



NEW SEQUENCES THAT CONVERGE TO A GENERALIZATION OF EULER'S CONSTANT

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Abstract

The purpose of the paper is to give some sequences that converge quickly to a generalization of Euler's constant, i.e., the limit of the sequence

$$\left(\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n-1}{a} \right)_{n \in \mathbb{N}},$$

where $a \in (0, +\infty)$.

1. Introduction

Euler's constant, being one of the most important constants in mathematics, was investigated by many mathematicians. Usually denoted by γ , this constant is the limit of the sequence $(D_n)_{n \in \mathbb{N}}$ defined by $D_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$, for each $n \in \mathbb{N}$. It is well-known that $\lim_{n \rightarrow \infty} n(D_n - \gamma) = \frac{1}{2}$ (see [1], [2], [3], [5, pp. 73–75], [7], [13, Problem 18, pp. 38, 197], [14], [21], [23], [24], [25], [26]).

In order to increase the slow rate of convergence of the sequence $(D_n)_{n \in \mathbb{N}}$ to γ , D. W. DeTemple considered in [4] the sequence $(R_n)_{n \in \mathbb{N}}$ defined by $R_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n + \frac{1}{2})$, for each $n \in \mathbb{N}$, and he proved that $\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}$, for each $n \in \mathbb{N}$.

L. Tóth used in [22] the sequence $(T_n)_{n \in \mathbb{N}}$ defined by $T_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n + \frac{1}{2} + \frac{1}{24n})$, for each $n \in \mathbb{N}$, and T. Negoi proved in [12] that $\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}$, for each $n \in \mathbb{N}$.

Let $a \in (0, +\infty)$. We consider the sequence $(y_n(a))_{n \in \mathbb{N}}$ defined by

$$y_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln \frac{a+n-1}{a},$$

for each $n \in \mathbb{N}$. The sequence $(y_n(a))_{n \in \mathbb{N}}$ is convergent (see, for example, [6, p. 453]; see also [15], [16], [17], [18], [19], [20] and some of the references therein) and

its limit, denoted by $\gamma(a)$, is a generalization of Euler’s constant. Clearly, $\gamma(1) = \gamma$. Numerous results regarding the generalization of Euler’s constant $\gamma(a)$ we have obtained in [15], [16], [17], [18], [19] and [20].

We mention the following representation of $\gamma(a)$ ([19, Theorem 2.2.4, p. 78]):

$$\gamma(a) = y_n - \frac{1}{2(a+n-1)} + \sum_{k=1}^m \frac{B_{2k}}{2k(a+n-1)^{2k}} - (2m+1)! \int_n^\infty \frac{P_{2m+1}(x)}{(a+x-1)^{2m+2}} dx,$$

for each $n \in \mathbb{N}$, any $m \in \mathbb{N}$, where B_{2k} is the Bernoulli number of index $2k$ and $P_{2m+1}(x) = (-1)^{m-1} \sum_{k=1}^\infty \frac{2 \sin(2k\pi x)}{(2k\pi)^{2m+1}}$, obtained by applying the Euler-Maclaurin summation formula ([6, p. 524], [5, p. 86]). If we take $a = 1$ in the above-mentioned representation, then we obtain a result presented, for example, in [6, pp. 527, 528], [5, pp. 88, 89].

Recent results regarding Euler’s constant have been obtained by C. Mortici in [9], [10], [11].

Also, we remind the following lemma (C. Mortici [8, Lemma]), which is a consequence of the the Stolz-Cesaro Theorem, the case $\frac{0}{0}$.

Lemma 1. *Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence of real numbers and $x^* = \lim_{n \rightarrow \infty} x_n$. We suppose that there exists $\alpha \in \mathbb{R}$, $\alpha > 1$, such that*

$$\lim_{n \rightarrow \infty} n^\alpha (x_n - x_{n+1}) = l \in \overline{\mathbb{R}}.$$

Then there exists the limit

$$\lim_{n \rightarrow \infty} n^{\alpha-1} (x_n - x^*) = \frac{l}{\alpha - 1}.$$

In Section 2 we present classes of sequences with the argument of the logarithmic term modified and that converge quickly to $\gamma(a)$.

2. Sequences That Converge to $\gamma(a)$

Theorem 2. *Let $a \in (0, +\infty)$. We specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n \in \mathbb{N}}$ from Introduction.*

(i) *We consider the sequence $(\alpha_{n,2}(a))_{n \in \mathbb{N}}$ defined by*

$$\begin{aligned} \alpha_{n,2}(a) = & \frac{1}{a} + \frac{1}{a+1} + \dots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} + \frac{1}{12(a+n-1)^2} \\ & - \ln \left(\frac{a+n-1}{a} + \frac{1}{120a(a+n-1)^3} \right), \end{aligned}$$

for each $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} n^6(\gamma(a) - \alpha_{n,2}(a)) = \frac{1}{252}.$$

(ii) We consider the sequence $(\beta_{n,2}(a))_{n \in \mathbb{N}}$ defined by

$$\beta_{n,2}(a) = \alpha_{n,2}(a) + \frac{1}{252(a+n-1)^6},$$

for each $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} n^8(\beta_{n,2}(a) - \gamma(a)) = \frac{121}{28800}.$$

Proof. (i) We have

$$\begin{aligned} & \alpha_{n+1,2}(a) - \alpha_{n,2}(a) \\ &= \frac{1}{2(a+n)} + \frac{1}{2(a+n-1)} + \frac{1}{12(a+n)^2} - \frac{1}{12(a+n-1)^2} \\ & \quad - \ln\left(a+n + \frac{1}{120(a+n)^3}\right) + \ln\left(a+n-1 + \frac{1}{120(a+n-1)^3}\right) \\ &= \frac{1}{2(a+n)} + \frac{1}{2(a+n)\left(1 - \frac{1}{a+n}\right)} + \frac{1}{12(a+n)^2} - \frac{1}{12(a+n)^2\left(1 - \frac{1}{a+n}\right)^2} \\ & \quad - \ln\left(1 + \frac{1}{120(a+n)^4}\right) + \ln\left(1 - \frac{1}{a+n} + \frac{1}{120(a+n)^4\left(1 - \frac{1}{a+n}\right)^3}\right), \end{aligned}$$

for each $n \in \mathbb{N}$. Set $\varepsilon_n := \frac{1}{a+n}$, for each $n \in \mathbb{N}$. Since $\varepsilon_n \in (-1, 1)$, $\frac{1}{120} \varepsilon_n^4 \in (-1, 1]$ and $-\varepsilon_n + \frac{1}{120} \cdot \frac{\varepsilon_n^4}{(1-\varepsilon_n)^3} \in (-1, 1]$, for each $n \in \mathbb{N} \setminus \{1\}$, using the series expansion ([6, pp. 171–179, p. 209]) we obtain

$$\begin{aligned} & \alpha_{n+1,2}(a) - \alpha_{n,2}(a) \\ &= \frac{1}{2} \varepsilon_n + \frac{1}{2} \cdot \frac{\varepsilon_n}{1-\varepsilon_n} + \frac{1}{12} \varepsilon_n^2 - \frac{1}{12} \cdot \frac{\varepsilon_n^2}{(1-\varepsilon_n)^2} \\ & \quad - \ln\left(1 + \frac{1}{120} \varepsilon_n^4\right) + \ln\left(1 - \varepsilon_n + \frac{1}{120} \cdot \frac{\varepsilon_n^4}{(1-\varepsilon_n)^3}\right) \\ &= \frac{1}{42} \varepsilon_n^7 + \frac{1}{12} \varepsilon_n^8 + \frac{679}{3600} \varepsilon_n^9 + \frac{279}{800} \varepsilon_n^{10} + O(\varepsilon_n^{11}), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n \rightarrow \infty} n^7(\alpha_{n+1,2}(a) - \alpha_{n,2}(a)) = \frac{1}{42}.$$

Now, according to Lemma 1, we get

$$\lim_{n \rightarrow \infty} n^6(\gamma(a) - \alpha_{n,2}(a)) = \frac{1}{252}.$$

(ii) We are able to write that

$$\begin{aligned} & \beta_{n,2}(a) - \beta_{n+1,2}(a) \\ &= \alpha_{n,2}(a) - \alpha_{n+1,2}(a) + \frac{1}{252(a+n-1)^6} - \frac{1}{252(a+n)^6} \\ &= \alpha_{n,2}(a) - \alpha_{n+1,2}(a) + \frac{1}{252(a+n)^6 \left(1 - \frac{1}{a+n}\right)^6} - \frac{1}{252(a+n)^6} \\ &= \alpha_{n,2}(a) - \alpha_{n+1,2}(a) + \frac{1}{252} \cdot \frac{\varepsilon_n^6}{(1 - \varepsilon_n)^6} - \frac{1}{252} \varepsilon_n^6 \\ &= \frac{121}{3600} \varepsilon_n^9 + \frac{121}{800} \varepsilon_n^{10} + O(\varepsilon_n^{11}), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n \rightarrow \infty} n^9(\beta_{n,2}(a) - \beta_{n+1,2}(a)) = \frac{121}{3600}.$$

Now, according to Lemma 1, we get

$$\lim_{n \rightarrow \infty} n^8(\beta_{n,2}(a) - \gamma(a)) = \frac{121}{28800}.$$

□

In the same manner as in the proof of Theorem 2, considering the sequence in each of the following parts, we get the indicated limit:

$$\begin{aligned} \delta_{n,2}(a) &= \beta_{n,2}(a) - \frac{121}{28800(a+n-1)^8}, \text{ for each } n \in \mathbb{N}, \\ & \lim_{n \rightarrow \infty} n^{10}(\gamma(a) - \delta_{n,2}(a)) = \frac{1}{132}; \\ \eta_{n,2}(a) &= \delta_{n,2}(a) + \frac{1}{132(a+n-1)^{10}}, \text{ for each } n \in \mathbb{N}, \\ & \lim_{n \rightarrow \infty} n^{12}(\eta_{n,2}(a) - \gamma(a)) = \frac{9950309}{471744000}; \\ \theta_{n,2}(a) &= \eta_{n,2}(a) - \frac{9950309}{471744000(a+n-1)^{12}}, \text{ for each } n \in \mathbb{N}, \\ & \lim_{n \rightarrow \infty} n^{14}(\gamma(a) - \theta_{n,2}(a)) = \frac{1}{12}; \\ \lambda_{n,2}(a) &= \theta_{n,2}(a) + \frac{1}{12(a+n-1)^{14}}, \text{ for each } n \in \mathbb{N}, \\ & \lim_{n \rightarrow \infty} n^{16}(\lambda_{n,2}(a) - \gamma(a)) = \frac{6250176017}{14100480000}; \\ \mu_{n,2}(a) &= \lambda_{n,2}(a) - \frac{6250176017}{14100480000(a+n-1)^{16}}, \text{ for each } n \in \mathbb{N}, \\ & \lim_{n \rightarrow \infty} n^{18}(\gamma(a) - \mu_{n,2}(a)) = \frac{43867}{14364}. \end{aligned}$$

We point out the pattern in forming the sequences from Theorem 2 and those mentioned above. For example, the general term of the sequence $(\mu_{n,2}(a))_{n \in \mathbb{N}}$ can be written in the form

$$\begin{aligned} \mu_{n,2}(a) = & \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} + \frac{B_2}{2} \cdot \frac{1}{(a+n-1)^2} \\ & - \ln \left(\frac{a+n-1}{a} - \frac{B_4}{4} \cdot \frac{1}{a(a+n-1)^3} \right) + \sum_{k=3}^8 \frac{c_{k,2}}{(a+n-1)^{2k}}, \end{aligned}$$

with

$$c_{k,2} = \begin{cases} \frac{B_{2k}}{2k}, & \text{if } k = 2p + 1, p \in \mathbb{N}, \\ \frac{B_{2k}}{2k} - \frac{2}{k} \left(\frac{B_4}{4} \right)^{\frac{k}{2}}, & \text{if } k = 2p + 2, p \in \mathbb{N}, \end{cases}$$

where B_{2k} is the Bernoulli number of index $2k$. Related to this remark, see also [16, Remark 3.4], [19, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\dots$ we obtain, for example:

$$\begin{aligned} \alpha_{2,2}(1) &= 0.5771654550\dots; & \alpha_{3,2}(1) &= 0.5772107618\dots; \\ \beta_{2,2}(1) &= 0.5772274589\dots; & \beta_{3,2}(1) &= 0.5772162053\dots; \\ \delta_{2,2}(1) &= 0.5772110473\dots; & \delta_{3,2}(1) &= 0.5772155649\dots; \\ \eta_{2,2}(1) &= 0.5772184455\dots; & \eta_{3,2}(1) &= 0.5772156932\dots; \\ \theta_{2,2}(1) &= 0.5772132959\dots; & \theta_{3,2}(1) &= 0.5772156535\dots; \\ \lambda_{2,2}(1) &= 0.5772183822\dots; & \lambda_{3,2}(1) &= 0.5772156709\dots; \\ \mu_{2,2}(1) &= 0.5772116186\dots; & \mu_{3,2}(1) &= 0.5772156606\dots \end{aligned}$$

As can be seen, $\mu_{3,2}(1)$ is accurate to eight decimal places in approximating γ .

Theorem 3. *Let $a \in (0, +\infty)$. We specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n \in \mathbb{N}}$ from Introduction.*

(i) *We consider the sequence $(\alpha_{n,3}(a))_{n \geq 2}$ defined by*

$$\begin{aligned} \alpha_{n,3}(a) = & \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} \\ & + \frac{1}{12(a+n-1)^2} - \frac{1}{120(a+n-1)^4} \\ & - \ln \left(\frac{a+n-1}{a} - \frac{1}{252a(a+n-1)^5} \right), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. Then

$$\lim_{n \rightarrow \infty} n^8 (\alpha_{n,3}(a) - \gamma(a)) = \frac{1}{240}.$$

(ii) We consider the sequence $(\beta_{n,3}(a))_{n \geq 2}$ defined by

$$\beta_{n,3}(a) = \alpha_{n,3}(a) - \frac{1}{240(a+n-1)^8},$$

for each $n \in \mathbb{N} \setminus \{1\}$. Then

$$\lim_{n \rightarrow \infty} n^{10}(\gamma(a) - \beta_{n,3}(a)) = \frac{1}{132}.$$

(iii) We consider the sequence $(\delta_{n,3}(a))_{n \geq 2}$ defined by

$$\delta_{n,3}(a) = \beta_{n,3}(a) + \frac{1}{132(a+n-1)^{10}},$$

for each $n \in \mathbb{N} \setminus \{1\}$. Then

$$\lim_{n \rightarrow \infty} n^{12}(\delta_{n,3}(a) - \gamma(a)) = \frac{174197}{8255520}.$$

Proof. (i) We have

$$\begin{aligned} & \alpha_{n,3}(a) - \alpha_{n+1,3}(a) \\ &= -\frac{1}{2(a+n-1)} - \frac{1}{2(a+n)} + \frac{1}{12(a+n-1)^2} - \frac{1}{12(a+n)^2} \\ & \quad - \frac{1}{120(a+n-1)^4} + \frac{1}{120(a+n)^4} \\ & \quad - \ln \left(a+n-1 - \frac{1}{252(a+n-1)^5} \right) + \ln \left(a+n - \frac{1}{252(a+n)^5} \right) \\ &= -\frac{1}{2(a+n) \left(1 - \frac{1}{a+n} \right)} - \frac{1}{2(a+n)} + \frac{1}{12(a+n)^2 \left(1 - \frac{1}{a+n} \right)^2} - \frac{1}{12(a+n)^2} \\ & \quad - \frac{1}{120(a+n)^4 \left(1 - \frac{1}{a+n} \right)^4} + \frac{1}{120(a+n)^4} \\ & \quad - \ln \left(1 - \frac{1}{a+n} - \frac{1}{252(a+n)^6 \left(1 - \frac{1}{a+n} \right)^5} \right) + \ln \left(1 - \frac{1}{252(a+n)^6} \right), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. Set $\varepsilon_n := \frac{1}{a+n}$, for each $n \in \mathbb{N} \setminus \{1\}$. Since $\varepsilon_n \in (-1, 1)$, $-\varepsilon_n - \frac{1}{252} \cdot \frac{\varepsilon_n^6}{(1-\varepsilon_n)^5} \in (-1, 1]$ and $-\frac{1}{252} \varepsilon_n^6 \in (-1, 1]$, for each $n \in \mathbb{N} \setminus \{1\}$, using the

series expansion ([6, pp. 171–179, p. 209]) we obtain

$$\begin{aligned} & \alpha_{n,3}(a) - \alpha_{n+1,3}(a) \\ &= -\frac{1}{2} \cdot \frac{\varepsilon_n}{1 - \varepsilon_n} - \frac{1}{2} \varepsilon_n + \frac{1}{12} \cdot \frac{\varepsilon_n^2}{(1 - \varepsilon_n)^2} - \frac{1}{12} \varepsilon_n^2 - \frac{1}{120} \cdot \frac{\varepsilon_n^4}{(1 - \varepsilon_n)^4} + \frac{1}{120} \varepsilon_n^4 \\ & \quad - \ln \left(1 - \varepsilon_n - \frac{1}{252} \cdot \frac{\varepsilon_n^6}{(1 - \varepsilon_n)^5} \right) + \ln \left(1 - \frac{1}{252} \varepsilon_n^6 \right) \\ &= \frac{1}{30} \varepsilon_n^9 + \frac{3}{20} \varepsilon_n^{10} + \frac{14}{33} \varepsilon_n^{11} + \frac{23}{24} \varepsilon_n^{12} + \frac{259573}{137592} \varepsilon_n^{13} + \frac{357653}{105840} \varepsilon_n^{14} + O(\varepsilon_n^{15}), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n \rightarrow \infty} n^9 (\alpha_{n,3}(a) - \alpha_{n+1,3}(a)) = \frac{1}{30}.$$

Now, according to Lemma 1, we get

$$\lim_{n \rightarrow \infty} n^8 (\alpha_{n,3}(a) - \gamma(a)) = \frac{1}{240}.$$

(ii) We are able to write that

$$\begin{aligned} & \beta_{n+1,3}(a) - \beta_{n,3}(a) \\ &= \alpha_{n+1,3}(a) - \alpha_{n,3}(a) - \frac{1}{240(a+n)^8} + \frac{1}{240(a+n-1)^8} \\ &= \alpha_{n+1,3}(a) - \alpha_{n,3}(a) - \frac{1}{240(a+n)^8} + \frac{1}{240(a+n)^8 \left(1 - \frac{1}{a+n}\right)^8} \\ &= \alpha_{n+1,3}(a) - \alpha_{n,3}(a) - \frac{1}{240} \varepsilon_n^8 + \frac{1}{240} \cdot \frac{\varepsilon_n^8}{(1 - \varepsilon_n)^8} \\ &= \frac{5}{66} \varepsilon_n^{11} + \frac{5}{12} \varepsilon_n^{12} + \frac{972403}{687960} \varepsilon_n^{13} + \frac{399103}{105840} \varepsilon_n^{14} + O(\varepsilon_n^{15}), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n \rightarrow \infty} n^{11} (\beta_{n+1,3}(a) - \beta_{n,3}(a)) = \frac{5}{66}.$$

Now, according to Lemma 1, we get

$$\lim_{n \rightarrow \infty} n^{10} (\gamma(a) - \beta_{n,3}(a)) = \frac{1}{132}.$$

(iii) We have

$$\begin{aligned} & \delta_{n,3}(a) - \delta_{n+1,3}(a) \\ &= \beta_{n,3}(a) - \beta_{n+1,3}(a) + \frac{1}{132(a+n-1)^{10}} - \frac{1}{132(a+n)^{10}} \\ &= \beta_{n,3}(a) - \beta_{n+1,3}(a) + \frac{1}{132(a+n)^{10} \left(1 - \frac{1}{a+n}\right)^{10}} - \frac{1}{132(a+n)^{10}} \\ &= \beta_{n,3}(a) - \beta_{n+1,3}(a) + \frac{1}{132} \cdot \frac{\varepsilon_n^{10}}{(1 - \varepsilon_n)^{10}} - \frac{1}{132} \varepsilon_n^{10} \\ &= \frac{174197}{687960} \varepsilon_n^{13} + \frac{174197}{105840} \varepsilon_n^{14} + O(\varepsilon_n^{15}), \end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n \rightarrow \infty} n^{13}(\delta_{n,3}(a) - \delta_{n+1,3}(a)) = \frac{174197}{687960}.$$

Now, according to Lemma 1, we get

$$\lim_{n \rightarrow \infty} n^{12}(\delta_{n,3}(a) - \gamma(a)) = \frac{174197}{8255520}.$$

□

In the same manner as in the proof of Theorem 3, considering the sequence in each of the following parts, we get the indicated limit:

$$\begin{aligned} \eta_{n,3}(a) &= \delta_{n,3}(a) - \frac{174197}{8255520(a+n-1)^{12}}, \text{ for each } n \in \mathbb{N} \setminus \{1\}, \\ \lim_{n \rightarrow \infty} n^{14}(\gamma(a) - \eta_{n,3}(a)) &= \frac{1}{12}; \end{aligned}$$

$$\begin{aligned} \theta_{n,3}(a) &= \eta_{n,3}(a) + \frac{1}{12(a+n-1)^{14}}, \text{ for each } n \in \mathbb{N} \setminus \{1\}, \\ \lim_{n \rightarrow \infty} n^{16}(\theta_{n,3}(a) - \gamma(a)) &= \frac{3617}{8160}; \end{aligned}$$

$$\begin{aligned} \lambda_{n,3}(a) &= \theta_{n,3}(a) - \frac{3617}{8160(a+n-1)^{16}}, \text{ for each } n \in \mathbb{N} \setminus \{1\}, \\ \lim_{n \rightarrow \infty} n^{18}(\gamma(a) - \lambda_{n,3}(a)) &= \frac{2785729949}{912171456}; \end{aligned}$$

$$\begin{aligned} \mu_{n,3}(a) &= \lambda_{n,3}(a) + \frac{2785729949}{912171456(a+n-1)^{18}}, \text{ for each } n \in \mathbb{N} \setminus \{1\}, \\ \lim_{n \rightarrow \infty} n^{20}(\mu_{n,3}(a) - \gamma(a)) &= \frac{174611}{6600}. \end{aligned}$$

We point out the pattern in forming the sequences from Theorem 3 and those mentioned above. For example, the general term of the sequence $(\mu_{n,3}(a))_{n \geq 2}$ can

be written in the form

$$\begin{aligned} \mu_{n,3}(a) &= \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} \\ &\quad + \frac{B_2}{2} \cdot \frac{1}{(a+n-1)^2} + \frac{B_4}{4} \cdot \frac{1}{(a+n-1)^4} \\ &\quad - \ln \left(\frac{a+n-1}{a} - \frac{B_6}{6} \cdot \frac{1}{a(a+n-1)^5} \right) + \sum_{k=4}^9 \frac{c_{k,3}}{(a+n-1)^{2k}}, \end{aligned}$$

with

$$c_{k,3} = \begin{cases} \frac{B_{2k}}{2k}, & \text{if } k = 3p + 1, p \in \mathbb{N}, \\ \frac{B_{2k}}{2k}, & \text{if } k = 3p + 2, p \in \mathbb{N}, \\ \frac{B_{2k}}{2k} - \frac{3}{k} \left(\frac{B_6}{6} \right)^{\frac{k}{3}}, & \text{if } k = 3p + 3, p \in \mathbb{N}, \end{cases}$$

where B_{2k} is the Bernoulli number of index $2k$. Related to this remark, see also [16, Remark 3.4], [19, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\dots$ we obtain, for example:

$$\begin{aligned} \alpha_{2,3}(1) &= 0.5772273253\dots; & \alpha_{3,3}(1) &= 0.5772162000\dots; \\ \beta_{2,3}(1) &= 0.5772110492\dots; & \beta_{3,3}(1) &= 0.5772155649\dots; \\ \delta_{2,3}(1) &= 0.5772184474\dots; & \delta_{3,3}(1) &= 0.5772156932\dots; \\ \eta_{2,3}(1) &= 0.5772132959\dots; & \eta_{3,3}(1) &= 0.5772156535\dots; \\ \theta_{2,3}(1) &= 0.5772183822\dots; & \theta_{3,3}(1) &= 0.5772156709\dots; \\ \lambda_{2,3}(1) &= 0.5772116186\dots; & \lambda_{3,3}(1) &= 0.5772156606\dots; \\ \mu_{2,3}(1) &= 0.5772232685\dots; & \mu_{3,3}(1) &= 0.5772156685\dots \end{aligned}$$

As can be seen, $\lambda_{3,3}(1)$ and $\mu_{3,3}(1)$ are accurate to eight decimal places in approximating γ .

Concluding, the following remark can be made. Let $a \in (0, +\infty)$ and $q \in \mathbb{N} \setminus \{1\}$. Let $n_0 = \min \left\{ n \in \mathbb{N} \mid a+n-1 - \frac{B_{2q}}{2q} \cdot \frac{1}{(a+n-1)^{2q-1}} > 0 \right\}$. We consider the sequence $(\alpha_{n,q}(a))_{n \geq n_0}$ defined by

$$\begin{aligned} \alpha_{n,q}(a) &= \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} + \sum_{k=1}^{q-1} \frac{B_{2k}}{2k} \cdot \frac{1}{(a+n-1)^{2k}} \\ &\quad - \ln \left(\frac{a+n-1}{a} - \frac{B_{2q}}{2q} \cdot \frac{1}{a(a+n-1)^{2q-1}} \right), \end{aligned}$$

for each $n \in \mathbb{N}$, $n \geq n_0$. Clearly, $\lim_{n \rightarrow \infty} \alpha_{n,q}(a) = \gamma(a)$. Based on the sequence $(\alpha_{n,q}(a))_{n \geq n_0}$, a class of sequences convergent to $\gamma(a)$ can be considered, namely

$$\{(\alpha_{n,q,r}(a))_{n \geq n_0} \mid r \in \mathbb{N}, r \geq q + 1\},$$

where

$$\alpha_{n,q,r}(a) = \alpha_{n,q}(a) + \sum_{k=q+1}^r \frac{c_{k,q}}{(a+n-1)^{2k}},$$

for each $n \in \mathbb{N}$, $n \geq n_0$, with

$$c_{k,q} = \begin{cases} \frac{B_{2k}}{2k}, & \text{if } k \in \{qp + 1, qp + 2, \dots, qp + q - 1\}, p \in \mathbb{N}, \\ \frac{B_{2k}}{2k} - \frac{q}{k} \left(\frac{B_{2q}}{2q}\right)^{\frac{k}{q}}, & \text{if } k = qp + q, p \in \mathbb{N}. \end{cases}$$

In this section we have obtained that the sequence $(\alpha_{n,2}(a))_{n \in \mathbb{N}}$ converges to $\gamma(a)$ with order 6 and that the sequence $(\alpha_{n,3}(a))_{n \geq 2}$ converges to $\gamma(a)$ with order 8. We have also obtained that the sequence $(\alpha_{n,2,r}(a))_{n \in \mathbb{N}}$ converges to $\gamma(a)$ with order $2(r + 1)$, for $r \in \{3, 4, 5, 6, 7, 8\}$, and that the sequence $(\alpha_{n,3,r}(a))_{n \geq 2}$ converges to $\gamma(a)$ with order $2(r + 1)$, for $r \in \{4, 5, 6, 7, 8, 9\}$.

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