#G04 INTEGERS 10 (2010), 849-878

ON SOLVING GAMES CONSTRUCTED USING BOTH SHORT AND LONG CONJUNCTIVE SUMS

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Received: 8/13/08, Revised: 5/24/10, Accepted: 9/5/10, Published: 12/6/10

Abstract

In a 1966 paper by C.A.B. Smith, the short and long conjunctive sums of games are defined and methods are described for determining the theoretical winner of a game constructed using one type of these sums. In this paper, we develop a method for determining the winner of a game constructed using arbitrary combinations of these sums.

1. Introduction

The conjunctive sum of two combinatorial games is a new game in which both component games are played concurrently. A move in the composite game consists of making a move in each component game. These games were originally introduced by Smith in [1]. Conway, in [3], notes that there are two distinct ending conditions for such sums: the short-rule and the long-rule. When playing with the short-rule, the composite game ends when the first component game ends. When using the long-rule, the composite ends when the last component game ends. There are well-known quantities called the remoteness and the suspense of a game that allow one to analyze games constructed from only short-rule conjunctive sums or only long-rule conjunctive sums. In this paper, we determine what information about a game is relevant for analyzing constructions using arbitrary combinations of short-rule and long-rule conjunctive sums.

In Section 2, we define a combinatorial game; introduce remoteness and suspense; discuss some simple games that we use later; and prove some basic results about these concepts. In Section 3, we give a more precise description of the problem we wish to solve and discuss an example of a game that could be analyzed with a solution. In Section 4, we solve the aforementioned problem by finding information about a game that allows us to analyze it under short and long conjunctive sums. In Section 5, we analyze the set of values that this information can actually take. In Section 6, we study the misère versions of games under short and long conjunctive sums. In Section 7, we discuss the addition of the operation we call concatenation. In Section 8, we discuss directions for further research.

2. Definitions, Notation and Preliminary Results

In this section we will provide definitions of what we mean by a game and give rigorous definitions of the shortened and continued conjunctive sums. We will then review the theory of Remoteness and Suspense of a game. These definitions and results (found in Sections 2.1, 2.3 and 2.4) are essentially recastings of results of Smith ([1]) into more modern terminology. It should be noted though that the notion of Remoteness dates back at least to Steinhaus ([2]) in 1925. In Section 2.5 we introduce a class of games that will prove very important in our later analysis and prove some basic results about them. Finally in Section 2.6, we use this theory to prove a pair of important lemmas.

Throughout this paper, unless stated otherwise, Greek letters represent ordinal numbers, uppercase Roman letters represent games and lowercase Roman letters represent members of an indexing set.

2.1. Games

We deal only with impartial games. These are games where in any position the options are the same for either player. Note that considering games under conjunctive sums this is the general case, since for each position of a game we can encode whose turn it is into the position. So for the purposes of this paper, we use the following definition of a game:

Definition. A game is defined recursively as a set of other games, also referred to as options or positions. The game is played by two players, First and Second. The two players alternate making moves, with First going first. On a turn, a player picks an option of the last position picked (or a position of the original game if it is the first move). The game ends once a player is unable to make a move (the current position is the empty set). The last player to make a move wins the game.

Notice that First wins a game G if and only if the game takes an odd number of moves to play.

Most of our results will depend on the axiom of foundation which states that if there is any property, P, such that whenever P holds for all elements of a set X, P also holds for X, then P holds for all sets. This essentially allows us to use induction on sets.

Definition. A game G is winning if First has a strategy by which he can guarantee that he will win. Otherwise, G is losing.

We now recall some basic facts about games.

Lemma 1. *A game is winning if and only if one of its options is losing. A game is losing if and only if Second has a winning strategy.*

Proof. This follows easily by induction. □

Corollary 2. *For any game G, some player has a winning strategy.*

Definition. If A and B are games, let $A =_w B$ denote that A and B are either both winning or both losing.

We define the game 0 to be $\emptyset$. We now define the short and long conjunctive sums of games recursively.

Definition. If A and B are games with options $\{A_i\}$ and $\{B_j\}$ respectively, the short-rule conjunctive sum, or short conjunctive sum, of A and B, denoted $A \wedge B$, is 0 if $A = 0$ or if $B = 0$ and $\{A_i \wedge B_j\}$ otherwise. The long-rule conjunctive sum, or long conjunctive sum, of A and B, denoted $A \triangle B$, is A if $B = 0$, B if $A = 0$ and $\{A_i \triangle B_j\}$ otherwise.

2.2. Ordinal Numbers

In this section we give a brief review of ordinal numbers and their properties. A reader already familiar with the ordinal numbers should feel free to skip this section.

Definition. An ordinal number is the order type of a well-ordered set.

The *finite ordinals* are the order types given by linear orderings on finite sets of a given size. So n is the well-ordering on $\{1, 2, \ldots, n\}$.

Note that if we have two well-ordered sets, it can be shown by induction that one is isomorphic to a subset of the other. This provides a natural linear order on the ordinal numbers. It is well-known that this ordering on the ordinal numbers is itself a well-ordering and furthermore, that it has the least-upper-bound property.

All ordinals α have a *successor*, $\alpha+1$. Ordinals which are not successors are called *limit ordinals*. It can be shown by induction that any ordinal can be obtained by taking finitely many successors of a limit ordinal.

2.3. Remoteness and Suspense

In this section we define the remoteness and suspense, which are ordinal numbers associated to a game. The idea behind these numbers is as follows. In a short conjunctive sum, it only matters who wins the first component game (to end). Hence if you are winning a component game, you should try to win it as quickly as possible. On the other hand if you are losing a component game, you should try to lose it as slowly as possible. Likewise, for long conjunctive sums, you should try to win component games slowly or lose them quickly. The remoteness and suspense are the (ordinal) lengths of time that the game should take to play using these strategies for playing short sums and long sums respectively.

We begin by defining remoteness. First we need to define a notion of what it means for the length of a game to correspond to being winning, for which the following definition suffices.

Definition. Let an ordinal number be R-even if it is either a finite even ordinal or a limit ordinal plus a finite even ordinal. Let an ordinal be R-odd if it is not R-even.

The idea here is that R-odd lengths correspond to winning games. Next we need to define an order on these lengths corresponding to how much First would like a subgame to be of that length.

Definition. We define a linear ordering $\geq_R$ on ordinal numbers by letting $\alpha \geq_R \beta$ if and only if one of the following hold:

- α is R-odd and β is R-even
- $\alpha \leq \beta$ and both are R-odd
- $\alpha \geq \beta$ and both are R-even

We can now recursively define the remoteness of a game.

Definition. Given a game $G = \{G_i\}$ we define the ordinal $R(G)$, called the *remoteness* of G, to be 0 if $G = 0$ and otherwise

$$R(G) = \begin{cases} \min_{R(G_i) \text{ is R-even}}(R(G_i)+1), & \text{if some } R(G_i) \text{ is R-even,} \\ \sup_i(R(G_i)+1), & \text{otherwise.} \end{cases}$$

In other words, $R(G)$ is the least upper bound in the R-ordering of the ordinals of the form $R(G_i)+1$.

We use an analogous approach to define suspense.

Definition. Let an ordinal number be S-even if it is either a finite even ordinal or a limit ordinal plus a finite odd ordinal. Let an ordinal be S-odd if it is not S-even.

Definition. We define a linear ordering $\geq_S$ on ordinal numbers by letting $\alpha \geq_S \beta$ if and only if one of the following hold:

- α is S-odd and β is S-even
- $\alpha \geq \beta$ and both are S-odd
- $\alpha \leq \beta$ and both are S-even

Definition. Given a game $G = \{G_i\}$ we define the ordinal $S(G)$, called the *suspense* of G, to be 0 if $G = 0$ and otherwise

$$S(G) = \begin{cases} \sup_{S(G_i) \text{ is S-even}}(S(G_i)+1), & \text{if some } S(G_i) \text{ is S-even,} \\ \min_i(S(G_i)+1), & \text{otherwise.} \end{cases}$$

In other words, $S(G)$ is the least upper bound in the S-ordering of ordinals of the form $S(G_i) + 1$.

2.4. The Properties of the Remoteness and Suspense

We now prove the properties of remoteness and suspense that allow them to be used to determine the winners of games constructed using short conjunctive sums and games constructed using long conjunctive sums respectively.

A game is winning if it takes an odd number of turns. Therefore, it should be winning if it has odd remoteness.

Lemma 3. *A game $G = \{G_i\}$ is winning if and only if its remoteness is R-odd.*

Proof. We proceed by induction. If $G = 0$, then $R(G) = 0$, which is R-even and G is losing. Assume that for all options G_i of G that G_i is winning if and only if $R(G_i)$ is R-odd. If G is winning, then at least one option is losing, hence $R(G_i)$ is R-even for some option, G_i. In this case $R(G) = (\min_{R(G_i) \text{ is R-even}} R(G_i)) + 1$, which is one more than an R-even number. Hence $R(G)$ is R-odd. If G is losing, then all options of G are winning, hence R-odd. So $R(G) = \sup_i(R(G_i) + 1)$, which is the sup of a set of R-even numbers. Thus $R(G)$ is either the largest of these numbers or a limit ordinal. In either case, $R(G)$ is R-even. This completes the proof. $\square$

Next we show how remoteness behaves under short conjunctive sums. The idea is that the subgames A and B should take $R(A)$ and $R(B)$ turns. Hence the whole game should take $\min(R(A), R(B))$ turns.

Lemma 4. *If A and B are games, then $R(A \wedge B) = \min(R(A), R(B))$.*

Proof. Let $A = \{A_i\}$ and $B = \{B_j\}$. We proceed by induction. Suppose that one of A or B is 0. Without loss of generality $B = 0$. Then $A \wedge B = 0$, so $R(A \wedge B) = 0$. Since $B = 0$, we have $R(B) = 0$, hence $R(A \wedge B) = \min(R(A), R(B))$.

Assume that $R(A_i \wedge B_j) = \min(R(A_i), R(B_j))$ for all i, j.

`Case 1:` $\min(R(A), R(B))$ is R-even

Without loss of generality $R(A) \leq R(B)$. Then $R(A)$ is R-even. Hence $R(A_i)$ is R-odd for all i. Now, $R(A) = \sup_i(R(A_i) + 1)$. Since $R(A) \leq R(B)$, for no j is $R(B_j)$ both R-even and less than $R(A)$. Hence all j for which $R(B_j)$ is R-even have $R(B_j) \geq R(A) > R(A_i)$ for all i. Hence, $\min(R(A_i), R(B_j)) = R(A_i \wedge B_j)$ is always R-odd. So,

$$R(A \wedge B) = \sup_{i,j}(\min(R(A_i), R(B_j)) + 1) \leq \sup_i(R(A_i) + 1) = R(A).$$

For each i, $\sup_j(R(B_j) + 1) \geq R(B) \geq R(A) \geq R(A_i) + 1$, so there exists a j with $R(B_j) \geq R(A_i)$. Hence $R(A \wedge B) = \sup_{i,j}(\min(R(A_i), R(B_j)) + 1) \geq \sup_i(R(A_i) + 1) = R(A)$. Hence $R(A) \geq R(A \wedge B) \geq R(A)$. So $R(A \wedge B) = R(A) = \min(R(A), R(B))$.

`Case 2:` $\min(R(A), R(B))$ is R-odd

Without loss of generality $R(A) \leq R(B)$. We have

$$R(A) = \min_{R(A_i) \text{ is R-even}} R(A_i) + 1.$$

Let A_k be an option such that $R(A_k) + 1 = R(A)$. Since $R(B) \geq R(A)$, there exists j such that $R(B_j) \geq R(A_k)$. Hence, $R(A_k \wedge B_j) = R(A_k)$ is R-even. Hence since some option of $A \wedge B$ is R-even, thus

$$R(A \wedge B) = \min_{i,j}(\min(R(A_i), R(B_j)) + 1)$$

where the minimum is taken over terms where $\min(R(A_i), R(B_j))$ is R-even. At least one such term equals $R(A_k)$, and all other such terms equal $R(A_i)$ or $R(B_j)$ for some i or j. Since $R(B_j) \geq R(A_k)$ whenever $R(B_j)$ is R-even and $R(A_i) \geq R(A_k)$ whenever $R(A_i)$ is R-even, we have $R(A) = R(A_k) + 1 \geq R(A \wedge B) \geq R(A_k) + 1 = R(A)$. Hence $R(A \wedge B) = R(A) = \min(R(A), R(B))$. □

We now state the analogous results for suspense. The proofs proceed along similar lines and are left to the reader.

Lemma 5. *A game $G = \{G_i\}$ is winning if and only if its suspense is S-odd.*

Next we show how suspense behaves under long conjunctive sums. The idea is that the subgames A and B should take $S(A)$ and $S(B)$ turns. Hence the whole game should take $\max(S(A), S(B))$ turns.

Lemma 6. *If A and B are games, then $S(A \triangle B) = \max(S(A), S(B))$.*

2.5. Some Important Games

We now define a set of games that will prove necessary in later work.

Definition. If $\alpha > \beta$ are ordinal numbers, define the game $[\alpha, \beta]$ recursively by $[\alpha, \beta] = \{[\gamma, \delta] : \alpha > \gamma \geq \beta, \gamma > \delta\}$.

Definition. If α is an ordinal, let $[\alpha] = [\alpha + 1, \alpha]$.

The idea is that we would like a game $[\alpha]$ that in some sense takes α moves to play, where α is an ordinal number. For finite ordinals, this is easy to produce, but for infinite ordinals we need to specify what we mean. In particular we want the structure of decisions made to determine the length of the game look like the process of starting at α and repeatedly picking a smaller ordinal number. The first coordinate of the subgame $[\alpha, \beta]$ can be thought of as keeping track of our place. On the other hand, we would like to guarantee that this process doesn't skip too many ordinals. This is why the second coordinate is used to ensure that both players can guarantee that the rate of descent is as slow as they desire. Note further that when making a move to $[\gamma, \delta]$ it is in one's interest to make δ as large as possible, since increasing the size of δ will only serve to decrease the number of options that the opponent has on their next turn.

Lemma 7. *If A has all the options of B, then $R(A) \geq_R R(B)$.*

Proof. The claim follows directly from the definition of remoteness. □

Lemma 8. *$R([\alpha])$ is α.*

Proof. We proceed by induction. If $\alpha = 0$, then $[\alpha + 1, \alpha] = [1, 0] = 0$ which has remoteness 0. Assume that the statement is true for all $[\beta]$ with $\beta < \alpha$.

`Case 1:` α is not a limit ordinal.
We have $[\alpha+1, \alpha] = \{[\alpha, \beta] : \beta < \alpha\}$. Since $[\alpha, \beta]$ has all of the options of $[\alpha, \alpha-1]$, Lemma 7 implies that $R([\alpha, \beta]) \geq_R R([\alpha, \alpha-1])$. Since $R([\alpha+1, \alpha])$ depends only on the sufficiently R-small remotenesses of its options, $R([\alpha+1, \alpha]) = R(\{[\alpha, \alpha-1]\}) = R([\alpha, \alpha-1]) + 1 = \alpha$, as desired.

`Case 2:` α is a limit ordinal.
We have $[\alpha + 1, \alpha] = \{[\alpha, \beta] : \beta < \alpha\}$. The inductive hypothesis implies that games of the form $[\alpha', \beta']$, with $\alpha' \leq \alpha$, can be losing only if $\alpha' = \beta' + 1$. This is because if $\alpha' > \beta' + 1$, $[\alpha', \beta']$ has a losing options of the form $[\gamma + 1, \gamma]$,

where $\alpha > \gamma + 1 \geq \beta$ and γ is R-even. Therefore, all options of $[\alpha + 1, \alpha]$ are winning, and so $[\alpha + 1, \alpha]$ is losing. The losing options of $[\alpha, \beta]$ are options of the form $[\gamma + 1, \gamma]$ with $\alpha > \gamma + 1 \geq \beta$ and these have options have remoteness γ. So $R([\alpha, \beta])$ is R-odd and between α and β. Hence the remoteness of any option of $[\alpha + 1, \alpha]$ is R-odd and can be an arbitrarily large ordinal less than α. Hence $R([\alpha + 1, \alpha]) = \sup_\beta(R([\alpha, \beta]) + 1) = \alpha$, as desired. □

We now show an important property of these games in relation to remoteness.

Lemma 9. *If G is a game, then $R(G)$ is one less than the smallest ordinal α such that $G \wedge [\alpha] \neq_w [\alpha]$.*

Proof. Let $R(G) = \beta$. By Lemmas 4 and 8 if $\alpha \leq \beta$ then $R(G \wedge [\alpha]) = \min(\beta, \alpha) = \alpha = R([\alpha])$, so by Lemma 3 $G \wedge [\alpha] =_w [\alpha]$. But if $\alpha = \beta + 1$, then $R(G \wedge [\alpha]) = \min(\beta + 1, \beta) = \beta$. Since $R([\alpha]) = \beta + 1$, we have $G \wedge [\alpha] \neq_w [\alpha]$. The lemma follows. □

We now state the analogous results for suspense. Again the proofs are similar.

Lemma 10. *If A has all the options of B, then $S(A) \geq_S S(B)$.*

Lemma 11. *$S([\alpha])$ is α if α is finite, and $\alpha + 1$ otherwise.*

Lemma 12. *If G is a game, then $S(G)$ is determined uniquely by the α for which $G \triangle [\alpha]$ is winning.*

We next determine the short and long conjunctive sums of these special games. Before we can do this though, we need a new definition of equivalence.

Definition. If A and B are games let $A = B$ mean that their underlying sets are the same. This is equivalent to saying that for every option A_i of A, there is an option B_j of B with $A_i = B_j$; and that for every option B_j of B, there is an option A_i of A with $A_i = B_j$.

It is immediate that:

Lemma 13. *If A, B and G are games with $A = B$, then:*

1. *$A =_w B$,*

2. *$A \wedge G = B \wedge G$,*

3. *$A \triangle G = B \triangle G$.*

Lemma 14. $[\alpha, \beta] \wedge [\gamma, \delta] = [\min(\alpha, \gamma), \min(\beta, \delta)]$.

Proof. We proceed by induction. If one of the components is $0 = [1, 0]$, (say $[\gamma, \delta]$ is) then the whole game is 0 and $[\min(\alpha, \gamma), \min(\beta, \delta)] = [1, 0] = 0$. Assume that the statement is true for all $\alpha' < \alpha$ and $\gamma' < \gamma$. Then the options of $[\alpha, \beta] \wedge [\gamma, \delta]$ are $[\alpha', \beta'] \wedge [\gamma', \delta'] = [\min(\alpha', \gamma'), \min(\beta', \delta')]$ under the restrictions that $\alpha > \alpha' \geq \beta$, $\gamma > \gamma' \geq \delta$, $\alpha' > \beta'$ and $\gamma' > \delta'$. These restrictions imply that $\min(\alpha, \gamma) > \min(\alpha', \gamma') \geq \min(\beta, \delta)$ and $\min(\alpha', \gamma') > \min(\beta', \delta')$. Also there are no further restrictions on $\min(\alpha', \gamma')$ and $\min(\beta', \delta')$. Hence the above restrictions are equivalent to $[\min(\alpha', \gamma'), \min(\beta', \delta')]$ being an option of $[\min(\alpha, \gamma), \min(\beta, \delta)]$. This proves the lemma. $\square$

Lemma 15. $[\alpha, \beta] \triangle [\gamma, \delta] = [\max(\alpha, \gamma), \max(\beta, \delta)]$.

Proof. The proof is analogous to that of Lemma 14. $\square$

Lastly, we show that if these games are played in a conjunctive sum, the order in which they end should be relative to the ordinal indexing the game.

Lemma 16. *Suppose that we have finitely many distinct ordinals α_i. Consider the games $[\alpha_i]$ being played in parallel, with First and Second taking turns making a move in each game. Either player can guarantee that:*

- *The game $[\alpha_i]$ ends before the game $[\alpha_j]$ if and only if $\alpha_i < \alpha_j$.*
- *This player wins all of the component games that are theoretically winning for them.*

Proof. We show that either player can maintain the following properties:

1. They are winning all of the component games that they were initially winning
2. For the component games that were originally $[\alpha_i]$ and $[\alpha_j]$ with $\alpha_i > \alpha_j$, we wish to ensure that the current positions $[\beta_i, \gamma_i]$ and $[\beta_j, \gamma_j]$ satisfy $\beta_i > \beta_j$ and that on the opponent's turn, $\gamma_i \geq \beta_j$.

These clearly hold in the initial position and clearly if they hold forever, we ensure the necessary conditions on play.

It is clear that if (1) and (2) hold on the opponent's turn, that they will hold at the beginning of the next turn. This is because the opponent cannot make a winning game losing and when picking options $[\beta_i', \gamma_i']$ and $[\beta_j', \gamma_j']$ of $[\beta_i, \gamma_i]$ and $[\beta_j, \gamma_j]$, he must have $\beta_i' \geq \gamma_i \geq \beta_j > \beta_j'$.

On our player's turn, we show that he can begin with the component game corresponding to the smallest α_i and work towards larger α_i picking moves in the component games that do not violate the invariants. First we note that subgames corresponding to winning α_i were losing on the previous turn, and hence must be in a position $[\beta_i, \gamma_i]$ with β_i R-even. Suppose that moves have been picked in all of the component games corresponding to α_k with $\alpha_k < \alpha_i$. We need to pick an option $[\beta_i', \gamma_i']$ of $[\beta_i, \gamma_i]$ that is winning if $[\alpha_i]$ was and has $\gamma_i' \geq \beta_j'$. But we know that $\beta_i > \beta_j > \beta_j'$. Hence since we can pick options $[\beta_i', \gamma_i']$ of $[\beta_i, \gamma_i]$ that are loosing if necessary and have γ_i' arbitrarily large subject to the condition that $\gamma_i' + 2 \leq \beta_i$, we can pick such an option with $\gamma_i' \geq \beta_j'$, thus maintaining our invariant. □

2.6. Two Important Lemmas

In this section we prove two important lemmas that will allow us to manipulate conjunctive sums of our simple games.

Lemma 17. *For games A and B,* $(A \wedge B) \bigtriangleup [\alpha, \beta] =_w (A \bigtriangleup [\alpha, \beta]) \wedge (B \bigtriangleup [\alpha, \beta])$.

The basic motivational idea behind Lemma 17 is that the simple game $[\alpha, \beta]$ can basically be played in only one way. If the two copies of this game on the right-hand-side are required to be played in the same way though, then the two games are the same.

Proof. `Case 1:` The player p who can win $[\alpha, \beta]$, is winning $(A \wedge B) \bigtriangleup [\alpha, \beta]$.

Player p must be able to win $(A \wedge B) \bigtriangleup [\alpha, \beta]$ in such a way that he is always winning the $[\alpha, \beta]$ component game. Indeed if he is winning both components, he can win by winning both independently and otherwise if he makes a losing move in the $[\alpha, \beta]$ component, he is losing the whole game. Since all losing positions of $[\alpha, \beta]$ are of the form $[\gamma + 1, \gamma]$, player p can win the game $(A \bigtriangleup [\alpha, \beta]) \wedge (B \bigtriangleup [\alpha, \beta])$ by ensuring the following:

- He is always winning the $[\alpha, \beta]$ components of the game and that these components have the same first part
- If the current position is $(X \bigtriangleup [\gamma, \delta]) \wedge (Y \bigtriangleup [\gamma, \epsilon])$, then he is winning the game $(X \wedge Y) \bigtriangleup [\gamma, \max(\delta, \epsilon)]$

This way p ensures that the $[\alpha, \beta]$ components of the game end at the same time, are won by him and that the first of the A and B components ends before this or is won by player p. Hence player p wins $(A \bigtriangleup [\alpha, \beta]) \wedge (B \bigtriangleup [\alpha, \beta])$.

`Case 2:` The player p not winning $[\alpha, \beta]$, is winning $(A \wedge B) \triangle [\alpha, \beta]$.

Call the game $(A \triangle [\alpha, \beta]) \wedge (B \triangle [\alpha, \beta])$ game Z. Player p can win game Z by doing the following. Have a game of $[\alpha, \beta]$ along side of game Z. Make moves for the opponent in the side game to ensure that the opponent is always winning the side game and that the first coordinate of the side game is at least as high as the first coordinate of the $[\alpha, \beta]$ components of game Z. Furthermore, this can be carried out by controlling the moves in the side game made on the other player's turn. Player p makes arbitrary moves in the $[\alpha, \beta]$ components of game Z on his turn and makes moves in the A and B components of game Z and the side game so that he is winning the short sum of A and B long summed with the side game (he can do this because he is winning $(A \wedge B) \triangle [\alpha, \beta]$). Hence the $[\alpha, \beta]$ components of game Z end before the side game. Since player p loses the side game, but wins the game consisting of the short sum of the A and B components of Z long summed with the side game, the first of the A or B components to end is won by player p and after both of the $[\alpha, \beta]$ components are over. Hence, playing this way, player p wins $(A \triangle [\alpha, \beta]) \wedge (B \triangle [\alpha, \beta])$. □

We now prove the analogous result interchanging short and long sums.

Lemma 18. *For games A and B,* $(A \triangle B) \wedge [\alpha, \beta] =_w (A \wedge [\alpha, \beta]) \triangle (B \wedge [\alpha, \beta])$.

Proof. The proof is analogous to that of Lemma 17. □

In particular, we obtain the following corollary:

Corollary 19.

$$(A \wedge B) \triangle [\alpha] =_w (A \triangle [\alpha]) \wedge (B \triangle [\alpha]);$$
$$(A \triangle B) \wedge [\alpha] =_w (A \wedge [\alpha]) \triangle (B \wedge [\alpha]).$$

3. Our Main Objective

The main objective of this paper is to solve games constructed using both short and long conjunctive sums, in the same way that suspense and remoteness allow us to solve games that are constructed using only one type of these sums. In particular we want to associate to each game G some relatively simple information $I(G)$ so that we can easily compute the following:

1. $I(0)$
2. $I(\{G_i\})$ from $I(G_i)$
3. $I(A \wedge B)$ from $I(A)$ and $I(B)$

4. $I(A \triangle B)$ from $I(A)$ and $I(B)$

5. who wins G from $I(G)$.

Note that remoteness has all of these properties but 4 and suspense has all but 3. Given this information, we would be able to solve games such as the following example.

Start with a rectangular array of nodes. On each turn, the array is separated into a number of rectangular subarrays. On each turn, the player whose turn it is applies one of the following operations to each subarray: either delete all the nodes in a row, potentially splitting it into two subarrays, or put a divider between adjacent columns, splitting it into two subarrays. The game ends when there is a path of deleted nodes from the top of the original array to the bottom of the original array that passes through no dividers.

Let us analyze the above game. Let $G(r, c)$ denote the version of the game with r rows and c columns where $r \geq 0$ and $c \geq 1$. We set $G(0, c) = 0$. Otherwise, deleting a row splits the game into a long conjunctive sum of $G(r', c)$ and $G(r'', c)$ where $r' + r'' = r - 1$, and r' and r'' are the number of rows in the two halves. If we add a divider, we end up with the short conjunctive sum of $G(r, c')$ and $G(r, c'')$, where c' and c'' are the number of columns on either side, which satisfy $c' + c'' = c$. Hence if $r > 0$, we have

$$G(r, c) = \{G(r', c) \triangle G(r'', c) : r' + r'' = r - 1\} \cup \{G(r, c') \wedge G(r, c'') : c' + c'' = c\}.$$

We can solve this game by computing $I(G(0, c)) = I(0)$, then using conditions 2, 3, and 4 to recursively compute $G(r, c)$ from $G(r', c')$ with $r' \leq r$, $c' \leq c$ and one of these inequalities strict. Once we have computed $I(G(r, c))$, we use condition 5 to determine who wins the game.

4. The Information Needed

4.1. The Definition of $I(G)$

We here define the information $I(G)$ that we associate with a game and show that it satisfies the conditions in Section 3.

Definition. For a game G, let $I(G)$ be the function that takes pairs of ordinal numbers to the set $\{\text{First}, \text{Second}\}$ by sending (α, β) to the winner of $(G \wedge [\alpha]) \triangle [\beta]$.

Note that this information must be obtainable from any classification of games under short and long conjunctive sum. In particular, this is in some sense the least amount of information needed.

4.2. Some Definitions Equivalent to $I(G)$

In this section we present some information to associate with games equivalent to knowing $I(G)$.

Lemma 20. *Knowing $I(G)$ is equivalent to knowing the winner of $(G \vartriangle [\alpha]) \wedge [\beta]$ for all α, β.*

Proof. By Corollary 19 and Lemma 14,

$$(G \vartriangle [\alpha]) \wedge [\beta] =_w (G \wedge [\beta]) \vartriangle ([\alpha] \wedge [\beta]) =_w (G \wedge [\beta]) \vartriangle ([\min(\alpha, \beta)]).$$

Hence the winner of $(G \vartriangle [\alpha]) \wedge [\beta]$ can be determined from $I(G)$. On the other hand, by Corollary 19 and Lemma 15,

$$(G \wedge [\alpha]) \vartriangle [\beta] =_w (G \vartriangle [\beta]) \wedge ([\alpha] \vartriangle [\beta]) =_w (G \vartriangle [\beta]) \wedge ([\max(\alpha, \beta)]).$$

Hence $I(G)$ can be determined from the winners of $(G \vartriangle [\alpha]) \wedge [\beta]$ for all α, β. □

We now define a more convenient equivalent piece of information.

Definition. For a game G, define functions, r_G and s_G, from the ordinal numbers to the ordinal numbers, by

$$r_G(\lambda) = R(G \vartriangle [\lambda]);$$
$$s_G(\lambda) = S(G \wedge [\lambda]).$$

Corollary 21. *Knowing $I(G)$ is equivalent to knowing $r_G(\omega)$ for all ω and to knowing $s_G(\omega)$ for all ω.*

Proof. The claim follows immediately from Lemmas 9, 12 and 20. □

4.3. Computing $I(0)$

In this section we show the Condition 1 from Section 3 is satisfied by $I(G)$.

Lemma 22. *$(0 \wedge [\alpha]) \vartriangle [\beta]$ is winning if and only if β is R-odd.*

Proof. The claim follows from the fact that $(0 \wedge [\alpha]) \vartriangle [\beta] = 0 \vartriangle [\beta] = [\beta]$. □

4.4. Computing $I(\{G_i\})$ from $I(G_i)$

In this section we show that Condition 2 from Section 3 is satisfied by $I(G)$. We consider three cases.

Lemma 23. $(\{G_i\} \wedge 0) \bigtriangleup [\alpha]$ *is winning if and only if* α *is R-odd.*

Proof. The claim follows from the fact that $(\{G_i\} \wedge 0) \bigtriangleup [\alpha] = [\alpha]$. □

Lemma 24. $(\{G_i\} \wedge [\alpha]) \bigtriangleup 0$ *is winning if and only if for some* G_i *and* $\alpha' < \alpha$, $G_i \wedge [\alpha'']$ *is losing for all* $\alpha > \alpha'' \geq \alpha'$.

Proof. $(\{G_i\} \wedge [\alpha]) \bigtriangleup 0 = \{G_i\} \wedge [\alpha]$. By Lemma 1, this game is winning if and only if some option, $G_i \wedge [\alpha, \alpha']$ is losing. This game has options equal to the union of the options of $G_i \wedge [\alpha'']$ for $\alpha > \alpha'' \geq \alpha'$. Therefore, by Lemma 1, it is losing if and only if $G_i \wedge [\alpha'']$ is losing for all such α''. □

Lemma 25. *If* $\alpha, \beta > 0$, *then* $(\{G_i\} \wedge [\alpha]) \bigtriangleup [\beta]$ *is winning if and only if there exists an index* i *and ordinals* $\alpha' < \alpha$ *and* $\beta' < \beta$ *such that* $(G_i \wedge [\alpha'']) \bigtriangleup [\beta'']$ *is losing for all* $\alpha > \alpha'' \geq \alpha'$ *and* $\beta > \beta'' \geq \beta'$.

Proof. By Lemma 1, $(\{G_i\} \wedge [\alpha]) \bigtriangleup [\beta]$ is winning if and only if some option, $(G_i \wedge [\alpha, \alpha']) \bigtriangleup [\beta, \beta']$ is losing. This option has exactly the same options as the union of the options of $(\{G_i\} \wedge [\alpha'']) \bigtriangleup [\beta'']$ taken over $\alpha > \alpha'' \geq \alpha'$ and $\beta > \beta'' \geq \beta'$. Therefore, by Lemma 1, $(G_i \wedge [\alpha, \alpha']) \bigtriangleup [\beta, \beta']$ is losing if and only if all of $(G_i \wedge [\alpha'']) \bigtriangleup [\beta'']$ are losing. □

4.5. Simplification of $I(G)$

Our original definition of $I(G)$ required that we know who wins $(G \wedge [\alpha]) \bigtriangleup [\beta]$ for all pairs of ordinals α, β. Here we show that we really only need to know the winner for all sufficiently small ordinals.

Lemma 26. *For every game,* G, *there exists an ordinal* γ *such that*

$$(G \wedge [\alpha]) \bigtriangleup [\beta] =_w [\beta]$$

if $\beta > \gamma$, *and*

$$(G \wedge [\alpha]) \bigtriangleup [\beta] =_w G \bigtriangleup [\beta]$$

if $\alpha > \gamma$.

The idea of this proof is that the game G cannot take longer than (the ordinal) γ moves for some γ.

Proof. We prove by induction that there is an ordinal γ so that for any $\delta > \gamma$ if games G and $[\delta]$ are played side by side then either player can guarantee that the copy of $[\delta]$ ends after the copy of G no matter how the copy of G is played, and so that they win the copy of $[\delta]$ if it was theoretically winning for them. We prove this by induction. If for each option G_i of G there is a corresponding γ_i, we pick $\gamma > \sup_i \gamma_i$. Then for $\delta > \gamma$ the first player can pick the option $[\delta, \delta']$ from $[\delta]$ so that:

- he is winning $[\delta, \delta']$ if he was winning $[\delta]$
- $\delta' \geq \sup_i \gamma_i$.

Second is now faced with the games G_i and $[\delta, \delta']$. But his move from $[\delta, \delta']$ must be an option of some game $[\epsilon]$ for $[\epsilon] > \delta' \geq \gamma_i$. Since by induction both First and Second can achieve the appropriate aims for the pair of games $G, [\epsilon]$ for any such ϵ and since some such $[\epsilon]$ will necessarily make the game winning for Second if and only if $[\delta]$ was originally winning for Second, our conditions are met.

This means that if $\beta > \gamma$, the player who is winning $[\beta]$ can guarantee that they win this subgame and that it ends after G does, thus winning the composite game.

If $\alpha > \gamma$, the player who is winning $G\Delta[\beta]$ can both guarantee that they win the latter of G and $[\beta]$ to end and that the $[\alpha]$ component ends after the G component, thus winning the composite game. $\square$

4.6. Some Technical Lemmas

Here we prove a few technical lemmas that will be used in the next two sections.

Lemma 27. *We have* $(G \wedge [\alpha]) \bigtriangleup [\alpha] =_w (G \bigtriangleup [\alpha]) \wedge [\alpha] =_w [\alpha]$.

The idea of this proof is that the subgames $[\alpha]$ should end at the same time therefore at the same time as the entire game.

Proof. Let p be the player who is winning $[\alpha]$. Player p is able to play the $[\alpha]$ subgames of $(G \wedge [\alpha]) \bigtriangleup [\alpha]$ or $(G \bigtriangleup [\alpha]) \wedge [\alpha]$, guaranteeing that:

1. He wins both subgames
2. Both subgames end at the same time.

Using this strategy, p can force a win in either of the other two composite games. $\square$

Corollary 28. *If* $\alpha \geq \beta$, *then*

$$(G \wedge [\beta]) \bigtriangleup [\alpha] =_w [\alpha];$$
$$(G \bigtriangleup [\alpha]) \wedge [\beta] =_w [\beta].$$

Proof. The claim follows immediately from Corollary 19 and Lemmas 14, 15 and 27. □

From the results above we get useful bounds on the sizes of $r_G(\alpha), s_G(\alpha)$.

Lemma 29. *For a game G and an ordinal α,*

$$r_G(\alpha) \geq \alpha;$$
$$s_G(\alpha) \leq \alpha.$$

Proof. By Corollary 28, we have $(G \triangle [\alpha]) \wedge [\beta] =_w [\beta]$ for $\beta \leq \alpha$. Hence by Lemma 9, we have $r_G(\alpha) \geq \alpha$.

Also, by Corollary 28, we have $(G \wedge [\alpha]) \triangle [\beta] =_w [\beta]$ for $\beta \geq \alpha$. If $S(G \wedge [\alpha]) = \gamma$ were greater than α, then we would have $(G \wedge [\alpha]) \triangle [\alpha] =_w (G \wedge [\alpha]) \triangle [\alpha + 1]$ by Lemmas 5 and 6. On the other hand, $[\alpha] \neq_w [\alpha + 1]$, leading to a contradiction. Hence $s_G(\alpha) \leq \alpha$. □

4.7. Computing $I(A \wedge B)$

In this section we show how to compute $I(A \wedge B)$ from $I(A)$ and $I(B)$, proving that I satisfies Condition 3 in Section 3. We will use the most convenient form of $I(G)$ given in Corollary 21.

Lemma 30. *For games A and B and an ordinal λ, we have*

$$r_{A \wedge B}(\lambda) = \min(r_A(\lambda), r_B(\lambda)).$$

The reason to suspect this is that Corollary 19 suggests that $(A \wedge B) \triangle [\lambda]$ and $(A \triangle [\lambda]) \wedge (B \triangle [\lambda])$ are very similar. If they were the same, the above result would follow from Lemma 4.

Proof. For an ordinal number $\kappa > \lambda$, Corollary 19 and Lemmas 14 and 15 imply that

$$\begin{aligned} ((A \wedge B) \triangle [\lambda]) \wedge [\kappa] &=_w (A \wedge B \wedge [\kappa]) \triangle [\lambda] \\ &=_w (A \triangle [\lambda]) \wedge (B \triangle [\lambda]) \wedge [\kappa]. \end{aligned}$$

For $\kappa \leq \lambda$, Lemma 29 implies that $r_{A \wedge B}(\lambda), r_A(\lambda), r_B(\lambda) \geq \kappa$.

Therefore, by considering the remotenesses, we find that

$$((A \wedge B) \triangle [\lambda]) \wedge [\kappa] =_w [\kappa] =_w (A \triangle [\lambda]) \wedge (B \triangle [\lambda]) \wedge [\kappa].$$

Therefore, we have

$$((A \wedge B) \triangle [\lambda]) \wedge [\kappa] =_w (A \triangle [\lambda]) \wedge (B \triangle [\lambda]) \wedge [\kappa]$$

for all κ. Hence by Lemma 9,

$$R((A \wedge B) \triangle [\lambda]) = \min(R(A \triangle [\lambda]), R(B \triangle [\lambda])),$$

or

$$r_{A \wedge B}(\lambda) = \min(r_A(\lambda), r_B(\lambda)).$$ □

4.8. Computing $I(A \triangle B)$

Analogously, we can compute $I(A \triangle B)$ from $I(A)$ and $I(B)$. In particular we have the following:

Lemma 31. *For games A and B and an ordinal λ, we have*

$$s_{A \triangle B}(\lambda) = \max(s_A(\lambda), s_B(\lambda)).$$

4.9. Computing the Winner of G

Here we show how to determine the winner of G from $I(G)$, showing that I satisfies Condition 5 of Section 3.

Lemma 32. *G is winning if and only if $r_G(0)$ is R-odd.*

Proof. By Lemma 3, G is winning if and only if $R(G) = R(G \triangle [0]) = r_G(0)$ is R-odd. □

4.10. Summary

So we have proved the following:

Theorem 33. *$I(G)$ can be computed from a sufficiently large ordinal γ (dependant on G) and the set of pairs of ordinals (α, β) so that $(G \wedge \alpha) \triangle \beta$ is winning (and hence can be stored as a set rather than a proper class). Furthermore, it is possible to compute the following:*

1. *$I(0)$*
2. *$I(\{G_i\})$ from $I(G_i)$*
3. *$I(A \wedge B)$ from $I(A)$ and $I(B)$*
4. *$I(A \triangle B)$ from $I(A)$ and $I(B)$*
5. *who wins G from $I(G)$*

Proof. The theorem follows from Lemmas 21, 22, 25, 26, 30, 31 and 32. □

5. Classification of r_G

In this section we classify the functions from the ordinal numbers to the ordinal numbers that can actually occur as functions r_G for some game G. We do this in three steps. First we state a number of conditions that r_G must satisfy. Next we describe a construction of game that we will use. Finally we use our construction to produce games with a given sequence of r_G that satisfies the conditions from the first step.

In particular we will show that a function r from ordinal numbers to ordinal numbers is r_G for some game G if and only if:

1. $r(\alpha) \geq \alpha$ for all α with equality for sufficiently large α
2. Letting α_+ and α_- be the smallest R-odd and R-even ordinals respectively so that $r(\alpha_\pm) = \alpha_\pm$, then

   For $\beta > \alpha_+$ and β R-odd, $r(\beta) = \beta$

   For $\beta > \alpha_-$ and β R-even, $r(\beta) = \beta$

   If $S([\alpha_+]) \geq_S S([\beta]) \geq_S S([\gamma]) \geq_S S([\alpha_-])$, then $r(\beta) \geq_R r(\gamma)$
3. Either $r(0) = r(1)$ or $r(\alpha) = \alpha$ for all α
4. Either $r(1) = r(2)$ or $r(\alpha) = \alpha$ for all R-odd α

We call these the *remoteness conditions*.

5.1. Conditions

First recall Corollary 29 which states that $r_G(\alpha) \geq \alpha$. By Lemma 26, equality holds for all sufficiently large α. Hence we can define

Definition. For a game G, let G_+ and G_- be the smallest R-odd and R-even ordinals, respectively, such that $r_G(G_+) = G_+$ and $r_G(G_-) = G_-$.

Next we would like to show that $G_\pm$ form a boundary between one type of behavior of r_G and another type.

Lemma 34. *If $\alpha \geq G_+$ and α is R-odd or $\alpha \geq G_-$ and α is R-even, then $r_G(\alpha) = \alpha$.*

Proof. By Corollary 29 and Lemma 9, it is enough to show that

$$(G \triangle [\alpha]) \wedge [\alpha + 1] =_w [\alpha].$$

Without loss of generality α is R-odd. Hence we need to show that $(G \triangle [\alpha]) \wedge [\alpha+1]$ is winning. Now by Corollary 19 and Lemma 14 we have

$$(G \triangle [\alpha]) \wedge [\alpha+1] =_w (G \wedge [\alpha+1]) \triangle [\alpha].$$

Since $S([\alpha]) \geq_S S([G_+])$, considering the suspense tells us that $(G \wedge [\alpha+1]) \triangle [\alpha]$ is winning if $(G \wedge [\alpha+1]) \triangle [G_+]$ is. But we have

$$(G \wedge [\alpha+1]) \triangle [G_+] = (G \triangle [G_+]) \wedge [\alpha+1],$$

which is winning because it has remoteness of G_+. □

For the next step we use the following technical lemma.

Lemma 35. *If $S([\alpha]) \geq_S S([\beta])$, then for $\gamma > \alpha, \beta$, $(G \triangle [\alpha]) \wedge [\gamma]$ is winning if $(G \wedge [\gamma]) \triangle [\beta]$ is winning.*

Proof. By Corollary 19 and Lemma 14, we have

$$(G \triangle [\alpha]) \wedge [\gamma] =_w (G \wedge [\gamma]) \triangle [\alpha];$$
$$(G \triangle [\beta]) \wedge [\gamma] =_w (G \wedge [\gamma]) \triangle [\beta].$$

Since $S([\alpha]) \geq_S S([\beta])$,

$$(G \triangle [\alpha]) \wedge [\gamma] \;\; \geq_S \;\; (G \wedge [\gamma]) \triangle [\beta].$$

Our result now follows from the definition of $\geq_S$ and Lemma 5. □

Lemma 36. *If $S([G_+]) \geq_S S([\alpha]) \geq_S S([\beta]) \geq_S S([G_-])$, then $r_G(\alpha) \geq_R r_G(\beta)$.*

Proof. `Case 1:` $\alpha \neq G_+$, $\beta \neq G_-$.
This implies that $r_G(\alpha) > \alpha$, $r_G(\beta) > \beta$.
`Case 1a:` $r_G(\alpha)$ is R-odd.
If $\beta > r_G(\alpha)$, then $r_G(\beta) > r_G(\alpha)$ and our result holds trivially. Otherwise, it suffices to show that $(G \triangle [\beta]) \wedge [r_G(\alpha) - 1]$ is losing. Using Lemma 35 with $\gamma = r_G(\alpha) - 1$ and the fact that $(G \triangle [\alpha]) \wedge [r_G(\alpha) - 1]$ is losing proves our result.
`Case 1b:` $r_G(\alpha)$ is R-even.
If $\alpha \geq r_G(\beta)$, then our result follows trivially from $r_G(\alpha) \geq r_G(\beta)$. Otherwise, since $G \triangle [\alpha]$ is losing and $S([\alpha]) \geq_S S([\beta])$, $G \triangle [\beta]$ is losing. Hence, it is sufficient to show that for all sufficiently large R-odd γ with $\gamma < r_G(\beta)$ that $(G \triangle [\alpha]) \wedge [\gamma]$ is winning. For "sufficiently large" we can substitute the condition

$\gamma \geq \max(\alpha, \beta)$. In this case our result follows from Lemma 35 applied to such γ and the fact that $(G \triangle [\beta]) \wedge [\gamma]$ is winning.

Case 2: $\alpha = G_+$.
Suppose for sake of contradiction that the lemma does not hold. Then by Case 1 of the lemma, we can find an R-odd β with $\beta < \alpha$ and $r_G(\beta) >_R \alpha$. The $r_G(\beta)$ is R-odd and less than α. Consider $r_G(r_G(\beta))$. On the one hand, since $r_G(\beta) > \beta$, Case 1 of the lemma implies that $r_G(r_G(\beta)) \geq_R r_G(\beta)$. On the other hand, since $r_G(\beta) < \alpha$, we have that $r_G(r_G(\beta)) > r_G(\beta)$ and thus $r_G(r_G(\beta)) <_R r_G(\beta)$, thus reaching a contradiction.

Case 3: $\beta = G_-$.
This is analogous to Case 2. □

We also have a few boundary cases to deal with.

Lemma 37. *$r_G(0) = r_G(1)$ unless $r_G(\alpha) = \alpha$ for all α.*

Proof. If $r_G(0) \neq r_G(1)$, $G \neq G \triangle [1]$, so it must be possible for G to end in fewer than one move. Hence $G = 0$. Hence $r_G(\alpha) = \alpha$. □

Lemma 38. *$r_G(1) = r_G(2)$ unless $G_+ = 1$.*

Proof. If $r_G(1) \neq r_G(2)$, then $G \triangle [1] \neq G \triangle [2]$, so either G is 0 or it is possible for G to end in one move. If G can end in one move, 0 must be an option of the initial position. Therefore, $(G \triangle [1]) \wedge [2]$ is winning because First can move to the game $(0 \triangle 0) \wedge [1] = 0$. Hence $r_G(1) = 1$, so $G_+ = 1$. □

Proposition 39. *For any game G, the function r_G satisfies the remoteness conditions.*

Proof. This follows from Corollary 29 and Lemmas 26, 34, 36, 37 and 38. □

5.2. A Class of Games

Here we define a class of games that we will use for our constructions in the next section. It will turn out that games in this class achieve almost all possible values of $I(G)$.

Definition. Let S be a set of closed sets of ordinal numbers greater than or equal to 2 (closed under the order topology of the ordinals) . Define the *Two-Choice-Game*, $G(S)$ as follows: For each α in an element of an element of S, construct the *Potential Game*, $[\alpha]$. On First's first move, he picks an element of S and makes a move in each of the potential games associated with elements of it. On Second's first move, he picks a potential game in the set that First picks and makes another move in it. They then play the game defined by the current position of the potential game chosen by Second.

Example. Suppose $S = \{\{3, 5\}, \{10\}\}$. First could pick $\{3, 5\}$ and make the moves [2] and [4] in the games [3] and [5] respectively. Then Second could pick the potential game corresponding to 3 and make the move [1]. First is now left with the game [1] and has to make the move to [0], at which point the game ends.

Two-Choice-Games are nice because it is easy to compute $I(G)$ for them. In particular, the meta-game of choosing which potential game is actually used, can be thought of as taking place before the rest of the game. We prove that:

Lemma 40. *For a Two-Choice-Game, $G(S)$, the game $(G(S) \triangle [\alpha]) \wedge [\beta]$ is winning for the player that can force the potential game that is played to correspond to an ordinal γ so that $([\gamma] \triangle [\alpha]) \wedge [\beta]$ is winning for that player.*

Proof. First, we may assume that $\beta \geq \alpha$, because otherwise

$$(G(S) \triangle [\alpha]) \wedge [\beta] =_w [\beta] =_w ([\gamma] \triangle [\alpha]) \wedge [\beta].$$

We prove this lemma by constructing a winning strategy to $(G(S) \triangle [\alpha]) \wedge [\beta]$ for the appropriate player. If this is Second, there is only one potential game in which he moves. Therefore, he can play to guarantee that for the γ chosen that $([\gamma] \triangle [\alpha]) \wedge [\beta]$ is winning for him, but can also play all potential games and the games $[\alpha]$ and $[\beta]$ in the way described in Lemma 16, thus guaranteeing that they win $(G(S) \triangle [\alpha]) \wedge [\beta]$. If First is winning he can employ a similar strategy. If neither α nor β are limit ordinals of the elements of the chosen elements of S, then First can still maintain the orders of each of the potential games with $[\alpha]$ and $[\beta]$ by sending then to $[\alpha, \alpha']$ and $[\beta, \beta']$ with no potential between α and α' or β and β'. On the other hand if either α or β are limits of such γ, then since this set is closed, either α or β is a limit ordinal and an element of the chosen element of S. In this case, First is not winning $([\gamma] \triangle [\alpha]) \wedge [\beta]$ since $\gamma = \alpha$ or $\gamma = \beta$ and is a limit ordinal (and hence losing). ☐

5.3. The Construction

Here we produce a construction showing that any sequence r satisfying the remoteness conditions can be realized as $r = r_G$ for some game G.

Theorem 41. *If r is a function from the ordinal numbers to the ordinal numbers, then there exists a game G with $r(\alpha) = r_G(\alpha)$ for all α if and only if r satisfies the remoteness conditions.*

Proof. We already have the only if part. Now we need to construct G.

`Case 1:` $r(\alpha) = \alpha$ for all α.

We can let $G = 0$.

`Case 2:` $r(0) = r(1) = r(2)$.

For ordinal numbers α with $\alpha \geq 2$ and $S([\alpha_+]) \geq_S S([\alpha]) \geq_S S([\alpha_-])$, we define the set E_α as follows:

- E_α contains $r(\alpha)$
- If α is R-odd, E_α contains all ordinals less than α that are at least 2
- If α is R-even, E_α contains $\alpha + 1$
- If $r(\alpha)$ is R-even, then E_α contains some ζ where ζ is larger than any $r(\beta)$ with $S([\alpha_+]) \geq_S S([\beta]) \geq_S S([\alpha_-])$
- E_α contains no other elements.

We let G be the Two-Choice-Game $G = G(\{E_\alpha\})$. We now show that for this G, $r(\alpha) = r_G(\alpha)$. We do this by using Lemma 40.

`Case 2a:` α is R-odd and $\alpha \geq \alpha_+$.

If First picks E_{α_+}, then he guarantees that the potential game γ picked has $\gamma \leq \alpha$. Therefore, $([\gamma] \triangle [\alpha]) \wedge [\alpha + 1]$ is winning. Therefore, $r_G(\alpha) \leq \alpha$, so by Corollary 29, $r_G(\alpha) = \alpha = r(\alpha)$.

`Case 2b:` α is R-even and $\alpha \geq \alpha_-$.

No matter which E_β First picks, Second can always pick a potential game corresponding to a γ with $\gamma < \alpha$. This is because he can make $\gamma = 2$ if β is R-odd, $\gamma = \beta + 1$ if $\beta < \alpha_-$ and $\gamma = r(\beta)$ if $\beta = \alpha_-$. Therefore by Lemma 40, Second can win $(G \triangle [\alpha]) \wedge [\alpha + 1]$, so $r_G(\alpha) \leq \alpha$. Hence by Corollary 29, $r_G(\alpha) = \alpha = r(\alpha)$.

`Case 2c:` α is R-odd and $1 < \alpha < \alpha_+$.

First can pick E_α on his first move. Then the possible potential games played correspond to the ordinals less than α, $r(\alpha)$, $\alpha - 1$ and possibly ζ. When any of these are taken in a long conjunctive sum with $[\alpha]$, the result is at least as R-large as $r(\alpha)$. This is because the possibilities for the max of α and γ are: $\alpha, \zeta, r(\alpha)$. Since ζ is only an option if $r_G(\alpha)$ is R-even, $r_G(\alpha)$ is the R-smallest. Hence we have by Lemma 40 that $r_G(\alpha) \geq_R r(\alpha)$.

On the other hand, Second can guarantee that when the potential game chosen is combined in a long conjunctive sum with $[\alpha]$, the result is at least as R-small as $r(\alpha)$. If First picks E_β with $S([\beta]) \leq_S S([\alpha])$, then Second may pick either $r(\beta)$ or ζ. $r(\alpha) \geq_R R([\max(\alpha, r(\beta))])$ unless $\alpha > r(\beta)$ and $r_G(\alpha)$ is R-odd, in which case $r(\alpha) \geq_R \zeta$. If, on the other hand, β is R-odd and $\beta > \alpha$, Second can pick the potential game $\alpha + 1$ which yields $\alpha + 1$ which is R-even and hence $\alpha + 1 \leq_R r(\alpha)$. Hence we have by Lemma 40 that $r_G(\alpha) \leq_R r(\alpha)$.

`Case 2d:` α is R-even and $0 < \alpha < G_-$.

First can pick E_α on his first move. Then the possible potential games played correspond to ordinals $\alpha + 1$, $r(\alpha)$ and ζ (the last only if $r(\alpha)$ is R-even). The maximums of these with α are $\alpha + 1, r(\alpha)$, and perhaps, ζ. All of these are at least as R-large as $r(\alpha)$, since $\alpha + 1$ is R-odd and at most $r(\alpha)$, and since $\zeta > r(\alpha)$ and only available if $r(\alpha)$ is R-even. Hence we have by Lemma 40 that $r_G(\alpha) \geq_R r(\alpha)$.

On the other hand, Second can guarantee that when the potential game chosen is taken in a long conjunctive sum with $[\alpha]$, the result is at least as R-small as $r(\alpha)$. Suppose First picks E_β on his first turn. If β is R-even and $\beta < \alpha$, Second may pick $\beta + 1$ whose max with α is α, which is R-smaller than $r(\alpha)$ because $\alpha \leq r(\alpha)$ and α is R-even. If β is R-odd, Second may pick 2 as the potential game, thus making this max α again. If β is R-even and $\beta \geq \alpha$, then Second may pick $r(\beta)$. Now, $\max(\alpha, r(\beta)) \leq_R r(\alpha)$ since $\alpha \leq_R r(\alpha)$ and $r(\beta) \leq_R r(\alpha)$ by the remoteness conditions. Hence we have by Lemma 40 that $r_G(\alpha) \leq_R r(\alpha)$.

`Case 2e:` α equals 0 or 1.

Since G cannot possibly end in fewer than two turns, $G \triangle [0] = G \triangle [1] = G \triangle [2]$. Hence $r_G(\alpha) = r_G(2) = r_G(2) = r_G(\alpha)$.

`Case 3:` $r(1) = r(0) = 1$.

Notice that if we replace $r(1)$, $r(0)$ by $r(2)$ and α_+ by 3, we have an r satisfying the remoteness conditions. Therefore, by Case 2, there exists a game G' with $r_{G'}$ equal to this new function. Define $G = G' \cup \{\emptyset\}$ (i.e. G' with the extra option of ending the game after one turn). Clearly, First can win $(G' \triangle [\alpha]) \wedge [\alpha + 1]$ for α R-odd by picking $\emptyset$ on his first move. Therefore, $r_G(1) = 1$. Also since $G \neq 0$, $r_G(0) = r_{G'}(1) = 1$. On the other hand, when playing $(G \triangle [\alpha]) \wedge [\beta]$ with α R-even and $\alpha \leq \beta$, it is never advantageous to move to $\emptyset$ on the first move. Hence for α R-even, $r_G(\alpha) = r_{G'}(\alpha) = r(\alpha)$. This completes the proof of the theorem. □

6. Misère Play

In this section, we discuss the strategy behind the misère play of games under conjunctive sum. The idea of misère play is that the game is played in the same way, only the objective is reversed (you are now trying to lose the game). In other words, the objective now is to be the first player unable to move. In this section we discuss the analysis of misère games under short and long conjunctive sums. We first introduce a new operation, and then adapt $I(G)$ to handle misère games as well as normal ones.

6.1. The Sequential Compound

Here we define the operation of sequential compound as defined in [4] and derive some of its basic properties.

Definition. If $A = \{A_i\}$ and B are games, let their *Sequential Compound* $A \to B$ be the game defined by $0 \to B = B$ and $A \to B = \{A_i \to B\}$ if $A \neq 0$.

In other words in the sequential compound, we have copies of each of the games and on a turn either make a move in A if it is not over, and otherwise makes move in B. Equivalently, you play A and when done, play B. In the following lemma we show the importance of concatenation to analysis of misère play.

Lemma 42. *The misère version of G is winning if and only if $G \to [1]$ is winning.*

Proof. When playing $G \to [1]$, after the subgame G ends, there is one more move made. Hence the winner of the subgame G is the loser of $G \to [1]$. Hence one can force a win in $G \to [1]$ only if one can force a loss in G, which is equivalent to forcing a win in the misère version of G. □

6.2. Analysis of Misère Play

Next, we follow some steps to show that all of the operations in the conditions in Section 3 can be done for misère play using $I(G \to [1])$.

Lemma 43. *$I(0 \to [1])$ can be computed.*

Proof. $I(0 \to [1]) = I(1)$. $([1] \triangle [\alpha]) \wedge [\beta] = [\min(\max(\alpha, 1), \beta)]$, hence it is winning if and only if $\min(\max(\alpha, 1), \beta)$ is R-odd. □

Lemma 44. *$I(\{G_i\} \to [1])$ can be computed from $I(G_i \to [1])$.*

Proof. By definition, $I(\{G_i\} \to [1]) = I(\{G_i \to [1]\})$. Hence by Lemma 25, it can be computed from $I(G_i \to [1])$. □

Lemma 45. *$I((A \wedge B) \to [1])$ can be computed from $I(A \to [1])$ and $I(B \to [1])$.*

Proof. Note that $(A \wedge B) \to [1] = (A \to [1]) \wedge (B \to [1])$. This is because in either case, the game ends one turn after the first of the subgames A and B end. Hence by Lemma 30 we can compute $I((A \wedge B) \to [1]) = I((A \to [1]) \wedge (B \to [1]))$ from $I(A \to [1])$ and $I(B \to [1])$. □

Lemma 46. *$I((A \triangle B) \to [1])$ can be computed from $I(A \to [1])$ and $I(B \to [1])$.*

Proof. Note that $(A \triangle B) \to [1] = (A \to [1]) \triangle (B \to [1])$. This is because in either case, the game ends one turn after the last of the subgames A and B end. Hence by Lemma 31 we can compute $I((A \triangle B) \to [1]) = I((A \to [1]) \triangle (B \to [1]))$ from $I(A \to [1])$ and $I(B \to [1])$. □

Combining the above we have

Theorem 47. *The information $I(G \to [1])$ associated with a game G satisfies the conditions in Section 3 for misère play.*

7. Short Games with Sequential Compound

It would be nice to have some information similar to $I(G)$ that allows us to solve games constructed using either type of conjunctive sum as well as sequential compound. Unfortunately, we only know of such a theory for a restricted class of games known as short games.

Definition. Call a game G *short* if there exists an integer N so that G will always end after at most N moves. We let $N(G)$ stand for the smallest such integer.

Shortness of a game is essentially a finiteness condition that excludes, for example, infinite ordinal games.

In this section, we use script Roman letters to represent sets and use Greek letters for finite ordinals or equivalently for non-negative integers.

Definition. For G a short game, let $J(G)$ be the set of all subsets $\mathcal{S} \subset \{0, 1, 2, \ldots\}$ so that First may force G to end in exactly a number of moves in $\mathcal{S}$.

Note that $J(G)$ is easy to compute for two-choice games. For example, we have that $J(\{\{[1]\}, \{[0], [2]\}\})$ contains the sets $\{3\}, \{2, 4\}$ and any supersets of these. We show that $J(G)$ contains all information necessary to determine the winner of short games constructed using concatenation and conjunctive sums. First we note that as a base case we know $J(0)$. Namely:

Lemma 48. $J(0) = \{\mathcal{S} : 0 \in \mathcal{S}\}$.

Proof. The game ends in 0 moves regardless of play. Hence First can force the game to end in some number of moves in $\mathcal{S}$ if and only if $0 \in \mathcal{S}$. $\square$

Additionally, we can compute J of a game that was constructed out of short games in any of these ways. In particular we prove that:

Theorem 49. *For A, B, G, G_i short games, the following hold:*

- *If $\{G_i\}$ is short, then $J(\{G_i\})$ can be determined from $J(G_i)$.*
- *$A \wedge B$ is short and $J(A \wedge B)$ can be determined from $J(A)$ and $J(B)$.*
- *$A \triangle B$ is short and $J(A \triangle B)$ can be determined from $J(A)$ and $J(B)$.*
- *$A \to B$ is short and $J(A \to B)$ can be determined from $J(A)$ and $J(B)$.*

This allows us to recursively compute $J(G)$. We note that this is enough to determine the winner of G. In particular:

Lemma 50. *If G is a short game, G is winning if and only if the set of odd numbers is in $J(G)$.*

Proof. G is winning if and only if First can cause it to end in an odd number of moves. This is the case if and only if the set of odd numbers is in $J(G)$. $\square$

We now proceed with the proofs of the statements in Theorem 49.

Definition. If $\mathcal{S}$ is a set of non-negative integers and α is an integer, then we let $\mathcal{S} - \alpha = \{\beta : \beta + \alpha \in S\}$.

Lemma 51. *If $G = \{G_i\}$ is short, then $\mathcal{S} \in J(G)$ if and only if for some G_i, no element of $J(G_i)$ is disjoint from $\mathcal{S} - 1$.*

Proof. By definition $\mathcal{S} \in J(G)$ if and only if First can force G to end in some number of moves in $\mathcal{S}$. This is equivalent to saying that First can pick some option G_i and then force the game to end in some number of extra moves in $\mathcal{S} - 1$. This is equivalent to saying that Second can force G_i to end in such a number of moves. Equivalently this says that First cannot force G_i to end in some number of moves in the compliment of $\mathcal{S} - 1$. Hence this is equivalent to all elements of $J(G_i)$ having nontrivial intersection with $\mathcal{S} - 1$. $\square$

Lemma 52. *If A and B are short games, then $G = A \wedge B$ is short, and $J(G)$ can be determined from $J(A)$ and $J(B)$.*

Proof. For the first claim, note that G must end in at most $\min(N(A), N(B))$ moves. For a given set $\mathcal{S}$, we need a way to determine from $J(A)$ and $J(B)$ whether $\mathcal{S} \in J(G)$. Given $\mathcal{S}$ we define an ordering $\geq_{R_\mathcal{S}}$ on nonnegative integers so that $\alpha \geq_{R_\mathcal{S}} \beta$ if one of the following holds:

- $\alpha \in \mathcal{S}$ and $\beta \notin \mathcal{S}$
- $\alpha \leq \beta$ and $\alpha, \beta \in \mathcal{S}$
- $\alpha \geq \beta$ and $\alpha, \beta \notin \mathcal{S}$

For any short game H, we consider First and Second playing the game H so that First is trying to maximize the length of the game under the $\geq_{R_\mathcal{S}}$ ordering and Second is trying to minimize it. We define $r_\mathcal{S}(H)$ to be the length of the game when First and Second play with these strategies. Note that because H is short, this value is well defined. We claim that $r_\mathcal{S}(G) = \min(r_\mathcal{S}(A), r_\mathcal{S}(B))$. This is because the length of G is maximized under the $R_\mathcal{S}$ ordering when the lengths of each of the component games are individually maximized. This in turn follows from noting that if $\alpha \geq_{R_\mathcal{S}} \alpha'$, then $\min(\alpha, \beta) \geq_{R_\mathcal{S}} \min(\alpha', \beta)$.

We note that $r_\mathcal{S}(A)$ is easily determined from $J(A)$ by noting that $r_\mathcal{S}(A)$ is the largest α under the $\geq_{R_\mathcal{S}}$ ordering, so that First can force the length of A to be at least as large as α. Hence from $J(A)$ and $J(B)$ one can determine $r_\mathcal{S}(A)$ and $r_\mathcal{S}(B)$ and, from those, $r_\mathcal{S}(G)$. Last we note that $\mathcal{S} \in J(G)$ if and only if First can force the length of G to be in $\mathcal{S}$. This is the case if and only if $r_\mathcal{S}(G) \in S$. Hence we can determine if $\mathcal{S}$ is in $J(G)$. □

Lemma 53. *If A and B are short games, then $G = A \triangle B$ is short, and $J(G)$ can be determined from $J(A)$ and $J(B)$.*

Proof. The proof is analogous to that of Lemma 52. □

Lemma 54. *If A and B are short, then so is $G = A \to B$. Furthermore $J(G)$ can be determined from $J(A)$ and $J(B)$.*

Proof. For the first claim, note that the number of turns it takes to play G is at most $N(A)+N(B)$. Suppose that while playing G it takes α turns to complete the play of A. First is still able to force the length of the full game G to lie in a set $\mathcal{S}$ if and only if he can force the remaining game B to take some number of turns lying in $\mathcal{S}-\alpha$. If α is even, this is possible if and only if $(\mathcal{S}-\alpha) \in J(B)$. Otherwise, since it is Second's turn when the game B starts, it is possible if and only if all elements of $J(B)$ have non-trivial intersection with $\mathcal{S} - \alpha$. Hence if we let $\mathcal{S}'$ be the set of all α so that either

- α is even and $(\mathcal{S} - \alpha) \in J(B)$, or
- α is odd and all elements of $J(B)$ have non-trivial intersection with $\mathcal{S} - \alpha$,

then $\mathcal{S} \in J(G)$ if and only if $\mathcal{S}' \in J(A)$. $\square$

Theorem 49 now follows immediately from Lemmas 48, 51, 52, 53, 54, and 50.

We note that $J(G)$ contains a lot more information than $I(G)$ does. Unfortunately this extra information is necessary in order to do the required computations. In particular we show that:

Theorem 55. *If G is a short game, $J(G)$ can be determined from knowing the winners of all games constructed by taking G and combining it with other short games using the operations $\wedge, \triangle$, and $\to$.*

Proof. For α a non-negative integer, we let $[[\alpha]]$ denote the game that takes exactly α moves to play.

We note that, since G cannot take more than $N(G)$ moves, $\mathcal{S} \in J(G)$ if and only if $(\mathcal{S} \setminus \{0, 1, \ldots, N(G)\}) \in J(G)$. Hence $\mathcal{S} \in J(G)$ if and only if $(\mathcal{S} \setminus \{0, 1, \ldots, \alpha\}) \in J(G)$ for all sufficiently large α. Therefore to determine $J(G)$ it suffices to determine which finite sets $\mathcal{S}$ are in $J(G)$.

Let $\mathcal{S}$ be a finite set of non-negative integers. Let α be an upper bound for $\mathcal{S} \cup \{N(G)\}$. Let $\mathcal{S}_e$ and $\mathcal{S}_o$ be the sets of even and odd integers in $\mathcal{S}$, respectively. Let $G_{\mathcal{S},\alpha}$ be the two-choice game defined by the following sets of integers:

- $\{\beta : 2\alpha - \beta \in \mathcal{S}_o\} \cup \{3\alpha + 2\}$, and
- $\{2, 2\alpha - \beta\}$ for some $\beta \in \mathcal{S}_e$.

Consider the game $H_{G,\mathcal{S},\alpha} = ((G \to G_{\mathcal{S},\alpha}) \wedge [[3\alpha + 2]]) \triangle [[\alpha + 2]]$. We ask the question of when First is able to force this game to end in either $\alpha + 2$ turns or in 2α turns.

Assuming that G takes an even number γ of turns to play, First starts off playing $G_{\mathcal{S},\alpha}$. If he picks the first set, Second can guarantee that the game ends in $3\alpha + 2$ turns. If First picks another set corresponding to a $\beta \in \mathcal{S}_e$, the composite game will end in either $\alpha + 2$ turns or $2\alpha - \beta + \gamma$ turns. Hence First can force H to end in the appropriate number of turns exactly when $\gamma \in \mathcal{S}_e$.

Suppose on the other hand that G ended in an odd number γ of moves. Second makes the first move in $G_{\mathcal{S},\alpha}$. If Second picks the first set, First will cause the game H to end in either $3\alpha + 2$ moves or in $2\alpha - \beta + \gamma$ moves for some $\beta \in \mathcal{S}_o$. If Second made any other move, First could cause H to end in $\alpha + 2$ moves. Hence, First can end H in an appropriate number of moves if and only if $\gamma \in \mathcal{S}_o$.

Putting the above together, if $\alpha > N(G)$, then $\{\alpha+2, 2\alpha\} \in J(H_{G,\mathcal{S},\alpha})$ if and only if $\mathcal{S} \in J(G)$.

Lastly we need a way to detect whether $\{\alpha+2, 2\alpha\} \in J(H_{G,\mathcal{S},\alpha})$. Consider the case where α is even. We claim that the game

$$((H_{G,\mathcal{S},\alpha} \to \{[[0]], [[\alpha-2]]\}) \wedge [[2\alpha+2]]) \triangle [[2\alpha]]$$

is winning exactly when this is the case. Note that First wins the above composite if and only if $H_{G,\mathcal{S},\alpha} \to \{[[0]], [[\alpha-2]]\}$ ends in exactly $2\alpha+1$ turns. Clearly this is possible to achieve exactly when $H_{G,\mathcal{S},\alpha}$ ended in either 2α moves or $\alpha+2$ moves.

Therefore, $\mathcal{S} \in J(G)$ if and only if $((H_{G,\mathcal{S},\alpha} \to \{[[0]], [[\alpha-1]]\}) \wedge [[2\alpha+2]]) \triangle [[2\alpha]]$ is winning for all sufficiently large even α. $\square$

We conclude our discussion of sequential compound by discussing exactly which sets are possible as values of $J(G)$. It is obvious that if $\mathcal{S} \in J(G)$ that any superset of $\mathcal{S}$ is in $J(G)$. We have also noted above that for sufficiently large α, at least $N(G)$, $\mathcal{S} \in J(G)$ if and only if $(\mathcal{S} \setminus \{0, 1, 2, \ldots, \alpha\}) \in J(G)$. There are two final restrictions coming from games that can end in a small number of moves. In particular, if G can end in 0 moves, $G = 0$. Therefore, if for some $\mathcal{S}$, $(\mathcal{S} \cup \{0\}) \in J(G)$ and $\mathcal{S} \notin J(G)$, then $J(G) = J(0)$. Additionally, if G can end in one turn, First can force it to. Hence if for some $\mathcal{S}$, $(\mathcal{S} \cup \{1\}) \in J(G)$, and $\mathcal{S} \notin J(G)$, then $\{1\} \in J(G)$.

It is not hard to show that any set of sets of natural numbers satisfying the above properties can be given as $J(G)$ for some short game G. In fact we can require that G be either a two-choice game, a two-choice game with the extra starting option 0, or 0.

8. Further Work

There are still several ways in which the theory of conjunctive sums of games could be extended. Probably, the most important of these is finding a formal duality between short and long conjunctive sums. Almost all of the results in this paper are true under a duality that sends short sum, remoteness, min, etc. to long sum, suspense, max, etc. In fact most of these dual results are also in the paper. It would be nice to see a way to formalize this duality.

Question 1. Is there a formal duality between short and long conjunctive sums, and if so, what is it?

The main problem that we see with developing a formal duality is that it would need to interchange the operations of min and max for ordinal numbers. The duality would need to extend the space of games somewhat since without doing so formal duality fails. For example, there exists a game, 0, so that $G \bigtriangleup 0 = G$ for all G, but no game X so that $G \wedge X = G$ for all G.

Next there is the problem of classifying the algebraic structure of short and long conjunctive sums given that we now know the appropriate equivalence.

Question 2. Is there a better way to think about the algebraic structure of games under $\bigtriangleup$ and $\wedge$ under the equivalence $A \sim B$ if $I(A) = I(B)$?

It is also unknown if there is a closed form solution to the example game we gave.

Question 3. For which r and c is $G(r, c)$ winning, where G is the game in Section 3?

$J(G)$ works only for short games. It would be nice to extend the theory to arbitrary games.

Question 4. Is there a new function, $J'(G)$, that satisfies the properties listed in Theorem 49, as well as being enough to determine the winner of G, but for all games rather than just short ones?

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