *For every positive integer k, and every function $f : \mathbb{N} \to \mathbb{N}$, there exists a (smallest) positive integer $B(k, f)$ such that every k-coloring of the interval $[1, B(k, f)]$ produces a monochromatic set $A = \{a_1 < a_2 < \cdots < a_n\}$ such that $|A| > f(d)$ where $d = \max_{1 \le i \le n-1}(a_{i+1} - a_i)$.*

The integer $\max_{1 \le i \le n-1}(a_{i+1} - a_i)$ is called the *gap size* of the set $A = \{a_1 < a_2 < \cdots < a_n\}$, and is denoted by $gs(A)$, so that Fact B asserts the existence of a monochromatic set A with $|A| > f(gs(A))$. (If $|A| = 1$, set $gs(A) = 1$.)

Fact A first appeared in [1]. Some applications appear in [2] and in [4]–[9]. Proofs of Fact A and Fact B are found in [7]. The book [4] contains a very short proof of Fact A.

Let id denote the identity function on $\mathbb{N}$. The inductive proof of Fact B in [7] gives the upper bound $B(k, id) < \lfloor k! \cdot e \rfloor$. This is the only previously known bound for any $B(k, f)$, and is mentioned in [3].

In Table 1, we give all the known values or the best lower bounds (known to date) for $B(k, id)$.

k	$B(k, id)$
1	2
2	5
3	13
4	35
5	≥ 74
6	≥ 143

Table 1: All Known Values/Lower Bounds of $B(k, id)$.

In this note we show that $k^{c \log k} \leq B(k, id) \leq k \cdot \left(2^k - k\right) + 1$, $k \geq 1$, for some $c > 0$.

Definition 1. Let A be a finite subset of $\mathbb{N}$. We say that A has *Property P* if $|B| \leq gs(B)$ for any subset B of A.

Theorem 2. *Let $A = \{a_1 < a_2 < \cdots < a_n\}$ be a subset of $\mathbb{N}$. Then the following are equivalent.*

(i) *A has Property P.*

(ii) *For each $1 \leq i < j \leq n$*

$$|[a_i, a_j]| \leq gs([a_i, a_j])$$

where $[a_i, a_j] = \{a_i, a_{i+1}, \cdots, a_j\}$.

Proof. (i)$\Rightarrow$ (ii) is true by definition.

(ii)$\Rightarrow$ (i) Assume that A does not have Property P, so that there exists a subset B of A such that

$$|B| > gs(B).$$

Let $i = \min\{k : a_k \in B\}$ and $j = \max\{k : a_k \in B\}$. Then

$$B \subseteq [a_i, a_j].$$

Since $a_i, a_j \in B$ and $B \subseteq [a_i, a_j]$,

$$gs([a_i, a_j]) \leq gs(B).$$

Hence

$$gs([a_i, a_j]) \leq gs(B) < |B| \leq |[a_i, a_j]|,$$

therefore (ii) does not hold. $\square$

Note that a finite set of positive integers has Property P if and only if any integer shift of it has Property P. This fact suggests the following definitions.

Definition 3. Let $A = \{a_1 < a_2 < \cdots < a_n\}$ be a subset $\mathbb{N}$. Then we define the difference sequence of A, $\mathbf{d}(A)$, as

$$\mathbf{d}(A) = (a_2 - a_1, a_3 - a_2, \cdots, a_n - a_{n-1}).$$

Definition 4. Let $\mathbf{d} = (d_1, d_2, ..., d_n) \in \mathbb{N}^n$. Then we say that $\mathbf{d}$ has Property P$'$ if

$$\max_{a \le i \le b} d_i \ge b - a + 2$$

for all $1 \le a \le b \le n$, i.e., any l consecutive numbers in $\mathbf{d}$ have maximum bigger than or equal to $l + 1$.

The following theorem gives the correspondence between Property P and Property P$'$.

Theorem 5. *A finite subset A of $\mathbb{N}$ has Property P if and only if $\mathbf{d}(A)$ has Property P$'$.*

Proof. Let $A = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{N}$ and let $\mathbf{d}(A) = (d_1, d_2, ..., d_{n-1})$ be the difference sequence of A where $d_i = a_{i+1} - a_i$ for $1 \le i \le n - 1$. Then

$$\begin{aligned} A \text{ has Property P} \quad &\Leftrightarrow \quad |[a_i, a_j]| \le gs([a_i, a_j]) \quad \forall i, j \text{ s.t. } 1 \le i < j \le n \\ &\Leftrightarrow \quad j - i + 1 \le \max_{i \le l \le j-1} a_{l+1} - a_l \quad \forall i, j \text{ s.t. } 1 \le i < j \le n \\ &\Leftrightarrow \quad t - i + 2 \le \max_{i \le l \le t} d_l \quad \forall i, j \text{ s.t. } 1 \le i \le t \le n - 1 \quad (t = j - 1) \\ &\Leftrightarrow \quad \mathbf{d}(A) \text{ has Property P}'. \end{aligned}$$

□

2. Upper Bound

In this section, we will show that

$$B(k, id) \le k \cdot \left(2^k - k\right) + 1$$

for all $k \ge 1$. ($B(k, id)$ is defined just after *Fact B* above.)

Definition 6. For a positive integer n, define

$$D_n = \left\{\mathbf{d} = (d_1, d_2, ..., d_n) \in \mathbb{N}^n : \mathbf{d} \text{ has Property P}'\right\}.$$

Lemma 7. *Let* $\mathbf{d} = (d_1, d_2, ..., d_n)$ *and* $\mathbf{d'} = (d'_1, d'_2, ..., d'_m)$ *for some positive integers* $n, m \in \mathbb{N}$, *and* $t \in \mathbb{N}$, $t > n+m+1$ *be arbitrary. For* $\mathbf{d''} = (d_1, d_2, ..., d_n, t, d'_1, d'_2, ..., d'_m)$

$\mathbf{d''}$ *has Property* P' *if and only if both* $\mathbf{d}$ *and* $\mathbf{d'}$ *have Property* P'.

Proof. The forward implication follows directly from the definition.

Now, assume both $\mathbf{d}$ and $\mathbf{d'}$ have Property P′ and let $1 \leq a \leq b \leq n + m + 1$ be arbitrary. Then

`Case 1`: $b \leq n$

$$\max_{a \leq i \leq b} d''_i = \max_{a \leq i \leq b} d_i \geq b - a + 2, \text{ since } \mathbf{d} \in D_n.$$

`Case 2`: $a \leq n + 1 \leq b$

$$\max_{a \leq i \leq b} d''_i \geq t \geq n + m + 2 \geq b - a + 2, \text{ since } d''_{n+1} = t.$$

`Case 3`: $a \geq n + 2$

$$\max_{a \leq i \leq b} d''_i = \max_{a \leq i \leq b} d'_i \geq b - a + 2, \text{ since } \mathbf{d'} \in D_m.$$

Therefore, $\mathbf{d''}$ has Property P′. □

Corollary 8. *Let* $\mathbf{d^i} \in D_{n_i}, 1 \leq i \leq m$ *for some* m *and* $n_1, n_2, ..., n_m$. *Let*

$$n = m - 1 + \sum_{i=1}^{m} n_i.$$

Then $\mathbf{d} = (\mathbf{d}^1, t_1, \mathbf{d}^2, t_2, \ldots, t_{m-1}, \mathbf{d}^m) \in D_n$ *for any* $t_i > n, 1 \leq i \leq m - 1$.

Corollary 9. *Let* $\mathbf{d} \in D_n$ *and* $m \geq 2$ *be arbitrary. Then*

$$\mathbf{d'} = (\mathbf{d}, t, \mathbf{d}, t, ..., t, \mathbf{d}) \in D_{m \cdot n + m - 1}$$

for any $t > m \cdot n + m - 1$, *where in* $\mathbf{d'}$, $\mathbf{d}$ *is repeated* m *times.*

For $\mathbf{d} = (d_1, d_2, ..., d_n) \in \mathbb{N}^n$, define

$$S(\mathbf{d}) = \sum_{i=1}^{n} d_i.$$

For a given set $A = \{a_1 < a_2 < ... < a_n\}$ of positive integers

$$S(d(A)) = a_n - a_1.$$

Now, define the function $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $F(0) = 0$ and

$$F(n) = \min_{\mathbf{d} \in D_n} S(\mathbf{d}).$$

for $n \geq 1$.

Note that

$$F(n) = \min \{a_{n+1} - a_1 : \{a_1 < a_2 < \cdots < a_{n+1}\} \text{ has Property P}\}$$

for all $n \geq 1$.

It is easy to check that $F(1) = 2$, $F(2) = 5$ and $F(3) = 8$.

The following two lemmas give a recursive definition for $F(n)$.

Lemma 10. *Let $n \in \mathbb{N}$. Then*

$$F(n) = n + 1 + F(n - m) + F(m - 1)$$

for some m in $[1, n]$.

Proof. Let $\mathbf{d} \in D_n$ be such that

$$F(n) = S(\mathbf{d}) = \sum_{i=1}^{n} d_i$$

By the definition of Property P$'$, $\max_{1 \leq i \leq n} d_i \geq n + 1$. And, by the minimality of $F(n)$, $\max_{1 \leq i \leq n} d_i \leq n + 1$. Therefore,

$$\max_{1 \leq i \leq n} d_i = n + 1$$

otherwise we could replace any d_i greater than $n+1$ with $n+1$ and the new sequence thus obtained would still be in D_n and have a smaller sum.

Assume $d_m = n + 1$. Then

$$\mathbf{d} = (d_1, d_2, ..., d_{m-1}, n + 1, d_{m+1}, d_{m+2}, ..., d_n).$$

Again by the minimality of $F(n)$ and Lemma 7,

$$\sum_{i=1}^{m-1} d_i = F(m - 1) \text{ and } \sum_{i=m+1}^{n} d_i = F(n - m).$$

Therefore,

$$F(n) = n + 1 + F(m - 1) + F(n - m)$$

for some m in $[1, n]$. □

Lemma 11. *We have*

$$F(n) = F(n-1) + \lfloor \log_2 n \rfloor + 2$$

$$F(n) = n + 1 + F\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + F\left(\left\lceil \frac{n-1}{2} \right\rceil\right)$$

for all $n \geq 2$.

Proof. We will prove both equalities by induction on n, at the same time.

It is clear that both equalities are true for $n = 2$ and $n = 3$.

Now assume that they are true for all $m < n$ for some $n > 3$.

So

$$\begin{array}{lll} & F(m) = F(m-1) + \lfloor \log_2 m \rfloor + 2 & \text{for all } m \in [1, n), \\ \text{which implies} & F(m) - F(m-1) = \lfloor \log_2 m \rfloor + 2 & \text{for all } m \in [1, n), \\ \text{which implies} & F(m) - F(m-1) \geq F(m-1) - F(m-2) & \text{for all } m \in [2, n). \end{array}$$

Hence, if $l < m < n$ then

$$F(m) - F(m-1) \geq F(l+1) - F(l)$$

which implies

$$F(m) + F(l) \geq F(m-1) + F(l+1). \tag{1}$$

Hence, if $m < n$,

$$\min_{0 \leq l \leq m} (F(l) + F(m-l)) = F\left(\left\lfloor \frac{m}{2} \right\rfloor\right) + F\left(\left\lceil \frac{m}{2} \right\rceil\right) \tag{2}$$

which follows by repeated application of (1).

By Lemma 7,

$$F(n) \leq n + 1 + F(m-1) + F(n-m) \tag{3}$$

for all m in $[1, n]$. And by Lemma 10,

$$F(n) = n + 1 + F(m-1) + F(n-m) \tag{4}$$

for some m in $[1, n]$.

Hence, by the minimality of $F(n)$, (3) and (4),

$$F(n) = n + 1 + \min_{1 \leq m \leq n} (F(m-1) + F(n-m))$$

and by (2)

$$F(n) = n + 1 + F\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + F\left(\left\lceil \frac{n-1}{2} \right\rceil\right). \tag{5}$$

Now we'll show that

$$F(n) = F(n-1) + \lfloor \log_2 n \rfloor + 2.$$

`Case 1`: Let $n = 2t$ for some $t \geq 2$. Then we have

$$F(2t) = 2t + 1 + F(t-1) + F(t) \text{ by (5)}$$

and

$$F(2t-1) = 2t + F(t-1) + F(t-1) \text{ by the induction hypothesis.}$$

Hence,

$$\begin{aligned} F(2t) - F(2t-1) &= 1 + F(t) - F(t-1) \\ &= 1 + \lfloor \log_2 t \rfloor + 2 \text{ (by the induction hypothesis)} \\ &= \lfloor \log_2 2t \rfloor + 2. \end{aligned}$$

`Case 2`: Let $n = 2t + 1$ for some $t \geq 2$. Then we have

$$\begin{aligned} F(2t+1) &= 2t + 2 + F(t) + F(t) \text{ by (5), and} \\ F(2t) &= 2t + 1 + F(t) + F(t-1) \text{ (by the induction hypothesis).} \end{aligned}$$

Hence,

$$\begin{aligned} F(2t+1) - F(2t) &= 1 + F(t) - F(t-1) \\ &= 1 + \lfloor \log_2 t \rfloor + 2 \text{ (by the induction hypothesis)} \\ &= \lfloor \log_2 2t \rfloor + 2 \\ &= \lfloor \log_2 (2t+1) \rfloor + 2, \text{ since } 2t+1 \text{ is odd.} \end{aligned}$$

Hence, in both cases

$$F(n) = F(n-1) + \lfloor \log_2 n \rfloor + 2.$$

□

Lemma 12. $F\left(2^k - 1\right) = k \cdot 2^k$ *for all* k *in* $\mathbb{N}$.

Proof. The equality is clear for $k = 1$.

Assume that the assumption is true for $k-1$, for some $k \geq 2$. Then

$$\begin{aligned} F\left(2^k - 1\right) &= 2 \cdot F\left(2^{k-1} - 1\right) + 2^k \text{ (from Lemma 11)} \\ &= 2 \cdot \left((k-1) \cdot 2^{k-1}\right) + 2^k \text{ (by the induction hypothesis)} \\ &= k \cdot 2^k \end{aligned}$$

□

We need two more lemmas to obtain an upper bound for $B(k, id)$ using the function $F(n)$.

Lemma 13. *$F\left(2^k - k\right) = k\left(2^k - k\right) + 1$ for all k in $\mathbb{N}$.*

Proof. Let k in $\mathbb{N}$ be given. Then

$$\begin{aligned} F(2^k - 1) &= F(2^k - k) + \sum_{i=1}^{k-1} \left(\left\lfloor \log_2(2^k - i) \right\rfloor + 2\right) \text{ (by Lemma 11)} \\ &= F(2^k - k) + (k-1)\left((k-1) + 2\right) \\ &= F(2^k - k) + (k^2 - 1). \end{aligned}$$

Hence,

$$\begin{aligned} F(2^k - k) &= F(2^k - 1) - (k^2 - 1) \\ &= k \cdot 2^k - (k^2 - 1) \\ &= k \cdot (2^k - k) + 1. \end{aligned}$$

□

Lemma 14. *Let $k \in \mathbb{N}$ be given and let $N \in \mathbb{N}$ be such that $B(k, id) > kN + 1$. Then $F(N) \leq kN$*

Proof. Assume that $B(k, id) > kN + 1$ for some $N \in \mathbb{N}$. Then there exists a k-coloring of $[1, kN+1]$ such that each color class has Property P. By the pigeon hole principle, at least one of the color classes has at least $N+1$ elements. Let C be this color class. Then

$$\mathbf{d}(C) = (d_1, d_2, ..., d_{|C|-1}) \in D_{|C|-1}$$

by Theorem 5.

So,

$$F(N) \leq F\left(|C| - 1\right) \leq S(\mathbf{d}(C)).$$

But since $C \subset [1, kN+1]$,

$$S(\mathbf{d}(C) \leq kN.$$

Hence

$$F(N) \leq kN.$$

$\square$

Theorem 15. *We have $B(k, id) \leq k(2^k - k) + 1$ for all $k \geq 1$.*

Proof. Let $k \geq 1$ be given and let $N = 2^k - k$.

If $B(k, id) > kN + 1$ then by Lemma 14 $F(N) \leq kN$. But this is a contradiction as $F(N) = kN + 1$ by Lemma 13. $\square$

3. Lower Bound

In what follows, we'll recursively construct a 2^s-coloring of the interval $[1, n_s]$ in such a way that all color classes will have Property P and therefore we will conclude that $B(2^s, id) \geq n_s$, where $n_s = 2^s \cdot \prod_{i=0}^{s-1}(2^i + 1)$. This coloring will be represented by a matrix M_s with 2^s rows and $\prod_{i=0}^{s-1}(2^i + 1)$ columns where the rows of M_s are the color classes of the coloring.

Let J_s denote the $2^s \times \prod_{i=0}^{s-1}(2^i + 1)$ matrix of all 1's.

Let $d(M_s)$ denote the difference sequence of the first row of M_s. For $s = 0$, let $n_0 = 1$ and $M_0 = [1]$. For $s = 1$, let

$$\begin{aligned}
n_1 &= 2^1 \cdot 2 = 4, \text{ and} \\
M_1 &= \begin{bmatrix} M_0 & M_0 + 2n_0 J_0 \\ M_0 + n_0 J_0 & M_0 + 3n_0 J_0 \end{bmatrix} \\
&= \begin{bmatrix} M_0 & M_0 + 2J_0 \\ M_0 + J_0 & M_0 + 3J_0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.
\end{aligned}$$

For $s = 2$, let

$$\begin{aligned} n_2 &= 2^2 \cdot 2 \cdot 3 = 24, \text{ and} \\ M_2 &= \begin{bmatrix} M_1 & M_1 + 2n_1J_1 & M_1 + 4n_1J_1 \\ M_1 + n_1J_1 & M_1 + 3n_1J_1 & M_1 + 5n_1J_1 \end{bmatrix} \\ &= \begin{bmatrix} M_1 & M_1 + 8J_1 & M_1 + 16J_1 \\ M_1 + 4J_1 & M_1 + 12J_1 & M_1 + 20J_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & 9 & 11 & 17 & 19 \\ 2 & 4 & 10 & 12 & 18 & 20 \\ 5 & 7 & 13 & 15 & 21 & 23 \\ 6 & 8 & 14 & 16 & 22 & 24 \end{bmatrix}. \end{aligned}$$

Clearly, M_0, M_1 and M_2 have the desired property.

Note that all the rows of M_1 and M_2 are obtained by shifting the first row of the corresponding matrix. And this will turn out to be true for each M_s so that each row of M_s has the same difference sequence as the first row of M_s. We will designate the common difference sequence as $\mathbf{d}(M_s)$.

Note that $\mathbf{d}(M_2) = (2, 6, 2, 6, 2)$, so by Theorem 5 and Definition 4, each row of M_2 has Property P.

Assume that we have constructed the coloring M_s of $[1, n_s]$ such that all the color classes (rows of M_s) have Property P.

We construct M_{s+1} as follows.

$$M_{s+1} = \begin{bmatrix} M_s & M_s + 2n_sJ_s & M_s + 4n_sJ_s & \cdots & M_s + 2^{s+1}n_sJ_s \\ M_s + n_sJ_s & M_s + 3n_sJ_s & M_s + 5n_sJ_s & \cdots & M_s + (2^{s+1} + 1)n_sJ_s \end{bmatrix}$$

Since each row of M_s is a shift of the first row of M_s, it is also true for M_{s+1}.

Then

$$\mathbf{d}(M_{s+1}) = (\mathbf{d}(M_s), t_s, \mathbf{d}(M_s), \cdots, t_s, \mathbf{d}(M_s)),$$

where $\mathbf{d}(M_s)$ is repeated 2^s+1 times, and

$$\begin{aligned} t_s &= (2n_s+1) - \max(M_s)_1 \\ &= (2^s+1)\left(\prod_{i=0}^{s-1}\left(2^i+1\right)-1\right) + 2^s + 1 \text{ (can be proven by induction on } s\text{)} \\ &= (2^s+1)\prod_{i=0}^{s-1}\left(2^i+1\right) \\ &= \prod_{i=0}^{s}\left(2^i+1\right) \end{aligned}$$

where $(M_s)_1$ denotes the first row of M_s.

Hence, by Corollary 9, $(M_s)_1$ has Property P and therefore all the color classes have Property P.

Therefore, we have

$$\begin{aligned} B(2^s, id) &\geq n_s \\ &= 2^s\prod_{i=0}^{s-1}(2^i+1) \\ &\geq 2^s \cdot 2^{\frac{s^2-s}{2}} \\ &= 2^{\frac{s^2+s}{2}} \\ &= \left(2^{s+1}\right)^{\frac{s}{2}}. \end{aligned}$$

Now, let k in $\mathbb{N}$ be given. Then

$$2^s \leq k < 2^{s+1}$$

for some $s \in \mathbb{N}$. So,

$$\begin{aligned} B(k, id) &\geq B(2^s, id) \\ &\geq \left(2^{s+1}\right)^{\frac{s}{2}} \\ &\geq k^{\frac{\log_2 k - 1}{2}} \\ &\geq k^{c\log k} \end{aligned}$$

for some $c > 0$.

Remark A slight modification of the above construction gives better lower bounds for $B(k, id)$, but it does not improve the asymptotic lower bound.

4. Upper Bound for $B(k, mx)$

In this section, we will give an upper bound for $B(k, f)$ where $f(x) = mx$ for some $m \in \mathbb{N}$. It will be analogous to what we did in Section 2.

Before we consider functions of this type, we will first prove a few theorems that are true for any increasing function.

Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be an arbitrary increasing function.

Definition 16. Let A be a finite subset of $\mathbb{N}$. We say that A has *Property P_f* if $|B| \leq f(gs(B))$ for any subset B of A.

Theorem 17. *Let $A = \{a_1 < a_2 < \cdots < a_n\}$ be a subset of $\mathbb{N}$. Then the following are equivalent.*

i. *A has Property P_f.*

ii. *For each $1 \leq i < j \leq n$,*

$$|[a_i, a_j]| \leq f(gs([a_i, a_j]))$$

where $[a_i, a_j] = \{a_i, a_{i+1}, \cdots, a_j\}$.

Proof. Analogous to the proof of Theorem 2. □

Definition 18. Let $\mathbf{d} = (d_1, d_2, ..., d_n) \in \mathbb{N}^n$. Then we say that $\mathbf{d}$ has Property P'_f if and only if, for all a, b such that $1 \leq a \leq b \leq n$ we have $\max_{a \leq i \leq b} d_i \geq f^{-1}(b - a + 2)$, i.e., any l consecutive numbers in $\mathbf{d}$ have maximum bigger than or equal to $f^{-1}(l + 1)$.

The following theorem gives the correspondence between Property P_f and Property P'_f.

Theorem 19. *A finite subset A of $\mathbb{N}$ has Property P_f if and only if $\mathbf{d}(A)$ has Property P'_f.*

Proof. Analogous to the proof of Theorem 5.

□

Let n be a positive integer. Define

$$D_{n,f} = \left\{\mathbf{d} = (d_1, d_2, ..., d_n) \in \mathbb{N}^n : \mathbf{d} \text{ has Property } \mathrm{P}'_f\right\}.$$

Now, define the function $F_f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ as

$$F_f(n) = \min_{\mathbf{d} \in D_{n,f}} S(\mathbf{d}).$$

Note that $F_f(n)$ equals $\min\left\{a_n - a_0 : \{a_0 < a_1 < \cdots < a_n\} \text{ has Property } \mathrm{P}_f\right\}$.

Theorem 20. *For every $n \geq 1$ and every increasing function f on $\mathbb{N}$,*

$$D_{n,f} = \left\{\left(\left\lceil f^{-1}(d_1)\right\rceil, \left\lceil f^{-1}(d_2)\right\rceil, ..., \left\lceil f^{-1}(d_n)\right\rceil\right) : (d_1, d_2, ..., d_n) \in D_n\right\}.$$

Proof.

$$\begin{aligned}
(d_1, d_2, ..., d_n) \in D_n &\implies \max_{a \leq i \leq b} d_i \geq b - a + 2 \quad \forall a, b \text{ s.t. } 1 \leq a \leq b \leq n \\
&\implies \max_{a \leq i \leq b} \left\lceil f^{-1}(d_i)\right\rceil \geq \max_{a \leq i \leq b} f^{-1}(d_i) \geq f^{-1}(b - a + 2) \\
&\qquad\qquad \forall a, b \text{ s.t. } 1 \leq a \leq b \leq n \\
&\implies \left(\left\lceil f^{-1}(d_1)\right\rceil, \left\lceil f^{-1}(d_2)\right\rceil, ..., \left\lceil f^{-1}(d_n)\right\rceil\right) \in D_{n,f}
\end{aligned}$$

$$\begin{aligned}
(d_1, d_2, ..., d_n) \in D_{n,f} &\implies \max_{a \leq i \leq b} d_i \geq f^{-1}(b - a + 2) \quad \forall a, b \text{ s.t. } 1 \leq a \leq b \leq n \\
&\implies \max_{a \leq i \leq b} f(d_i) = f\left(\max_{a \leq i \leq b} d_i\right) \geq b - a + 2 \\
&\qquad\qquad \forall a, b \text{ s.t. } 1 \leq a \leq b \leq n \\
&\implies (f(d_1), f(d_2), ..., f(d_n)) \in D_n
\end{aligned}$$

□

Theorem 21. *Let k in $\mathbb{N}$ be given. Then if there exists an N in $\mathbb{N}$ such that $F_f(N) > kN$ then $B(k, f) \leq kN + 1$.*

Proof. Analogous to the proof of Theorem 14. □

In the rest of this section, we will only consider linear functions on $\mathbb{N}$. For ease of notation, we will write $F_m(n)$, $B_m(n)$ and $D_{n,m}$ for $F_f(n)$, $B(n, f)$ and $D_{n,f}$, respectively, if $f(x) = mx$ for some $m \in \mathbb{N}$.

Lemma 22. *Let m and n be two given positive integers. Then*

$$F_m(n) \geq \frac{1}{m} F(n).$$

Proof. We have that

$$\begin{aligned} F_m(n) &= \min_{\mathbf{d}\in D_{n,m}} S(\mathbf{d}) \\ &= \min_{\mathbf{d}\in D_n} \sum_{i=1}^{n} \left\lceil \frac{d_i}{m} \right\rceil, \text{ by Theorem 20} \\ &\geq \frac{1}{m} \min_{\mathbf{d}\in D_n} \sum_{i=1}^{n} d_i \\ &= \frac{1}{m} F(n). \end{aligned}$$

□

Lemma 23. $F_m\left(2^{mk} - mk\right) \geq k\left(2^{mk} - mk\right) + 1.$

Theorem 24. *Let k and m be two positive integers Then*

$$B_m(k) \leq k(2^{mk} - mk) + 1.$$

Proof. Analogous to the proof of Theorem 15. □

5. Conclusion

Remark The method used in Section 3 to obtain a lower bound for $B(2^s, id)$ can be extended in the obvious way to obtain the following lower bound for $B(2^s, mx)$ for any positive integer m and s.

$$B(2^s, mx) \geq n_s = m2^s \prod_{i=0}^{s-1} (m2^i + 1).$$

Therefore, for any positive integer k,

$$B(k, mx) \geq (mk)^{clogk}$$

for some $c > 0$.

There is a big gap between the lower and upper bounds established for $B(k, id)$. The known values suggests that the upper bound is a better estimate. In fact, it seems like

$$B(k, id) = k \cdot \left(2^{k-1}\right) + O(k).$$

It would be nice to have proven this.

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