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AN OBSERVATION ON THE EXTENSION OF ABEL'S LEMMA

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Abstract

We prove a general formula for a certain class of sums of tails. A new proof of some known identities is given, and a general identity that appears to be the first of its kind is established.

1. Introduction and Main Results

In the work of Andrews, Jiménez-Urroz and Ono [2], it was shown that sums over the differences of q-products and truncated q-products have very interesting applications to both partitions and generating functions for values of L-functions (see [5, 8] as well). Series of this type are more casually termed *sums of tails* [5]. One elegant example, given in Ramanujan's "lost" notebook [4], is:

$$\sum_{n=0}^{\infty}((-q)_\infty - (-q)_n) = (-q)_\infty D(q) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q)_n}, \tag{1}$$

where

$$D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n},$$

and where the q-Pochhammer symbol [7] $(\xi)_n = (\xi; q)_n := \prod_{j=0}^{n-1}(1 - \xi q^j)$ has been used. In [1], Andrews gave the first proof of (1) and a related identity found in Ramanujan's "lost" notebook [4]. Zagier [9] also studied series of this type after Ramanujan, and his work gave some of the motivation for the results contained in [2].

Proposition 2.1 of [1] is the key formula to obtaining these special sums of tails identities, and its proof requires use of Abel's lemma, which states that if $\lim_{n\to\infty} a_n = L$, then $\lim_{t\to 1^-}(1-t)\sum_{n=0}^{\infty} a_n t^n = L$. For a proof of Proposition 2.1, which we now state, see [4], Chapter 7, "Special Identities."

(Proposition 2.1. [2]) *Suppose that* $f(z) = \sum_{n=0}^{\infty} \alpha(n) z^n$ *is analytic for* $|z| < 1$. *If* α *is a complex number for which*

$$\text{(i)} \qquad \sum_{n=0}^{\infty} (\alpha - \alpha(n)) < +\infty$$

and

$$\text{(ii)} \qquad \lim_{n\to\infty} n(\alpha - \alpha(n)) = 0,$$

then

$$\lim_{z\to 1^-} \frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty} (\alpha - \alpha(n)).$$

Andrews and Freitas [3] have found how to generalize Proposition 2.1 of [2] to the p-th derivative. Two examples from [3], which are of interest to our study, are:

$$\sum_{n=0}^{\infty} \left(\frac{1}{(q)_\infty} - \frac{1}{(q)_n} \right)^2 = \frac{1}{(q)_\infty^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{q^{n(n+1)/2}}{(q)_n} \frac{1-(q)_n}{1-q^n}, \tag{2}$$

$$\sum_{n=0}^{\infty} \left(\frac{(t)_\infty}{(q)_\infty} - \frac{(t)_n}{(q)_n} \right)^2 = \frac{(t)_\infty^2}{(q)_\infty^2} \sum_{n=1}^{\infty} \frac{(q/t)_n t^n}{(q)_n} \left(\frac{(q)_n}{(t)_n} - 1 \right) \frac{1}{1-q^n}. \tag{3}$$

Here (2) is [3, pg.148, eq.(iii)] and a corollary of (3), and (3) is [3, pg.148, eq.(ii)]. As it turns out, these two identities are also a consequence of our main theorem, and can be viewed as a corollary of Proposition 2.1.

Theorem 1 *Assuming the hypothesis of Proposition 2.1, we have for each integer* $r \geq 1$,

$$\sum_{n=0}^{\infty} (\alpha - \alpha(n))^r = -\sum_{k=1}^{r} \binom{r}{k} (\alpha)^{r-k} (-1)^k \sum_{n=0}^{\infty} (\alpha^k - \alpha^k(n)). \tag{4}$$

Throughout this note we will be primarily concerned with applications to q-series, and so we will not run into any problems with our choices of $\alpha(n)$.

2. Proof of Main Theorem

The proof of Theorem 1 essentially relies on Proposition 2.1, which is why one can view Theorem 1 as a corollary of Proposition 2.1. First, we need the binomial theorem [7, p. 25]:

$$(x+y)^r = \sum_{k=0}^{r} \binom{r}{k} x^{r-k} y^k. \tag{5}$$

Set $\alpha = 0$, $\alpha(n) = \eta_n(r) := (\lambda - \lambda_n)^r$, and assume $\eta_n(r)$ satisfies (i) and (ii), where λ_n also satisfies the hypothesis of Proposition 2.1 and $\lim_{n\to\infty} \lambda_n = \lambda$. Following the proofs in [1], set ϵ to be the differential operator $\epsilon = \lim_{t\to 1^-} \frac{d}{dt}$. Then, using the fact that $\alpha = \lim_{n\to\infty} \eta_n(r) = 0$, by Proposition 2.1, it follows that

$$\begin{aligned}
-\sum_{n=0}^{\infty} (\lambda - \lambda_n)^r &= \epsilon(1-t) \sum_{n=0}^{\infty} \eta_n(r) t^n = \epsilon(1-t) \sum_{n=0}^{\infty} (\lambda - \lambda_n)^r\, t^n \\
&= \epsilon(1-t) \sum_{n=0}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (\lambda)^{r-k} (-\lambda_n)^k t^n \\
&= \sum_{k=0}^{r} \binom{r}{k} (\lambda)^{r-k} (-1)^k \epsilon(1-t) \sum_{n=0}^{\infty} (\lambda_n)^k t^n \\
&= \sum_{k=1}^{r} \binom{r}{k} (\lambda)^{r-k} (-1)^k \sum_{n=0}^{\infty} (\lambda^k - \lambda_n^k).
\end{aligned}$$

In the fifth line we have used Proposition 2.1 and the fact that if $k = 0$, then $\lambda_n^0 = 1$ and $\epsilon(1-t) \sum_{n\geq 0} t^n = 0$.

3. Proof of (3)

Here we provide proofs of (2) and (3) using Theorem 1. Clearly (2) is just the case $t = 0$ of (3), so we will just prove, in detail, the identity (3).

Proposition 2 *Identity (3) is valid.*

Proof. If we take $r = 2$ in Theorem 1 we get

$$\sum_{n=0}^{\infty} (\alpha - \alpha(n))^2 = 2\alpha \sum_{n=0}^{\infty} (\alpha - \alpha(n)) - \sum_{n=0}^{\infty} \left(\alpha^2 - \alpha(n)^2\right). \tag{6}$$

Now taking $\alpha := (t)_\infty/(q)_\infty$ and $\alpha(n) := (t)_n/(q)_n$, we find α satisfies (i) and (ii) of Proposition 2.1, and further

$$\sum_{n=0}^{\infty}\left(\frac{(t)_\infty}{(q)_\infty}-\frac{(t)_n}{(q)_n}\right)^2 = 2\frac{(t)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\left(\frac{(t)_\infty}{(q)_\infty}-\frac{(t)_n}{(q)_n}\right) - \sum_{n=0}^{\infty}\left(\left(\frac{(t)_\infty}{(q)_\infty}\right)^2-\left(\frac{(t)_n}{(q)_n}\right)^2\right). \tag{7}$$

In [3, Corollary 4.3 (vi)] we find

$$\sum_{n=0}^{\infty}\left(\left(\frac{(t)_\infty}{(q)_\infty}\right)^2-\left(\frac{(t)_n}{(q)_n}\right)^2\right) = -\left(\frac{(t)_\infty}{(q)_\infty}\right)^2\sum_{n=1}^{\infty}\frac{(q/t)_n t^n}{(q)_n(1-q^n)}\left(\frac{(q)_n}{(t)_n}+1\right). \tag{8}$$

By Corollary 4.2 of [2] with $a=q$ we have

$$\sum_{n=0}^{\infty}\left(\frac{(t)_\infty}{(q)_\infty}-\frac{(t)_n}{(q)_n}\right) = -\frac{(t)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(q/t)_n t^n}{(q)_n(1-q^n)}. \tag{9}$$

Since $\alpha = (t)_\infty/(q)_\infty$, we find by (6) that (3) readily follows after inserting (9) and (8). □

4. The Case $r > 2$

The slightly more difficult task with Theorem 1 is finding identities for $r > 2$. We consider an identity for general $r > 2$, for a particular choice of $\alpha(n)$. First we need a new result from a paper by Fang [6]. Throughout this section we will use standard notation for the q-binomial coefficients

$$\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} := \frac{(1-q)(1-q^2)\cdots(1-q^{m_0})}{(1-q)(1-q^2)\cdots(1-q^{m_1})(1-q)(1-q^2)\cdots(1-q^{m_0-m_1})}.$$

Theorem 3 (Fang [6, Corollary 6.1]) *For $0 \le m_{t+1} \le m_t \le \cdots \le m_1 \le m_0 = m$, where t, m, and the m_t are non-negative integers, and $|x| < 1$, we have*

$$(q;q)_\infty^{t+2}\sum_{n=0}^{\infty}\frac{(c;q)_n x^n}{(q;q)_n^{t+3}} = \frac{(cx;q)_\infty}{(x;q)_\infty}\sum_{m=0}^{\infty}\frac{(x;q)_m(-1)^m q^{m(m+1)/2}}{(q,cx;q)_m} \tag{10}$$

$$\times \sum_{m_1,m_2,\cdots,m_{t+1}}\begin{bmatrix} m_0 \\ m_1 \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\begin{bmatrix} m_2 \\ m_3 \end{bmatrix}\cdots\begin{bmatrix} m_t \\ m_{t+1} \end{bmatrix} q^{\sum_{i=0}^{t} m_{i+1}(m_{i+1}-m_i)}.$$

Now set $\alpha := 1/(q)_\infty^{t+3}$, $\alpha(n) := 1/(q)_n^{t+3}$, and put

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^{t+3}}.$$

Using Proposition 2.1, Equation (10) with $c = 0$ and $x = z$, we find that

$$\sum_{n=0}^{\infty} \left(\frac{1}{(q)_\infty^{t+3}} - \frac{1}{(q)_n^{t+3}} \right) = \epsilon(1-z) f(z) = \epsilon(1-z) \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^{t+3}}$$

$$= \epsilon(1-z) \frac{1}{(q;q)_\infty^{t+2} (z;q)_\infty} \sum_{m=0}^{\infty} \frac{(z;q)_m (-1)^m q^{m(m+1)/2}}{(q;q)_m}$$

$$\times \sum_{0 \le m_{t+1} \le m_t \le \cdots \le m_1 \le m_0 = m} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \end{bmatrix} \cdots \begin{bmatrix} m_t \\ m_{t+1} \end{bmatrix} q^{\sum_{i=0}^{t} m_{i+1}(m_{i+1} - m_i)}$$

$$= \frac{1}{(q)_\infty^{t+3}} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + \frac{1}{(q)_\infty^{t+3}} \epsilon \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2} (z;q)_m}{(q;q)_m}$$

$$\times \sum_{0 \le m_{t+1} \le m_t \le \cdots \le m_1 \le m_0 = m} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \end{bmatrix} \cdots \begin{bmatrix} m_t \\ m_{t+1} \end{bmatrix} q^{\sum_{i=0}^{t} m_{i+1}(m_{i+1} - m_i)}.$$

Finally, observing that $\epsilon(z;q)_m/(q;q)_m = -1/(1-q^m)$, we find the last equation produces the following lemma.

Lemma 4 *For each non-negative integer t, we have*

$$\sum_{n=0}^{\infty} \left(\frac{1}{(q)_\infty^{t+3}} - \frac{1}{(q)_n^{t+3}} \right) = \frac{1}{(q)_\infty^{t+3}} \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{(1-q^m)} \right. \tag{11}$$

$$\left. \times \sum_{0 \le m_{t+1} \le m_t \le \cdots \le m_1 \le m_0 = m} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \end{bmatrix} \cdots \begin{bmatrix} m_t \\ m_{t+1} \end{bmatrix} q^{\sum_{i=0}^{t} m_{i+1}(m_{i+1} - m_i)} \right).$$

We can now obtain the following new result.

Theorem 5 *For any integer $r > 2$, we have*

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_n}\right)^r = r(q)_\infty^{-r}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\frac{1}{2}r(r-1)(q)_\infty^{-r}$$

$$\times\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{1-q^n}\right)$$

$$-(q)_\infty^{-r}\sum_{k=0}^{r-3}\binom{r}{k+3}(-1)^{k+3}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{m=1}^{\infty}\frac{(-1)^m q^{m(m+1)/2}}{(1-q^m)}\right.$$

$$\left.\times\sum_{\substack{0\le m_{k+1}\le m_k\le\cdots\\ \le m_1\le m_0=m}}\begin{bmatrix}m_0\\ m_1\end{bmatrix}\begin{bmatrix}m_1\\ m_2\end{bmatrix}\begin{bmatrix}m_2\\ m_3\end{bmatrix}\cdots\begin{bmatrix}m_k\\ m_{k+1}\end{bmatrix}q^{\sum_{i=0}^{k}m_{i+1}(m_{i+1}-m_i)}\right). \quad (12)$$

Proof. First observe that, from Theorem 1, we can write for $r > 2$,

$$\sum_{n=0}^{\infty}(\alpha-\alpha(n))^r = r\alpha^{r-1}\sum_{n=0}^{\infty}(\alpha-\alpha(n))-\frac{1}{2}r(r-1)\alpha^{r-2}\sum_{n=0}^{\infty}\left(\alpha^2-\alpha(n)^2\right)$$

$$-\sum_{k=3}^{r}\binom{r}{k}(\alpha)^{r-k}(-1)^k\sum_{n=0}^{\infty}(\alpha^k-\alpha^k(n)). \quad (13)$$

Choosing $\alpha(n) := 1/(q)_n$ and $\alpha := 1/(q)_\infty$ in (13) we obtain

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_n}\right)^r = r(q)_\infty^{-(r-1)}\sum_{n=0}^{\infty}\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_n}\right)$$

$$-\frac{1}{2}r(r-1)(q)_\infty^{-(r-2)}\sum_{n=0}^{\infty}\left(\frac{1}{(q)_\infty^2}-\frac{1}{(q)_n^2}\right)$$

$$-\sum_{k=0}^{r-3}\binom{r}{k+3}\left(\frac{1}{(q)_\infty}\right)^{r-(k+3)}(-1)^{k+3}\sum_{n=0}^{\infty}\left((q)_\infty^{-(k+3)}-(q)_n^{-(k+3)}\right). \quad (14)$$

Since (11) holds for each non-negative integer t, we insert it into the third line of (14) to get

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_n}\right)^r = r(q)_\infty^{-r}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\frac{1}{2}r(r-1)(q)_\infty^{-r}$$
$$\times\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{1-q^n}\right)$$
$$-\sum_{k=0}^{r-3}\binom{r}{k+3}(\frac{1}{(q)_\infty})^{r-(k+3)}(-1)^{k+3}\frac{1}{(q)_\infty^{k+3}}$$
$$\times\Bigg(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{m=1}^{\infty}\frac{(-1)^m q^{m(m+1)/2}}{(1-q^m)}$$
$$\times\sum_{\substack{0\le m_{k+1}\le m_k\le\cdots\\ \le m_1\le m_0=m}}\begin{bmatrix}m_0\\m_1\end{bmatrix}\begin{bmatrix}m_1\\m_2\end{bmatrix}\begin{bmatrix}m_2\\m_3\end{bmatrix}\cdots\begin{bmatrix}m_k\\m_{k+1}\end{bmatrix}q^{\sum_{i=0}^{k}m_{i+1}(m_{i+1}-m_i)}\Bigg), \quad (15)$$

upon invoking [2, pg.19, eq.(6.7)], and [2, pg.19, eq.(5.4)]. Lastly, canceling out the product $(q)_\infty^{-k-3}$, gives the theorem. □

5. Concluding Remarks

The results contained herein suggest that there should be some further interesting consequences of Proposition 2.1. One way we could obtain a generalization of Theorem 1 is choosing a different $\eta_n(r)$ in our proof. Namely, we could choose

$$\eta_n(r)=\prod_{i=1}^{r}(\alpha_i-\alpha_i(n)),$$

where at least some of the $\alpha_i(n)$ are different for each i with $1\le i\le r$. (Choosing all the $\alpha_i(n)$ to be equal to each other for each i gives Theorem 1.) The trick here is to ensure that $\lim_{n\to\infty}\eta_n(r)=0$. The difficult task in applications of Theorem 1 to q-series is finding the simpler, more common expressions for integers $r>2$, for the right hand side of (4). We can, however, obtain more results like Theorem 5 using Fangs' more general identity [6, pg.1404, Theorem 6.1]. It would be interesting to see some applications of Proposition 2.1 outside of q-series.

References

[1] G. E. Andrews, *Ramanujan's Lost Notebook V: Eulers partition identity,* Advances in Mathematics 61 (1986), 156-164.

[2] G. E. Andrews, J. Jiménez-Urroz and K. Ono, *q-series identities and values of certain L-functions*, Duke Math. Journal 108 (2001), 395-419.

[3] G.E. Andrews, and P. Freitas, *Extension of Abel's lemma with q-series implications*, Ramanujan J. 10 (2005), 137-152.

[4] G. E. Andrews, and B. C. Berndt, *Ramanujan's lost notebook, Part II.* Springer, New York, 2009. xii+418 pp.

[5] K. Bringmann and B. Kane, *New identities involving sums of the tails related to real quadratic fields,* Special issue of the Ramanujan Journal in honor of G.E. Andrews's 70th birthday, The Ramanujan J., accepted for publication.

[6] J.-P. Fang, *q-Differential operator identities and applications,* J. Math. Anal. Appl., 332 (2007), 1393-1407.

[7] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge Univ. Press, Cambridge, 1990.

[8] A. E. Patkowski *Divisors, partitions and some new q-series identities,* Colloq. Math. 117 (2009), 289-294.

[9] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function,* Topology 40 (2001), pages 945-960.