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ON THE NUMBER OF FACTORIZATIONS OF AN INTEGER

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Abstract

Let $f(n)$ be the number of unordered factorizations of a positive integer n as a product of factors > 1. In this paper, we show that the number of distinct values of $f(n)$ below x is at most $\exp(9(\log x)^{2/3})$ for all $x \geq 1$.

1. Introduction

Let $f(n)$ be the number of unordered factorizations of a positive integer n as a product of factors > 1. For example, $f(12) = 4$ since the factorizations of 12 are $12,\ 2 \cdot 6,\ 3 \cdot 4,\ 2 \cdot 2 \cdot 3$. This function has already been extensively investigated in several papers.

For any real number $x \geq 1$ put $\mathcal{F}(x) = \{f(n) \leq x\}$. The authors of [1] say that they could prove that $\#\mathcal{F}(x) = x^{o(1)}$ as $x \to \infty$ but did not supply details. The bound

$$\#\mathcal{F}(x) = x^{O(\log\log\log x/\log\log x)}$$

appears in [2]. Here, we improve this estimate. Our result is the following.

Theorem 1. *The inequality*

$$\#\mathcal{F}(x) \leq \exp(9(\log x)^{2/3})$$

holds for all $x \geq 1$.

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2. The Proof of Theorem 1

Observe first that $\#\mathcal{F}(x) \le x$, and the inequality $x \le \exp(9(\log x)^{2/3})$ holds for all $1 \le x \le \exp(9^3) =: x_0$. So from now on, we assume that $x > x_0$.

Next, note that it suffices to count the values of $f(n) \le x$, where $n \in \mathcal{N}$ and

$$\mathcal{N} = \{2^{\alpha_1} \cdot 3^{\alpha_2} \cdots p_k^{\alpha_k} \cdot p_{k+1} \cdots p_{k+\ell} : \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_k \ge 2,\ \ell \ge 0\}.$$

Here, p_j is the jth prime. It is also clear that if $a > 1$ and $b > 1$ are coprime, then $f(ab) \ge f(a)f(b)$ since the function which associates to two factorizations $\mathbf{d} = (d_1, d_2, \ldots, d_s)$ and $\mathbf{e} = (e_1, e_2, \ldots, e_t)$ of a and b, respectively, where

$$d_1 \cdot d_2 \cdots d_s = a, \qquad \text{with } 2 \le d_1 \le d_2 \le \cdots \le d_s,$$

$$e_1 \cdot e_2 \cdots e_t = b, \qquad \text{with } 2 \le e_1 \le e_2 \le \cdots \le e_t,$$

the factorization $(d_1, \ldots, d_s, e_1, \ldots, e_t)$ of ab is clearly injective when a and b are coprime. Since also $f(p^\alpha) = p(\alpha)$, where $p(n)$ is the partition function of n, and $f(m) = B_{\omega(m)}$ when m is squarefree, where $\omega(m)$ is the number of distinct prime factors of m and B_s is the sth Bell number, we get that

$$\begin{aligned} f(2^{\alpha_1} \cdot 3^{\alpha_2} \cdot \cdots \cdot p_k^{\alpha_k} p_{k+1} \cdots p_{k+\ell}) &\ge f(2^{\alpha_1}) \cdot f(3^{\alpha_2}) \cdots f(p_k^{\alpha_k}) f(p_{k+1} \cdots p_{k+\ell}) \\ &\ge p(\alpha_1) \cdot p(\alpha_2) \cdots p(\alpha_k) B_\ell. \end{aligned}$$

Assuming that $f(n) \le x$, we then get that

$$x \ge p(\alpha_1) \cdot p(\alpha_2) \cdots p(\alpha_k) B_\ell. \tag{1}$$

We need effective lower bounds on $p(n)$ and B_t. By Corollary 3.1 in [3], we have

$$p(n) > \frac{e^{2\sqrt{n}}}{14} \qquad \text{for all} \quad n \ge 1.$$

From the above inequality, it follows immediately that

$$p(n) \ge \exp(c_1\sqrt{n}) \qquad \text{holds for all} \quad n \ge 2 \quad \text{with} \quad c_1 := (\log 2)/\sqrt{2}. \tag{2}$$

Indeed, for $n = 2,\ 3$ this can be checked directly, while for the remainig n this follows from the fact that the inequality

$$\frac{e^{2\sqrt{n}}}{14} > e^{c_1\sqrt{n}} \qquad \text{holds for all} \qquad n \ge 4.$$

For the Bell number, we use the Dobinski formula to get that inequality

$$B_\ell = \frac{1}{e}\sum_{k \ge 0} \frac{k^\ell}{k!} \ge \frac{3^\ell}{6e} > e^{\ell-3} \qquad \text{holds for all} \qquad \ell \ge 0. \tag{3}$$

It suffices to find an upper bound on the number of vectors $(\alpha_1, \alpha_2, \ldots, \alpha_k, \ell)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 2$ and $\ell \geq 0$ satisfying inequality (1). Taking logarithms and using the above lower bounds (2) and (3) for $p(n)$ and B_ℓ, respectively, we get that inequality (1) leads to

$$c_2 \log x \geq \sqrt{\alpha_1} + \sqrt{\alpha_2} + \cdots + \sqrt{\alpha_k} \qquad \text{and} \quad \log x \geq \ell - 3,$$

where $c_2 := 1/c_1$. Taking integer parts, we get

$$\lfloor c_2 \log x \rfloor \geq \lfloor \sqrt{\alpha_1} \rfloor + \lfloor \sqrt{\alpha_2} \rfloor + \cdots + \lfloor \sqrt{\alpha_k} \rfloor \qquad \text{and} \qquad \lfloor \log x \rfloor + 3 \geq \ell. \tag{4}$$

So, let us fix some number m and count the number of solutions of

$$m = \lfloor \sqrt{\alpha_1} \rfloor + \lfloor \sqrt{\alpha_2} \rfloor + \cdots + \lfloor \sqrt{\alpha_k} \rfloor, \qquad \text{with} \qquad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 1. \tag{5}$$

Write

$$m = a_1 + a_2 + \cdots + a_k, \qquad \text{with} \qquad a_1 \geq a_2 \geq \cdots \geq a_k \geq 1. \tag{6}$$

Write also

$$\alpha_i = a_i^2 + 2b_i, \qquad \text{with} \qquad 0 \leq b_i \leq 2a_i, \qquad i = 1, \ldots, k. \tag{7}$$

All solutions of (5) arise as a pair consisting of a vector of the form $(a_1, \ldots, a_k)$ with $a_1 \geq \cdots \geq a_k$ as in (6), the number of them being counted by the partition function $p(m)$, together with another vector of the form $(b_1, \ldots, b_k)$ such that relations (7) are satisfied. Now we ask: from a solution such as in (6), how many distinct solutions to (5) can we get via relations (7)?

To make it more clear, let us argue by means of an example. Let us say that 7 occurs t times in (6). Then we have $a_i = a_{i+1} = \ldots = a_{i+t-1}$ for some $i = 1, 2, \ldots, k - t + 1$. Then $\alpha_i, \ldots, \alpha_{i+t-1} \in [49, 63]$ are integers subject to $\alpha_i \geq \alpha_{i+1} \geq \cdots \geq \alpha_{i+t-1}$. Since there are 15 integers in $[49, 63]$, to count the number of such possibilities, it suffices to count the number of ways of writing $t = \lambda_1 + \cdots + \lambda_{15}$ with nonnegative integers $\lambda_i \geq 0$, and this number is $\binom{t+14}{14}$. Then we choose the first λ_1 values of the a_j's to be 63, the following λ_2 values of the a_j's to be 62, and so on until we get to the last λ_{15} values of the a_j's which we set to be 49, where here $j \in \{i, i+1, \ldots, i+t-1\}$.

In general, this shows that if the partition of n given in (6) is given by

$$a := 1^{h_1} 2^{h_2} \cdots m^{h_m}, \tag{8}$$

then the number of solutions to (6) arising from this particular partition for n is

$$w(a) := \prod_{j=1}^{n} \binom{h_j + 2j}{2j}. \tag{9}$$

This suggests that if we give to each partition $a = (a_1, \ldots, a_k)$ of m with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ the *weight* $w(a)$ being the number of corresponding solutions to (5) arising from this partition, then the number of solutions to (5) is

$$q(m) = \sum_{a \in \mathcal{P}(m)} w(a),$$

where we write $\mathcal{P}(m)$ for the set of partitions of m. Observe now that if we take

$$f(z) = (1-z)^{-3}(1-z^2)^{-5} \cdots (1-z^m)^{-(2m+1)} \cdots$$

then one can easily check using formula (9) that

$$f(z) = 1 + \sum_{m \geq 1} q(m) z^m. \tag{10}$$

We next prove that $q_N \leq \exp(5N^{2/3})$ holds for all $N \geq 1$. For this, take $B := N^{1/3}$ and put $z := e^{-1/B}$. Then

$$\begin{aligned} q(N)z^N &\leq 1 + \sum_{m \geq 1} q(m) z^m = \prod_{m \geq 1} (1-z^m)^{-(2m+1)} \\ &= \exp\left(\sum_{m \geq 1} -(2m+1)\log(1-z^m)\right) < \exp\left(\sum_{m \geq 1} (2m+1) z^m\right) \\ &< \exp((1-z)^{-2}) < \exp(4B^2). \end{aligned}$$

Since also

$$z^N = \exp(-N/B) = \exp(-N^{2/3}),$$

we get that $q(N) < \exp(5N^{2/3})$, which is what we wanted to prove.

Returning to our original problem, note that ℓ appearing in inequality (4) can take at most $\lfloor \log x \rfloor + 4$ values. Hence,

$$\#\mathcal{F}(x) \leq (\log x + 4) \sum_{0 \leq N \leq c_2 \log x} q(N) < (c_2 \log x + 1)(\log x + 4)\exp(c_3(\log x)^{2/3}),$$

where $c_3 := 5c_2^{2/3}$. Since $c_2 < 3$ and $c_3 < 8.05$, we get that

$$\#\mathcal{F}(x) \leq (3y+1)(y+4)\exp(8.05y^{2/3}), \qquad \text{where} \quad y := \log x.$$

One verifies that the inequality

$$(3y+1)(y+4)\exp(8.05y) < \exp(9y) \qquad \text{holds for all} \quad y \geq 6.$$

Since for us $y = \log x \geq \log x_0 = 9^3 > 6$, we conclude that $\#\mathcal{F}(x) \leq \exp(9(\log x)^{2/3})$ holds for all $x \geq x_0$, which is what we wanted to prove.

3. Comments and Remarks

Along the way of the proof of Theorem 1, we note that we also proved the following result which we record since it might be of independent interest.

Proposition 2. *Let $q(m)$ be the number of representations of*

$$m = \lfloor \sqrt{\alpha_1} \rfloor + \lfloor \sqrt{\alpha_2} \rfloor + \cdots + \lfloor \sqrt{\alpha_k} \rfloor$$

with positive integers $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Then the inequality

$$q(m) \leq \exp(5m^{2/3})$$

holds for all positive integers m.

References

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