A COMBINATORIAL PROOF OF GUO'S MULTI-GENERALIZATION OF MUNARINI'S IDENTITY

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Abstract

We give a combinatorial proof of Guo's multi-generalization of Munarini's identity, answering a question of Guo.

1. Introduction

Simons [7] proved a binomial coefficient identity which is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(1+x)^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}x^k. \tag{1}$$

Several different proofs of (1) were given in [1, 5, 8]. Using Cauchy's integral formula as in [5], Munarini [4] obtained the following generalization:

$$\sum_{k=0}^{n} \binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(-1)^{n-k}(x+y)^k y^{n-k} = \sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\beta+k}{k}x^k y^{n-k}, \tag{2}$$

where α, β, x and y are indeterminates. It is clear that (2) reduces to (1) when $\alpha = \beta = n$ and $y = 1$. Shattuck [6] and Chen and Pang [2] provided two interesting combinatorial proofs of (2).

Recently, Guo [3] obtained the following multinomial coefficient generalization of

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(2):

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}(-1)^{|\mathbf{n}|-|\mathbf{k}|}\binom{\beta-\alpha+|\mathbf{n}|}{\mathbf{n}-\mathbf{k}}\binom{\beta+|\mathbf{k}|}{\mathbf{k}}(\mathbf{x}+\mathbf{y})^{\mathbf{k}}\mathbf{y}^{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\binom{\alpha}{\mathbf{n}-\mathbf{k}}\binom{\beta+|\mathbf{k}|}{\mathbf{k}}\mathbf{x}^{\mathbf{k}}\mathbf{y}^{\mathbf{n}-\mathbf{k}}, \tag{3}$$

where $\mathbf{n}=(n_1,\ldots,n_m)\in\mathbb{N}^m$, $|\mathbf{n}|=n_1+\cdots+n_m$, the *multinomial coefficient* $\binom{x}{\mathbf{n}}$ is defined by

$$\binom{x}{\mathbf{n}} = \begin{cases} \dfrac{x(x-1)\cdots(x-|\mathbf{n}|+1)}{n_1!\cdots n_m!}, & \text{if } \mathbf{n}\in\mathbb{N}^m,\\ 0, & \text{otherwise,}\end{cases}$$

and $\mathbf{x}+\mathbf{y}=(x_1+y_1,x_2+y_2,\ldots,x_m+y_m)$, $\mathbf{x}^{\mathbf{a}}=x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$ for $\mathbf{x}=(x_1,x_2,\ldots,x_m)$, $\mathbf{y}=(y_1,y_2,\ldots,y_m)\in\mathbb{C}^m$ and $\mathbf{a}=(a_1,a_2,\ldots,a_m)\in\mathbb{N}^m$.

In this paper we shall give an involutive proof of (3), answering a question of Guo [3]. Our proof is motivated by Shattuck's work [6].

2. The Involutive Proof

Notice that both sides of (3) are polynomials in $\alpha,\beta,x_1,\ldots,x_m$ and $y_1,\ldots,y_m$. We may consider only the case of positive integers with $\beta\geq\alpha$. We first understand the unsigned quantity in the sum of the left-hand side of (3). Let $\Gamma=\{a,b_1,\ldots,b_m\}$ be an alphabet. We construct the weighted words $w=w_1\cdots w_{\beta+|\mathbf{n}|}$ on Γ as follows:

i) Choose $\mathbf{k}=(k_1,\ldots,k_m)\in\mathbb{N}^m$ with $0\leq k_i\leq n_i$ for $i=1,2,\ldots,m$;

ii) Let a subword of $w_1\cdots w_{\beta-\alpha+|\mathbf{n}|}$ be a permutation of the multiset $\{b_1^{n_1-k_1},\ldots,b_m^{n_m-k_m}\}$, with each b_i weighted y_i and also circled;

iii) Let all the other w_i's be a permutation of the multiset $\{a^\beta,b_1^{k_1},\ldots,b_m^{k_m}\}$, with each b_i weighted x_i or y_i and each a weighted 1.

We call such a weighted word w a *configuration*, and define its weight as the product of the weights of all the w_i's. Here is an example for $\beta=4,\alpha=2,\mathbf{n}=(2,2)$ and $\mathbf{k}=(2,1)$ (the configuration has weight $x_1x_2y_1y_2$):

a	b_2	a	b_1	$\textcircled{b_2}$	a	b_1	a	
1	x_2	1	y_1	y_2	1	x_1	1	.

Notice that the circled letters can occur only in the first $\beta - \alpha + |\mathbf{n}|$ positions, but not in the last α positions. It is not hard to see that the sum of the weights of the configurations defined above is equal to $\binom{\beta-\alpha+|\mathbf{n}|}{\mathbf{n}-\mathbf{k}}\binom{\beta+|\mathbf{k}|}{\mathbf{k}}(\mathbf{x}+\mathbf{y})^{\mathbf{k}}\mathbf{y}^{\mathbf{n}-\mathbf{k}}$ for any $\mathbf{k} \in \mathbb{N}^m$.

Let S be the set of all configurations just defined and let $\varphi : S \to S$ be the involution defined as follows. If the configuration $w \in S$ contains at least one letter w_j with weight y_i in the first $\beta-\alpha+|\mathbf{n}|$ positions, then let $\varphi(w)$ be the configuration obtained from w by choosing the first letter w_j with weight y_i and circling it (if it is not circled) or uncirling it (if it is circled). If the configuration $w \in S$ does not contain letters w_j with weight y_i in the first $\beta-\alpha+|\mathbf{n}|$ positions, then let $\varphi(w) = w$. For the above example, we have

$$\begin{array}{cccccccccccccccccc} a & b_2 & a & b_1 & \textcircled{b_2} & a & b_1 & a & \stackrel{\varphi}{\longleftrightarrow} & a & b_2 & a & \textcircled{b_1} & \textcircled{b_2} & a & b_1 & a & \\ 1 & x_2 & 1 & y_1 & y_2 & 1 & x_1 & 1 & & 1 & x_2 & 1 & y_1 & y_2 & 1 & x_1 & 1 & . \end{array}$$

Let $\mathrm{Fix}(\varphi) := \{w|\varphi(w) = w,\, w \in S\}$. For each $w \in \mathrm{Fix}(\varphi)$, notice that every letter w_j with weight y_i is in the right α positions. The total weight of the configurations in $\mathrm{Fix}(\varphi)$ is equal to the right-hand side of (3). This is because if the subwords with elements weighted y_i in $w_{\beta-\alpha+|\mathbf{n}|+1}\cdots w_{\beta+|\mathbf{n}|}$ is a permutation of the multiset $\{b_1^{n_1-k_1}, \ldots, b_m^{n_m-k_m}\}$, then there are $\binom{\beta+|\mathbf{k}|}{\mathbf{k}}$ possible ways to choose the remaining subwords of w, where each b_i is weighted x_i. This proves (3).

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