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ON THE DISTANCE BETWEEN SMOOTH NUMBERS

Jean-Marie De Koninck[1]
Département de mathématiques et de statistique, Université Laval, Québec, Québec, Canada
jmdk@mat.ulaval.ca

Nicolas Doyon
Département de mathématiques et de statistique, Université Laval, Québec, Québec, Canada
nicolas.doyon@mat.ulaval.ca

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Abstract

Let $P(n)$ stand for the largest prime factor of $n \geq 2$ and set $P(1) = 1$. For each integer $n \geq 2$, let $\delta(n)$ be the distance to the nearest $P(n)$-smooth number, that is, to the nearest integer whose largest prime factor is no larger than that of n. We provide a heuristic argument showing that $\sum_{n \leq x} 1/\delta(n) = (4\log 2 - 2 + o(1))x$ as $x \to \infty$. Moreover, given an arbitrary real-valued arithmetic function f, we study the behavior of the more general function $\delta_f(n)$ defined by $\delta_f(n) = \min_{1 \leq m \neq n,\ f(m) \leq f(n)} |n - m|$ for $n \geq 2$, and $\delta_f(1) = 1$. In particular, given any positive integers $a < b$, we show that $\sum_{a \leq n < b} 1/\delta_f(n) \geq 2(b-a)/3$ and that if $f(n) \geq f(a)$ for all $n \in [a, b[$, then $\sum_{a<n<b} \delta_f(n) \leq (b-a)\log(b-a)/(2\log 2)$.

1. Introduction

A *smooth number* (or a *friable number*) is a positive integer n whose largest prime factor is "small" compared to n. Hence, given an integer $B \geq 2$, we say that an integer n is B*-smooth* if all its prime factors are $\leq B$.

Let $P(n)$ stand for the largest prime factor of n (with $P(1) = 1$).

For each integer $n \geq 2$, let $\delta(n)$ be the distance to the nearest $P(n)$-smooth number, that is, to the nearest integer whose largest prime factor is no larger than that of n. In other words,

$$\delta(n) := \min_{\substack{1 \leq m \neq n \\ P(m) \leq P(n)}} |n - m|.$$

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Equivalently, if we let

$$\Psi(x, y) := \#\{n \le x : P(n) \le y\},$$

then $\delta(n)$ is the smallest positive integer δ such that either one of the following two equalities occur:

$$\Psi(n + \delta, P(n)) - \Psi(n, P(n)) = 1, \qquad \Psi(n, P(n)) - \Psi(n - \delta, P(n)) = 1.$$

For convenience, we set $\delta(1) = 1$. In particular,

$$\delta(2^a) = 2^{a-1} \quad \text{for each integer } a \ge 1. \tag{1}$$

The first 40 values of $\delta(n)$ are

1,1,1,2,1,2,1,4,1,1,1,3,1,1,1,8,1,2,1,2,1,1,1,3,1,1,3,1,1,2,1,16,1,1,1,4,1,1,1,4.

We call $\delta(n)$ the *index of isolation* of n and we say that an integer n is *isolated* if $\delta(n) \ge 2$ and *non-isolated* if $\delta(n) = 1$. Finally, an integer n is said to be *very isolated* if $\delta(n)$ is "large".

It follows from (1) that the most isolated number $\le x$ is the largest power of 2 not exceeding x, which implies in particular that $\delta(n) \le n/2$ for all $n \ge 2$.

Remark. One might think, as a rule of thumb, that the smaller $P(n)$ is, the larger $\delta(n)$ will be, that is, that "smooth numbers have a large index of isolation". But this is not true for small values of n: for instance, $n = 11\,859\,211$ has a small $P(n)$ and nevertheless $\delta(n) = 1$, since

$$\begin{aligned} n &= 11\,859\,211 = 7 \cdot 13 \cdot 19^4, \\ n - 1 &= 11\,859\,210 = 2 \cdot 3^4 \cdot 5 \cdot 11^4. \end{aligned}$$

However, for large values of n, one can say that smooth numbers do indeed have a large index of isolation. Indeed, one can prove (see Lemma 3) that, given $B \ge 3$ fixed, there exist a constant $c = c(B) > 0$ and a number $n_0 = n_0(B)$ such that

$$\delta(n) > \frac{n}{(\log n)^c} \qquad \text{for all } B\text{-smooth integers } n \ge n_0.$$

2. Preliminary Observations and Results

It is clear that $\delta(p) = 1$ for each prime p and also that if p is odd, then $\delta(p^2) = 1$. Each of the following also holds:

$$\begin{aligned}
\delta(2p) &= 1 \quad \text{for} \quad p \geq 5,\\
\delta(3p) &= 1 \quad \text{for} \quad p \geq 3,\\
\delta(4p) &\leq 2 \quad \text{for} \quad p \geq 5,\\
\delta(5p) &= 1 \quad \text{for} \quad p \geq 2,\\
\delta(6p) &\leq 2 \quad \text{for} \quad p \geq 3,\\
\delta(7p) &\leq 2 \quad \text{for} \quad p \geq 2,\\
\delta(8p) &\leq 3 \quad \text{for} \quad p \geq 7,\\
\delta(9p) &\leq 3 \quad \text{for} \quad p \geq 2,\\
\delta(10p) &\leq 2 \quad \text{for} \quad p \geq 2.
\end{aligned}$$

The above are easily proven. For instance, to prove the second statement, observe that if $p \equiv 1 \pmod 4$, then $3p+1 \equiv 0 \pmod 4$, in which case $P(3p+1) < p$, while if $p \equiv 3 \pmod 4$, then $3p-1 \equiv 0 \pmod 4$, in which case $P(3p-1) < p$, so that in both cases $\delta(3p) = 1$.

Observe also that given any prime number p, if a is an integer such that $P(a) \leq p$, then $\delta(ap) \leq a$, because

$$P(ap - a) = P(a(p-1)) \leq \max(P(a), P(p-1)) \leq p.$$

If follows from this simple observation that

$$\delta(n) = \delta(aP(n)) \leq a = \frac{n}{P(n)} \qquad (n \geq 2). \tag{2}$$

Moreover, one can easily show that if $P(n)^2 | n$, then $\delta(n) \leq \dfrac{n}{P(n)^2}$.

Definition. For each integer $n \geq 1$, let

$$\Delta(n) := \sum_{d|n} \delta(d).$$

Trivially we have $\Delta(n) \geq \tau(n)$.

Lemma 1. *If n is a power of 2, then $\Delta(n) = n$. On the other hand, for all $n > 1$ such that $P(n) \geq 3$, we have*

$$\Delta(n) < n. \tag{3}$$

Proof. The first assertion is obvious since for each integer $\alpha \geq 1$, we have $\Delta(2^\alpha) = \sum_{i=1}^{\alpha} 2^{i-1} = 2^\alpha$.

Now consider the case when n is not a power of 2. First, it is easy to show that if $P(n) = 3$, then (3) holds. Indeed, if $n = 2^\alpha \cdot 3^\beta$ for some integers $\alpha \geq 0$ and $\beta \geq 1$, then in light of (2), we have

$$\begin{aligned}
\Delta(n) &\leq \sum_{d|n} \frac{d}{P(d)} = 1 + \sum_{\substack{d|n \\ P(d)=2}} \frac{d}{P(d)} + \sum_{\substack{d|n \\ P(d)=3}} \frac{d}{P(d)} \\
&= 1 + \frac{1}{2}\sum_{i=1}^{\alpha} 2^i + \frac{1}{3}\sum_{\substack{d|n \\ 3|d}} d \\
&= 1 + 2^\alpha - 1 + \frac{1}{3}(\sigma(n) - \sigma(2^\alpha)) \\
&= 2^\alpha + \frac{1}{3}\left((2^{\alpha+1}-1)\frac{3^{\beta+1}-1}{2} - (2^{\alpha+1}-1)\right) \\
&= n - \frac{3^\beta}{2} + \frac{1}{2} \leq n - 1 < n,
\end{aligned}$$

which proves that (3) holds if $P(n) = 3$.

Hence, from here on, we shall assume that $P(n) \geq 5$. We shall use induction on the number of distinct prime factors of n in order to prove that

$$\sum_{d|n} \frac{d}{P(d)} < n. \tag{4}$$

First observe that the above inequality is true if $\omega(n) = 1$. Indeed, in this case, we have $n = p^b$. It is clear that

$$\sum_{d|n} \frac{d}{P(d)} = 1 + 1 + p + p^2 + \ldots + p^{b-1} = 1 + \frac{p^b - 1}{p - 1} < 1 + p^b - 1 = p^b = n,$$

which will clearly establish (3).

Let us now assume that the result holds for all n such that $\omega(n) = r - 1$ and prove that it does hold for n such that $\omega(n) = r$. Take such an integer n with k being the unique positive integer such that $P(n)^k \| n$. We then have

$$\begin{aligned}\sum_{d\mid n}\frac{d}{P(d)} &= \sum_{d\mid n/P(n)^k}\frac{d}{P(d)}+\sum_{i=1}^{k}\sum_{d\mid n/P(n)^k}\frac{dP(n)^i}{P(dP(n)^i)}\\ &= \sum_{d\mid n/P(n)^k}\frac{d}{P(d)}+\sum_{d\mid n/P(n)^k} d+P(n)\sum_{d\mid n/P(n)^k} d\\ &\qquad +\ldots+P(n)^{k-1}\sum_{d\mid n/P(n)^k} d\\ &= \sum_{d\mid n/P(n)^k}\frac{d}{P(d)}+(1+P(n)+\ldots+P(n)^{k-1})\sigma(n/P(n)^k). \qquad (5)\end{aligned}$$

Using the identity $\sigma\left(\frac{n}{P(n)^k}\right)=\frac{\sigma(n)}{1+P(n)+P(n)^2+\ldots+P(n)^k}$ and the induction argument, it follows from (5) that

$$\sum_{d\mid n}\frac{d}{P(d)}<\frac{n}{P(n)^k}+\frac{1+P(n)+\ldots+P(n)^{k-1}}{1+P(n)+\ldots+P(n)^k}\sigma(n)<\frac{n}{P(n)}+\frac{\sigma(n)}{P(n)}. \qquad (6)$$

On the other hand, it is clear that for any integer $n>1$ with $P(n)\geq 5$ and $\omega(n)\geq 2$,

$$\frac{\sigma(n)}{n}<\prod_{p\mid n}\frac{p}{p-1}<\frac{3}{4}P(n).$$

Using this in (6), we obtain, since $P(n)\geq 5$, that $\sum_{d\mid n}\frac{d}{P(d)}<\frac{n}{5}+\frac{3}{4}n<n$, which completes the proof of (4) and thus of (3). □

Lemma 2. *The following are true:*

(i) $\#\{n\leq x:\delta(n)=1\}\geq\frac{x}{2}$ *for all* $x\geq 2$*;*

(ii) $\frac{x}{2}\leq\sum_{n\leq x}\frac{1}{\delta(n)}<x$ *for all* $x\geq 4$*;*

(iii) $\delta(n)=\delta(m)=k\Longrightarrow|n-m|\geq k$*;*

(iv) $|m-n|\geq\min(\delta(m),\delta(n))$*;*

(v) $c_k(x):=\#\{n\leq x:\delta(n)=k\}\leq\frac{x}{k}$*;*

(vi) $\#\{n\leq x:\delta(n)\geq y\}\leq\frac{x}{y}$.

Proof. Since it is clear that if $\delta(n) \geq 2$, then $\delta(n-1) = \delta(n+1) = 1$, (i) follows immediately, along with (ii).

On the other hand, (iii) follows from the definition of $\delta(n)$.

From (iii), we easily deduce (iv) and (v).

To prove (vi), we proceed as follows. Fix $1 < y \leq x$ and assume that $k := \#\{n \leq x : \delta(n) \geq y\} > x/y$. Let $y \leq n_1 < n_2 < \ldots < n_k$ be the integers $n_i \leq x$ such that $\delta(n_i) \geq y$. By the Pigeonhole Principle, there exist n_r and n_s with $1 \leq r < s \leq k$ such that $n_s - n_r < y$. Without any loss in generality, one can assume that $P(n_r) \leq P(n_s)$, in which case we have $\delta(n_s) \leq n_s - n_r < y$, a contradiction. □

Lemma 3. *Let $B \geq 3$ be a fixed integer. Then there exist a constant $c = c(B) > 0$ and a number $n_0 = n_0(B)$ such that*

$$\delta(n) > \frac{n}{(\log n)^c} \qquad \text{for all } B\text{-smooth integers } n \geq n_0.$$

Proof. The result follows almost immediately from an estimate of Tijdeman [7] who showed, using the theory of logarithmic forms of Baker (see Theorem 3.1 in the book of Baker [1]), that if $n_1 < n_2 < \ldots$ represents the sequence of B-smooth numbers, then there exist positive constants $c_1(B)$ and $c_2(B)$ such that

$$\frac{n_i}{(\log n_i)^{c_1(B)}} \ll n_{i+1} - n_i \ll \frac{n_i}{(\log n_i)^{c_2(B)}},$$

where $c_2(B) \leq \pi(B) \leq c_1(B)$. Observe that Langevin [6] later provided explicit values for the constants $c_1(B)$ and $c_2(B)$. □

3. Probabilistic Results

In 1978, Erdős and Pomerance [4] showed that the lower density of those integers n for which $P(n) < P(n+1)$ (or $P(n) > P(n+1)$) is positive. Most likely, this density is $\frac{1}{2}$, but this fact remains an open problem. In 2001, Balog [2] showed that the number of integers $n \leq x$ with

$$P(n-1) > P(n) > P(n+1) \tag{7}$$

is $\gg \sqrt{x}$ and observed that "undoubtedly" the density of those integers n such that (7) holds is equal to $\frac{1}{6}$.

To establish our next result, we shall make the following reasonable assumption.

Hypothesis A. Fix an arbitrary integer $k \geq 2$ and let n be a large number. Let $a_1, a_2, \ldots, a_k$ be any permutation of the numbers $0, 1, 2, \ldots, k-1$. Then,

$$\text{Prob}[P(n+a_1) < P(n+a_2) < \ldots < P(n+a_k)] = \frac{1}{k!}.$$

Theorem 4. *Assuming Hypothesis A and given any integer $k \geq 1$, the expected proportion of integers n for which $\delta(n) = k$ is equal to $\frac{2}{4k^2-1}$. In particular, the proportion of non-isolated numbers is $\frac{2}{3}$.*

Proof. Fix k. Let E_k be the expected proportion of integers n for which $\delta(n) = k$. Given a large integer n, the probability that $\delta(n) > k$ is equal to the probability that

$$\min(P(n \pm 1), \ldots, P(n \pm k)) > P(n).$$

Under Hypothesis A, this probability is equal to $\frac{1}{2k+1}$. This implies that

$$E_k = P(\delta(n) > k-1) - P(\delta(n) > k) = \frac{1}{2k-1} - \frac{1}{2k+1} = \frac{2}{4k^2-1},$$

which completes the proof of the theorem. $\square$

Remark. Let $S_k(x) := \#\{n \leq x : \delta(n) = k\}$ and choose $x = 10^6$. Then, we obtain the following numerical evidence.

k	1	2	3	4	5
$a = S_k(x)$	664 084	134 239	57 089	32 185	20 145
$b = [x \cdot 2/(4k^2-1)]$	666 666	133 333	57 142	31 746	20 202
a/b	0.996	1.006	0.999	1.013	0.997

Theorem 5. *Assuming Hypothesis A,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} \frac{1}{\delta(n)} = 2(2\log 2 - 1) \approx 0.7725.$$

Proof. According to Theorem 4 (proved assuming Hypothesis A),

$$\frac{1}{x}\#\{n \leq x : \delta(n) = k\} = (1 + o(1))\frac{2}{4k^2-1} \qquad (x \to \infty).$$

Therefore, given a fixed large integer N, we have, as $x \to \infty$,

$$\begin{aligned}
\sum_{n \le x} \frac{1}{\delta(n)} &= \sum_{k=1}^{N} \frac{1}{k} \sum_{\substack{n \le x \\ \delta(n)=k}} 1 + \sum_{k=N+1}^{[x/2]} \frac{1}{k} \sum_{\substack{n \le x \\ \delta(n)=k}} 1 \\
&= \sum_{k=1}^{N} \frac{2x}{k(4k^2-1)}(1+o(1)) + O\left(\sum_{k>N} \frac{x}{k^2} \right) \qquad (8) \\
&= (1+o(1))xT_1(N) + O(xT_2(N)),
\end{aligned}$$

say, where we used Lemma 2(v). First, one can show that

$$T_1(N) = \sum_{k=1}^{N} \frac{2}{k(4k^2-1)} = 2(2\log 2 - 1) + O\left(\frac{1}{N}\right) \qquad (N \to \infty). \tag{9}$$

To prove (9), we proceed as follows. Assume for now that $N \equiv 3 \pmod 4$. Then, using the estimate $\sum_{k=1}^{N} \frac{1}{k} = \log N + \gamma + O(1/N)$ as $N \to \infty$ (where γ is Euler's constant), we have

$$\begin{aligned}
\frac{1}{2}\,T_1(N) &= -\sum_{k=1}^{N} \frac{1}{k} + \sum_{k=1}^{N} \frac{1}{2k-1} + \sum_{k=1}^{N} \frac{1}{2k+1} \\
&= -\frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{N} \\
&\quad +2\left(\frac{1}{N+2} + \frac{1}{N+4} + \ldots + \frac{1}{2N-1} \right) + \frac{1}{2N+1} \\
&= \log 2 - 1 + O(1/N) + 2 \sum_{j=(N+1)/2}^{N-1} \frac{1}{2j+1} + \frac{1}{2N+1} \\
&= \log 2 - 1 + O(1/N) + \log 2 + O(1/N) \\
&= 2\log 2 - 1 + O(1/N),
\end{aligned}$$

which proves (9). A similar argument holds if $N \equiv 0, 1 \text{ or } 2 \pmod 4$, thus establishing (9).

On the other hand,

$$\sum_{k>N} \frac{1}{k^2} < \int_N^\infty \frac{1}{t^2}\,dt = \frac{1}{N},$$

so that

$$T_2(N) < \frac{1}{N}. \tag{10}$$

Now let $\varepsilon > 0$ be arbitrarily small and let $N = [1/\varepsilon] + 1$. We then have, using (9) and (10) in (8),

$$\sum_{n \le x} \frac{1}{\delta(n)} = 2(2\log 2 - 1)x + O(\varepsilon x) + O(\varepsilon x) \qquad (x \to \infty),$$

which completes the proof of the theorem. □

Remark. Using a computer, one obtains that

$$\frac{1}{10^9} \sum_{n \le 10^9} \frac{1}{\delta(n)} = 0.7719\ldots$$

4. The Isolation Index With Respect to a Given Function

The function δ can also be defined relatively to any real-valued arithmetic function f as

$$\delta_f(n) := \min_{\substack{1 \le m \ne n \\ f(m) \le f(n)}} |n - m| \qquad (n \ge 2)$$

with $\delta_f(1) = 1$.

Examples

- Let $f(n)$ be any monotonic function. Then, $\delta_f(n) = 1$, $\forall n \ge 2$.
- Let $f(n) = \omega(n) := \sum_{p|n} 1$. Then, the first 40 values of $\delta_\omega(n)$ are:

  1,1,1,1,1,1,1,1,1,1,2,1,2,1,1,1,1,1,2,1,1,1,2,1,2,1,2,1,2,1,1,1,1,1,1,1,4,1,1,1.

- Let $f(n) = \Omega(n) := \sum_{p^\alpha \| n} \alpha$. Then, the first 40 values of $\delta_\Omega(n)$ are:

  1,1,1,1,2,1,2,1,1,1,2,1,2,1,1,1,2,1,2,1,1,1,4,1,1,1,1,1,2,1,2,1,1,1,1,1,4,1,1,1.

- Let $f(n) = \tau(n) := \sum_{d|n} 1$. Then, the first 40 values of $\delta_\tau(n)$ are:

  1,1,1,1,2,1,2,1,2,1,2,1,2,1,1,1,2,1,2,1,1,1,4,1,2,1,1,1,2,1,2,1,1,1,1,1,4,1,1,1.

Remark 6. It turns out that Hypothesis A holds unconditionally when one replaces the function $P(n)$ by the function $\omega(n)$ or $\Omega(n)$ or $\tau(n)$, and therefore that the equivalent of Theorem 4 for either of these three functions is true without any conditions, that is, that for any fixed positive integer k,

$$\frac{1}{x}\#\{n \le x : \delta_\omega(n) = k\} = \frac{2}{4k^2-1} + o(1) \qquad (x \to \infty),$$

the same being true for $\Omega(n)$ or $\tau(n)$ in place of $\omega(n)$.

To prove our claim, we first need to prove the following two propositions.

Proposition 7. *Let $a_1, a_2, \ldots, a_k$ be any distinct integers and let $z_1, z_2, \ldots, z_k$ be arbitrary real numbers. Then,*

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x : \frac{\omega(n+a_j) - \log\log n}{\sqrt{\log\log n}} < z_j, 1 \le j \le k\right\} = \prod_{1\le j\le k} \Phi(z_j), \qquad (11)$$

where $\Phi(z) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{z} e^{-t^2/2}dt.$

Proof. Given a real-valued vector $(t_1, t_2, \ldots, t_k)$, consider the function

$$H(n) = \sum_{j=1}^{k} t_j \omega(n+a_j).$$

We will now apply Proposition 2 and Theorem 1 of Granville and Soundararajan [5], where instead of considering the function

$$f_p(n) = \begin{cases} 1 - \frac{1}{p} & \text{if } p|n, \\ -\frac{1}{p} & \text{otherwise,} \end{cases}$$

we use the function

$$f_p(n) = \begin{cases} t_r - \frac{1}{p}\sum_{j=1}^{k} t_j & \text{if } p|n+a_r, \\ -\frac{1}{p}\sum_{j=1}^{k} t_j & \text{otherwise,} \end{cases}$$

where it is clear that, except for a finite number of primes p, each prime p divides $n + a_r$ for at most one a_r.

Using this newly defined function $f_p(n)$ and following exactly the same steps as in the proof of Proposition 2 of Granville and Soundararajan, we obtain that

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x : \frac{H(n) - \sum_{j=1}^{k} t_j \log\log n}{\sqrt{\log\log n}} < z\right\} = \Phi\left(\frac{z}{\sqrt{\sum_{j=1}^{k} t_j^2}}\right). \qquad (12)$$

In other words, $H(n)$ has a Gaussian distribution with mean value $\sum_{j=1}^{k} t_j \log\log n$ and standard deviation $\sqrt{\sum_{j=1}^{k} t_j^2 \cdot \log\log n}$. Because of the moments of a Gaussian distribution, statement (12) is equivalent to

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\left(\frac{H(n)-\sum_{j=1}^{k}t_j\log\log n}{\sqrt{\log\log n}}\right)^m = G(m)\left(\sum_{j=1}^{k}t_j^2\right)^{m/2} \qquad (m=1,2,\ldots), \tag{13}$$

where, for each positive integer m,

$$G(m)=\begin{cases}\prod_{1\le j\le m/2}(2j-1) & \text{if } m \text{ is odd,}\\ 0 & \text{if } m \text{ is even.}\end{cases}$$

By expanding the left-hand side of (13) using the multinomial theorem, we may rewrite it (for each positive integer m) as

$$\sum_{\substack{0\le u_i\le m,\ i=1,\ldots,k\\ u_1+\cdots+u_k=m}}\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\frac{m!}{u_1!\cdots u_k!}\prod_{1\le j\le k}t_j^{u_j}\left(\frac{\omega(n+a_j)-\log\log n}{\sqrt{\log\log n}}\right)^{u_j}. \tag{14}$$

By considering (14) as a function of $t_1,t_2,\ldots,t_k$ and comparing the coefficients with those on the right-hand side of (13), we obtain, for each positive integer m,

$$\frac{m!}{u_1!\cdots u_k!}\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\prod_{1\le j\le k}\left(\frac{\omega(n+a_j)-\log\log n}{\sqrt{\log\log x}}\right)^{u_j} = G(m)\frac{(m/2)!}{(u_1/2)!\cdots(u_k/2)!}$$

or equivalently

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\prod_{1\le j\le k}\left(\frac{\omega(n+a_j)-\log\log n}{\sqrt{\log\log x}}\right)^{u_j} = \prod_{1\le j\le k}G(u_j). \tag{15}$$

Since the right-hand side of (15) corresponds to the centered moments of a multivariate independent Gaussian distribution, the validity of (11) follows, thereby completing the proof. $\square$

Proposition 8. *Let g stand for any of the functions ω, Ω or τ. Let $a_1,a_2,\ldots,a_k$ be any permutation of the integers $0,1,\ldots,k-1$. Then,*

$$\lim_{x\to\infty}\frac{1}{x}\#\{n\le x: g(n+a_j)<g(n+a_{j+1}), j=1,\ldots,k-1\}=\frac{1}{k!}.$$

Proof. In the case $g=\omega$, the result follows from Proposition 7, namely by simple integration of (11). As for $g=\Omega$, observe that it is easy to show that

$$\lim_{K\to\infty}\lim_{x\to\infty}\frac{1}{x}\#\{n\le x:\Omega(n)-\omega(n)>K\}=0. \tag{16}$$

Hence, using (16) and integrating (11), Proposition 7 holds for $g=\Omega$. Finally, since Proposition 7 holds for $g=\omega$ and $g=\Omega$, the inequality $2^{\omega(n)}\le\tau(n)\le 2^{\Omega(n)}$ implies that it also holds for $g=\tau$, thus completing the proof. $\square$

Theorem 9. *Let $f(n) = \omega(n)$ or $\Omega(n)$ or $\tau(n)$. Then,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \frac{1}{\delta_f(n)} = 2(2\log 2 - 1).$$

Proof. In light of Remark 6 and of Proposition 8, the result is immediate. □

Theorem 10. *Let $a < b$ be positive integers. For any real-valued arithmetic function f and any interval $I = [a, b[$ of length $N = b - a$,*

$$\sum_{n\in I} \frac{1}{\delta_f(n)} \ge \frac{2}{3}N - \frac{2}{3}.$$

Proof. We conduct the proof using induction on N. First observe that Theorem 10 holds for small values of N. For instance, if $N = 1$,

$$\sum_{n\in I} \frac{1}{\delta_f(n)} = \frac{1}{\delta_f(a)} > 0 = \frac{2}{3} - \frac{2}{3}.$$

If $N = 2$, then since at least one of $\delta_f(n)$ and $\delta_f(n+1)$ must be 1, it follows that

$$\sum_{n\in I} \frac{1}{\delta_f(n)} = \frac{1}{\delta_f(a)} + \frac{1}{\delta_f(a+1)} > 1 > \frac{2}{3} = \frac{2}{3}\cdot 2 - \frac{2}{3}.$$

We will now assume that the result holds for every integer smaller than N and prove that it must therefore hold for N. We shall do this by distinguishing three possible cases:

(i) either $\delta_f(a) = 1$ or $\delta_f(b-1) = 1$;

(ii) Case (i) is not satisfied and there exists a positive integer $k \in]a, b-2[$ such that both $\delta_f(k) = 1$ and $\delta_f(k+1) = 1$;

(iii) neither of the two previous cases holds.

In Case (i), we can assume without any loss of generality that $\delta_f(a) = 1$, in which case

$$\sum_{n\in I} \frac{1}{\delta_f(n)} = 1 + \sum_{a+1\le n<b} \frac{1}{\delta_f(n)}.$$

Using our induction hypothesis we get that

$$1 + \sum_{a+1\le n<b} \frac{1}{\delta_f(n)} \ge 1 + \frac{2}{3}(N-1) - \frac{2}{3} = \frac{2}{3}N - \frac{1}{3} > \frac{2}{3}N - \frac{2}{3},$$

proving the theorem in Case (i).

Suppose now that Case (ii) is satisfied. Then, there exists an integer $k \in]a, b-2[$ such that $\delta_f(k) = \delta_f(k+1) = 1$, in which case

$$\sum_{n\in I} \frac{1}{\delta_f(n)} = \sum_{a\le n<k} \frac{1}{\delta_f(n)} + 2 + \sum_{k+2\le n<b} \frac{1}{\delta_f(n)}.$$

Again, using our induction hypothesis we have that the right-hand side is larger or equal to

$$\frac{2}{3}(k-a) - \frac{2}{3} + 2 + \frac{2}{3}(b-k-2) - \frac{2}{3} = \frac{2}{3}N - \frac{2}{3},$$

proving the theorem in Case (ii).

We now consider Case (iii). In this situation, N has to be odd, since the sum starts with the term $1/\delta_f(a) < 1$ and ends with the term $1/\delta_f(b-1) < 1$, because every second value of $\delta_f(n)$ must be 1. Assume that a is odd, in which case $b-1$ is odd. The case a and $b-1$ even can be treated in a similar way. Hence, is at odd integers n that $\delta_f(n) > 1$, in which case we must have both $f(n-1) > f(n)$ and $f(n+1) > f(n)$. Now recall the definition $\delta_f(n) := \min_{\substack{1\le m\ne n\\ f(m)\le f(n)}} |m-n|$; now since the integer m at which this minimum occurs must be odd (since for the other m's, the even ones, we have $\delta_f(m) = 1$), it follows that for an odd $n = 2j+1$, we have

$$\delta_f(2j+1) = 2 \min_{\substack{1\le k\ne j\\ f(2k+1)\le f(2j+1)}} |k-j|,$$

so that we may write

$$\begin{aligned}
\sum_{\substack{n\in[a,b[\\ n \text{ odd}}} \frac{1}{\delta_f(n)} &= \sum_{j\in[\frac{a-1}{2},\frac{b}{2}[} \frac{1}{\delta_f(2j+1)} \\
&= \frac{1}{2} \sum_{j\in[\frac{a-1}{2},\frac{b}{2}[} \frac{1}{\min_{\substack{1\le k\ne j\\ f(2k+1)\le f(2j+1)}} |k-j|} \\
&= \frac{1}{2} \sum_{j\in[\frac{a-1}{2},\frac{b}{2}[} \frac{1}{\min_{\substack{1\le k\ne j\\ g(k)\le g(j)}} |k-j|} \\
&= \frac{1}{2} \sum_{j\in[\frac{a-1}{2},\frac{b}{2}[} \frac{1}{\delta_g(j)},
\end{aligned} \tag{17}$$

where we have set $g(j) := f(2j+1)$. Now since the interval $[\frac{a-1}{2}, \frac{b}{2}[$ is of length $\dfrac{N+1}{2} < N$, we can apply our induction hypothesis and write that the last expression in (17) is no larger than

$$\frac{1}{2}\cdot\frac{2}{3}\left(\frac{N+1}{2} - 1\right).$$

It follows from this that

$$\sum_{n\in I}\frac{1}{\delta_f(n)} \geq \frac{N-1}{2} + \frac{1}{2}\cdot\frac{2}{3}\left(\frac{N+1}{2}-1\right) = \frac{2}{3}(N-1),$$

as needed to be proved. This completes the proof of the theorem. □

In the statement of Theorem 10, is there any hope that one could replace the constant $\frac{2}{3}$ by a larger one? The answer is no, as the following result shows.

Theorem 11. *Let $\alpha_{0,q}(n), \alpha_{1,q}(n), \alpha_{2,q}(n), \dots, \alpha_{k,q}(n)$ be the digits of n when written in base q, that is,*

$$n = \sum_{j=0}^{k} \alpha_{j,q}(n) q^j.$$

Then the function $f = g_q$ defined by

$$g_q(n) = \sum_{j=0}^{k} \alpha_{j,q}(n) q^{k-j} \tag{18}$$

has the following property:

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\frac{1}{\delta_f(n)} = \frac{q}{q+1}. \tag{19}$$

Remark. Observe that $g_q(n)$ is the number obtained by writing the basis q digits of n in reverse order. In a sense, the result claims that the function g_2 is the one that provides the minimal value for the sum of the reciprocals of the index of isolation.

Remark 12. Let n be written in basis $q \geq 2$, that is,

$$n := \sum_{j=0}^{k} \alpha_{j,q}(n) q^j.$$

Let m be the smallest integer such that $\alpha_{m,q}(n)$ is greater than zero. Then, under the assumption that n is not a perfect power of q, it is easy to verify that

$$\delta_{g_q}(n) = q^m.$$

On the other hand, if n is a perfect power of q, say $n = q^k$, we have

$$\delta_{g_q}(n) = q^{k-1}(q-1).$$

Proof of Theorem 11. We shall only consider the case $q = 2$, since the general case can be treated similarly. We observe that $\delta_{g_2}(n) = 1$ if and only if n is odd. More generally, in light of Remark 12, we have

$$\delta_{g_2}(n) = 2^k \text{ if and only if } \frac{n}{2^k} \text{ is an odd integer} \qquad (k \geq 0).$$

We can therefore write

$$\sum_{n \leq x} \frac{1}{\delta_{g_2}(n)} = \sum_{k=0}^{[\log x/\log 2]} \frac{1}{2^k} \cdot \# \left\{ n \leq x : \frac{n}{2^k} \equiv 1 \pmod 2 \right\}.$$

It is easy to see that

$$\# \left\{ n \leq x : \frac{n}{2^k} \equiv 1 \pmod 2 \right\} = \frac{x}{2^{k+1}} + O(1),$$

so that

$$\sum_{n \leq x} \frac{1}{\delta_f(n)} = \sum_{k=0}^{[\log x/\log 2]} \frac{x}{2^{2k+1}} + O\left(\sum_{k \geq 0} \frac{1}{2^k} \right) = \frac{2}{3}x + O(1),$$

which proves (19) in the case $q = 2$, thus completing the proof of the theorem. □

Theorem 13. *For real numbers y, w such that $\frac{2}{3} \leq y \leq w \leq 1$, there exists an arithmetic function f such that*

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \frac{1}{\delta_f(n)} = y \quad \text{and} \quad \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \frac{1}{\delta_f(n)} = w. \tag{20}$$

Proof. The function $1/\delta_{g_2}$ has a mean value of $\frac{2}{3}$, while it is clear that the mean value of the reciprocal of the index of isolation of any monotone function h is 1. We shall construct a function that behaves piecewise like g_2 and piecewise like h so that the mean value of the reciprocal of its isolation index will be a pondered mean of $\frac{2}{3}$ and 1. We first define real numbers $s, t \in [0, 1]$ in such a way that $s + (1 - s)\frac{2}{3} = y$ and $t + (1 - t)\frac{2}{3} = w$. Now consider the intervals

$$I_j := [2^{2^j}, 2^{2^{j+1}}[\qquad (j = 1, 2, 3, \ldots).$$

For j even and for each integer $m \in [1, 2^{2^j - j}(2^{2^j} - 1)]$, define the families of subintervals $K_{j,m}$ and $L_{j,m}$ as follows:

$$K_{j,m} := [2^{2^j} + (m - 1)2^j, 2^{2^j} + (m - 1 + s) \cdot 2^j[$$

and

$$L_{j,m} := [2^{2^j} + (m - 1 + s)2^j, 2^{2^j} + m2^j[,$$

so that

$$I_j = \bigcup_{m=1}^{2^{2^j-j}(2^{2^j}-1)} (K_{j,m} \cup L_{j,m}).$$

For j odd, we replace the number s by t in the definition of the subintervals $K_{j,m}$ and $L_{j,m}$. For simplicity, we define implicitly $A_{j,m}, B_{j,m}, C_{j,m}, D_{j,m}$ as

$$K_{j,m} = [A_{j,m}, B_{j,m}[$$

and

$$L_{j,m} = [C_{j,m}, D_{j,m}[.$$

We are now ready to define a function f satisfying (20). We define f piecewise in the following manner. If $n \in K_{j,m}$, then set

$$f(n) = n - A_{j,m},$$

while if $n \in L_{j,m}$, set

$$f(n) = g_2\left(n - C_{j,m}\right).$$

Assume first that j is even. Then,

$$\begin{aligned}
\sum_{n\in I_j} \frac{1}{\delta_f(n)} &= \sum_m \sum_{n\in K_{j,m}} \frac{1}{\delta_f(n)} + \sum_m \sum_{n\in L_{j,m}} \frac{1}{\delta_f(n)} \\
&= \sum_m \left(B_{j,m} - A_{j,m} + O(1)\right) + \sum_m \frac{2}{3}\left(D_{j,m} - C_{j,m} + O(1)\right) \\
&= \sum_m \big(s2^j + O(1)\big) + \sum_m \left(\frac{2}{3}(1-s)2^j + O(1)\right) \\
&= \sum_m \big(y2^j + O(1)\big) \\
&= y\left(2^{2^{j+1}} - 2^{2^j}\right) + O\left(\frac{2^{2^{j+1}}}{2^j}\right). \qquad (21)
\end{aligned}$$

If j is odd, we obtain in a similar fashion

$$\sum_{n\in I_j} \frac{1}{\delta_f(n)} = w\left(2^{2^{j+1}} - 2^{2^j}\right) + O\left(\frac{2^{2^{j+1}}}{2^j}\right). \qquad (22)$$

Let x be a large real number. Let j^* be the largest integer such that $2^{2^{j^*}} < x$ and let m^* be the largest integer such that $D_{j^*,m^*} \le x$. We then have

$$\begin{aligned}
\sum_{n\le x} \frac{1}{\delta_f(n)} &= \sum_{n<2^{2^{j^*-1}}} \frac{1}{\delta_f(n)} + \sum_{n\in I_{j^*-1}} \frac{1}{\delta_f(n)} + \sum_{2^{2^{j^*}}\le n\le x} \frac{1}{\delta_f(n)} \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3, \qquad (23)
\end{aligned}$$

say. On the one hand, we trivially have

$$\Sigma_1 = O\left(2^{2^{j^*-1}}\right), \tag{24}$$

while assuming $j^* - 1$ even, we have, using (21),

$$\Sigma_2 = y \cdot 2^{2^{j^*}} + O\left(\frac{2^{2^{j^*}}}{2^{j^*}}\right). \tag{25}$$

Finally,

$$\Sigma_3 = \sum_{m \le m^*} \sum_{n \in L_{j^*,m}} \frac{1}{\delta_f(n)} + \sum_{m \le m^*} \sum_{n \in K_{j^*,m}} \frac{1}{\delta_f(n)} + O\left(2^{j^*}\right),$$

which yields, in light of (22) (since j^* is odd),

$$\Sigma_3 = w\left(x - 2^{2^{j^*}}\right) + O\left(2^{j^*} + m^*\right). \tag{26}$$

Using (24), (25) and (26) in (23), we get

$$\sum_{n \le x} \frac{1}{\delta_f(n)} = y 2^{2^{j^*}} + w\left(x - 2^{2^{j^*}}\right) + O\left(\frac{x}{\log x}\right),$$

which completes the proof of the theorem. □

5. The Mean Value of the Index of Isolation

While the mean value of the reciprocal of the index of isolation gives information on the local behavior of a function f, the mean value of the index of isolation itself gives information on the very isolated numbers.

Theorem 14. *Let f be a real-valued arithmetic function. Let a and b be two positive integers such that $b - a = N$. Suppose furthermore that for all $m \in [a, b[$, $f(m) \ge f(a)$. Then*

$$\sum_{n \in]a,b[} \delta_f(n) \le \frac{N \log N}{2 \log 2}. \tag{27}$$

Proof. We prove Theorem 14 by using induction on N. The result holds for $N = 1$, because the left-hand side of (27) is 0 (since the interval of summation contains no integers), yielding the inequality $0 \le 0$. It also holds for $N = 2$, because in this case we have $\delta_f(a+1) = 1$, yielding the inequality $1 \le 1$. So, let us assume that (27) is true for all intervals $]a, b[$ with $a - b = N$ for some integer $N > 1$. We shall prove that under the hypothesis that $b-a = N+1$ and that for all $m \in]a, b[$, $f(m) \geq f(a)$ and $f(m) \geq f(b)$, we have

$$\sum_{a+1 \leq n \leq b-1} \delta_f(n) \leq \frac{N}{2} \frac{\log N}{\log 2}. \tag{28}$$

We prove (28) by using induction on N. Clearly the result is true for $N = 1$ and $N = 2$. Assume that it is true for $N-1$ and let us prove that it is true for N. Define n^* as an integer such that $n^* \in]a, b[$ and such that for all $m \in]a, b[$, $f(m) \geq f(n^*)$. We thus have

$$\sum_{a+1 \leq n \leq b-1} \delta_f(n) \leq \min(n^*-a-1, b-n^*-1) + \sum_{a+1 \leq n \leq n^*-1} \delta_f(n) + \sum_{n^*+1 \leq n \leq b-1} \delta_f(n).$$

Using our induction hypothesis, we have

$$\sum_{a+1 \leq n \leq n^*-1} \delta_f(n) \leq \frac{n^* - a}{2} \frac{\log(n^* - a)}{\log 2}$$

and

$$\sum_{n^*+1 \leq n \leq b-1} \delta_f(n) \leq \frac{b - n^*}{2} \frac{\log(b - n^*)}{\log 2}.$$

Without any loss of generality, we can assume that $n^* \in]a, a + (b - a)/2]$, so that

$$\sum_{a+1 \leq n \leq b-1} \delta_f(n) \leq n^* - a - 1 + \frac{n^* - a}{2} \frac{\log(n^* - a)}{\log 2} + \frac{b - n^*}{2} \frac{\log(b - n^*)}{\log 2}. \tag{29}$$

Assuming for now that n^* is a real variable, and taking the derivative of the right-hand side of (29) with respect to n^*, we obtain

$$1 + \frac{\log(n^* - a)}{2 \log 2} - \frac{\log(b - n^*)}{2 \log 2}.$$

Since the second derivative is positive, the right-hand side of (29) reaches its maximum value at the end points, that is, either when $n^* = a + 1$ or $n^* = (b + a)/2$. In the first case, we get

$$\sum_{a+1 \leq n \leq b-1} \delta_f(n) \leq \frac{(N - 1) \log(N - 1)}{2 \log 2} \leq \frac{N \log N}{2 \log 2}.$$

In the second case, that is, when $n^* = (b + a)/2$, we obtain

$$\begin{aligned} \sum_{a+1 \leq n \leq b-1} \delta_f(n) &\leq \frac{N}{2} - 1 + \frac{N \log(N/2)}{2 \log 2} \\ &= \frac{N \log N}{2 \log 2} - 1 \leq \frac{N \log N}{2 \log 2}, \end{aligned}$$

thus proving (28) in all cases.

We are now ready to complete the proof of Theorem 14, that is, to remove the condition $f(m) \geq f(b)$. For this, we use induction.

Let $n_0 \in]a, b[$ be an integer such that for all $m \in]a, b[$, $f(m) \geq f(n_0)$. We can write

$$\sum_{a+1 \leq n \leq b-1} \delta_f(n) \leq \sum_{a+1 \leq n \leq n_0-1} \delta_f(n) + n_0 - a - 1 + \sum_{n_0+1 \leq n \leq b-1} \delta_f(n).$$

Using our induction hypothesis, we obtain

$$\sum_{a+1 \leq n \leq n_0-1} \delta_f(n) \leq \frac{(n_0 - a)\log(n_0 - a)}{2\log 2}$$

and

$$\sum_{n_0+1 \leq n \leq b-1} \delta_f(n) \leq \frac{(b - n_0)\log(b - n_0)}{2\log 2}.$$

From the last three estimates, it follows that

$$\sum_{a+1 \leq n \leq b-1} \delta_f(n) \leq n_0 - a - 1 + \frac{(n_0 - a)\log(n_0 - a)}{2\log 2} + \frac{(b - n_0)\log(b - n_0)}{2\log 2}. \quad (30)$$

Proceeding as we did to estimate the right-hand side of (29), we obtain that the right-hand side of (30) is less than $\dfrac{N \log N}{2\log 2}$, which completes the proof of the theorem. □

Theorem 15. *As $x \to \infty$,*

$$\sum_{n \leq x} \delta_\omega(n) \ll x \log\log x.$$

Proof. For each $x \geq 2$,

$$\begin{aligned} \sum_{n \leq x} \delta_\omega(n) &= \sum_{d} \sum_{\substack{n \leq x \\ \omega(n)=d}} \delta_\omega(n) \\ &= \sum_{d \leq 10 \log\log x} \sum_{\substack{n \leq x \\ \omega(n)=d}} \delta_\omega(n) + \sum_{d > 10 \log\log x} \sum_{\substack{n \leq x \\ \omega(n)=d}} \delta_\omega(n). \end{aligned} \quad (31)$$

Clearly, for any fixed $d \geq 1$,

$$\sum_{\substack{n \leq x \\ \omega(n)=d}} \delta_\omega(n) \leq 2x.$$

Therefore,

$$\sum_{d\le 10\log\log x}\ \sum_{\substack{n\le x\\ \omega(n)=d}} \delta_\omega(n) \le 20x\log\log x. \tag{32}$$

Let x be large and fixed, and consider the set $S := \{n \le x : \omega(n) > 10\log\log x\}$. Write S as the union of disjoint intervals $S = I_1 \cup I_2 \cup \ldots \cup I_k$, and let ℓ_j stand for the length of the interval I_j. We have from Theorem 14 that

$$\sum_{n\in I_j} \delta_\omega(n) \le \frac{\ell_j \log \ell_j}{2\log 2}. \tag{33}$$

On the other hand, one can show that

$$\sum_{j=1}^{k} \ell_j = \#S = \sum_{\substack{n\le x\\ \omega(n)>10\log\log x}} 1 = o\left(\frac{x}{\log x}\right) \qquad (x\to\infty) \tag{34}$$

(see, for instance, relation (18) in De Koninck, Doyon and Luca [3]).

It follows from (33) and (34) that

$$\sum_{n\in S} \delta_\omega(n) \le \sum_{j=1}^{k} \frac{\ell_j\log\ell_j}{2\log 2} \le \frac{\log x}{2\log 2}\cdot o\left(\frac{x}{\log x}\right) = o(x) \qquad (x\to\infty). \tag{35}$$

Substituting (32) and (35) in (31) completes the proof of the theorem. □

Remark. Using a computer, one can observe that, for $x = 10^9$,

$$\frac{1}{x\log\log x}\sum_{n\le x} \delta_\omega(n) \approx 0.60.$$

Theorem 16. *Let the function g_q be defined as in (18). Then*

$$\sum_{1\le n\le N} \delta_{g_q}(n) = \frac{(q-1)N\log N}{q\log q} + O(N),$$

so that the function g_2 is the function for which the sums of the index of isolation is maximal.

Proof. Let n be written in base $q \ge 2$, that is,

$$n := \sum_{j=0}^{k} \alpha_{j,q}(n) q^j.$$

In light of Remark 12, we get, letting k_0 be the largest integer such that $q^{k_0} \leq N$,

$$\begin{aligned}
\sum_{n\leq N} \delta_{g_q}(n) &= \sum_{\substack{n\leq N\\ n\neq q^k}} \delta_{g_q}(n) + \sum_{\substack{n\leq N\\ n=q^k}} \delta_{g_q}(n) \\
&= \sum_{m\leq \log N/\log q} q^m \cdot \#\{n \leq N : \delta_{g_q}(n) = q^m\} + \sum_{q^k\leq N} q^{k-1}(q-1) \\
&= \sum_{m\leq \log N/\log q} q^m \cdot \left(\frac{q-1}{q^{m+1}}N + O(1)\right) + (q-1)\sum_{q^k\leq N} q^{k-1} \\
&= \frac{q-1}{q}N\frac{\log N}{\log q} + O(N) + (q-1)\frac{q^{k_0}-1}{q-1} \\
&= \frac{q-1}{q}N\frac{\log N}{\log q} + O(N),
\end{aligned}$$

thus completing the proof of the theorem. □

6. Computational Data and Open Problems

If $n = n_k$ stands for the smallest positive integer n such that

$$\delta(n) = \delta(n+1) = \ldots = \delta(n+k-1) = 1, \tag{36}$$

then we have the following table:

k	1	2	3	4	5	6	7	8	9	10
n_k	1	1	1	91	91	169	2737	26 536	67 311	535 591

k	11	12	13	14	15
n_k	3 021 151	26 817 437	74 877 777	657 240 658	785 211 337

Some open problems concerning the sequence n_k, $k = 1, 2, 3, \ldots$, are the following:

1. Prove that n_k exists for each integer $k \geq 16$.
2. Estimate the size of n_k as a function of k. Also, is it true that $n_k \leq k!$ for each integer $k \geq 5$?
3. Prove that for any fixed $k \geq 3$, there are infinitely many integers n such that (36) is satisfied. The fact that the matter is settled for $k = 2$ follows immediately from the Balog result stated at the beginning of Section 3.

Interesting questions also arise from the study of the function $\Delta(n)$ first mentioned in Section 2. For instance, let m_k stand for the smallest number m for which

$$\Delta(m) = \Delta(m+1) = \ldots = \Delta(m+k-1). \tag{37}$$

Then

- $m_2 = 14$, with $\Delta(14) = \Delta(15) = 4$;
- $m_3 = 33$, with $\Delta(33) = \Delta(34) = \Delta(35) = 4$;
- $m_4 = 2\,189\,815$, with $\Delta(m_4 + i) = 12$ for $i = 0, 1, 2, 3$;
- $m_5 = 7\,201\,674$, with $\Delta(m_5 + i) = 14$ for $i = 0, 1, 2, 3, 4$;
- if m_6 exists, then $m_6 > 1\,500\,000\,000$.

Specific questions are the following:

1. Prove that m_k exists for each integer $k \geq 6$.
2. Estimate the size of m_k as a function of k.
3. Prove that for any fixed $k \geq 3$, there are infinitely many integers m such that (37) is satisfied ?

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