A REMARK ON THE BOROS-MOLL SEQUENCE

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Abstract

We generalize a result of Prodinger in a recent issue of *Integers* about the oscillatory behavior of a double summation related to the 2-adic valuation of the Boros-Moll sequence.

1. Introduction

In order to evaluate the quartic integral

$$\int_0^\infty \frac{\mathrm{d}x}{(x^4 + 2ax^2 + 1)^{m+1}}$$

Boros and Moll introduced in [7] the sequence

$$d_{l,m} := 2^{-2m} \sum_{1 \le k \le m} 2^k \binom{2M-2k}{m-k} \binom{m+k}{m} \binom{k}{l},$$

defined for integers l, m with $0 \le l \le m$. These numbers were seen to be the quotients of positive integers divided by powers of 2.

Several papers were then devoted to the combinatorial and arithmetic properties of (a variation of) this sequence and of its 2-adic valuation (e.g., [8, 13, 9, 5, 14, 15]). The purpose of this note is to generalize the result given in the recent paper [15], which appeared in *Integers*. In that paper the author studies, for the values $l = 3$ and $l = 5$, the oscillatory behavior of the double sum $\sum_{1 \le n < N} \sum_{1 \le m \le n} f_l(m) = \sum_{1 \le k < n} f_l(k)(n-k)$, where $f_l(m)$ is the 2-adic valuation of a certain subsequence of the sequence $(l!m!2^{m+l} d_{l,m})$. We extend the result to any odd value of $l \ge 3$. We also prove that the oscillatory term that involves a continuous periodic function is closely related to a function studied by Delange in [11] and in particular is nowhere differentiable.

2. Notation and a Basic Property

Definition 1. The sequences $d_{l,m}$ and $A_{l,m}$ are defined for integers l, m satisfying $0 \le l \le m$ by

$$d_{l,m} := 2^{-2m} \sum_{1 \le k \le m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

and

$$A_{l,m} := l!m!2^{m+l} d_{l,m} = l!m!2^{-(m-l)} \sum_{1 \le k \le m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}.$$

Remark 2. As proven in [9], the numbers $A_{l,m}$ are integers.

Definition 3. The sequence $f_l(m)$ is defined for positive integers l, m by

$$f_l(m) := \nu_2(A_{l,l+(m-1).2^{1+\nu_2(l)}})$$

where $\nu_2(k)$ is the 2-adic valuation of the integer k, i.e., the largest exponent j such that 2^j divides k.

Remark 4. This definition is given in [17]. The study of $f_l(m)$ is simpler than, but equivalent to, the study of $\nu_2(A_{l,m})$.

It happens that $\nu_2(A_{l,m})$ and thus $f_l(m)$ have a simple expression in terms of the sequence $j \to s_2(j)$, the sum of the binary digits of the integer j, as proven in a somewhat hidden place.

Theorem 5. (see Corollary 1.3 of [6]) *The* 2*-adic valuation of* $A_{l,m}$*, for* $l > 0$ *is given by*

$$\nu_2(A_{l,m}) = 3l - s_2(m+l) + s_2(m-l).$$

Corollary 6. *We have the relation*

$$f_l(m) = 3l - s_2(l + 2^{\nu_2(l)}(m-1)) + s_2(m-1).$$

In particular, for each odd integer l*,*

$$f_l(m) = 3l + s_2(m-1) - s_2(l+m-1).$$

Proof. It suffices to use Theorem 5, Definition 3 above, and to note that for any integer x we have $s_2(2x) = s_2(x)$. □

3. A Summatory Function

The author of the paper [15] proves an asymptotic expansion of the double summatory function of the sequences $f_3(m)$ and $f_5(m)$, that shows a remarkable oscillatory behavior.

Theorem 7. (Theorems 1 and 2 of [15]) *There exist two periodic continuous functions ϕ and ψ such that*

$$\sum_{1\le k<n} f_3(k)(n-k) = \frac{9n^2}{2} - \frac{3n}{2}\log_2 n - \frac{3n}{2}\log_2 \pi - \frac{7n}{4} + \frac{3n}{2\log 2} + n\phi(\log_2 n) + O(n^{3/4})$$

and

$$\sum_{1\le k<n} f_5(k)(n-k) = \frac{15n^2}{2} - \frac{5n}{2}\log_2 n - \frac{5n}{2}\log_2 \pi - \frac{5n}{4} + \frac{5n}{2\log 2} + n\psi(\log_2 n) + O(n^{3/4}).$$

Remark 8. Actually the expansions given in [15] are not correct. The expansions above are taken from the corrected version [16]. Also note that the Fourier series of ϕ (resp. ψ) is explicitly (resp. implicitly) given in [15].

The author of [15] indicates that the same method, i.e., the general principles described in [12] and applied to the Dirichlet series $\sum \frac{f_l(n)}{n^s}$, would work for any odd $l \ge 3$, but that the Dirichlet series for $l \ge 7$ become more cumbersome. We will see here that, for this Theorem 7, the seminal 1975 paper of Delange [11] suffices, and that it even gives more. (Note that the paper of Delange uses only "elementary" methods.) Let us first recall the theorem of Delange in [11].

Theorem 9. (Delange, [11]) *Let $q \ge 2$ be an integer. Let $s_q(n)$ denote the sum of the base q digits of the integer n. Then, there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$, periodic with period 1 and nowhere differentiable, such that*

$$\sum_{0\le n<m} s_q(n) = \frac{q-1}{2} m\log_q m + mF(\log_q m).$$

Furthermore the Fourier series of F is given by

$$F \sim \sum_{k\in\mathbb{Z}} c_k e^{2ik\pi x}$$

with

$$c_0 = \frac{q-1}{2\log q}(\log 2\pi - 1) - \frac{q+1}{4}$$

and for $k \neq 0$

$$c_k = i\frac{q-1}{2k\pi}\left(1 + \frac{2ik\pi}{\log q}\right)^{-1} \zeta\left(\frac{2ik\pi}{\log q}\right).$$

We will prove the following result, which extends the main result in [15].

Theorem 10. *We have the following asymptotic expansion for $l \geq 3$ odd:*

$$\sum_{1\leq n<N}\sum_{1\leq m\leq n} f_l(m) = \frac{3lN^2}{2} - \frac{lN\log N}{2\log 2} + d_0N + lNF\left(\frac{\log N}{\log 2}\right) + O(\log N),$$

where F is Delange's function. In particular, F is continuous, periodic with period 1, and nowhere differentiable. Its Fourier series is given by

$$G(x) \sim \sum_{k\in\mathbb{Z}\setminus\{0\}} c'_k e^{2ik\pi x}$$

with

$$c'_k = \frac{1}{2ik\pi}\left(1+\frac{2ik\pi}{\log 2}\right)^{-1}\zeta\left(\frac{2ik\pi}{\log 2}\right)$$

and

$$d_0 := \sum_{1\leq m\leq l-1} s_2(m) - \frac{5l}{4} - \frac{l\log\pi}{2\log 2} + \frac{l}{2\log 2}.$$

Proof. We note that the double summation for f_l is given by

$$\sum_{1\leq n<N}\sum_{1\leq m\leq n} f_l(m) = \sum_{1\leq m<N} f_l(m)\sum_{m\leq n<N} 1 = \sum_{1\leq k<n} f_l(k)(n-k).$$

Using Corollary 6 we have

$$\begin{aligned}\sum_{1\leq n<N}\sum_{1\leq m\leq n} f_l(m) &= \sum_{1\leq n<N}\sum_{1\leq m\leq n}(3l+s_2(m-1)-s_2(l+m-1))\\ &= \frac{3lN(N-1)}{2} + \sum_{1\leq n<N} W_n\end{aligned}$$

where

$$W_n := \sum_{1\leq m\leq n} s_2(m-1) - \sum_{1\leq m\leq n} s_2(l+m-1) = \sum_{1\leq m\leq n-1} s_2(m) - \sum_{1\leq m\leq n} s_2(l+m-1).$$

Now, if $l<n$, we have

$$\begin{aligned}W_n &= \begin{cases}\displaystyle\sum_{1\leq m\leq l-1} s_2(m) + \sum_{l\leq m\leq n-1} s_2(m)\\ \displaystyle -\sum_{1\leq m\leq n-l} s_2(l+m-1) - \sum_{n-l+1\leq m\leq n} s_2(l+m-1)\end{cases}\\ &= \sum_{1\leq m\leq l-1} s_2(m) - \sum_{n-l+1\leq m\leq n} s_2(l+m-1)\\ &= \sum_{1\leq m\leq l-1} s_2(m) - \sum_{0\leq k\leq l-1} s_2(n+k).\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{1\le n<N} W_n &= \sum_{l<n<N} W_n + O(1)\\
&= \sum_{l<n<N}\ \sum_{1\le m\le l-1} s_2(m) - \sum_{l<n<N}\ \sum_{0\le k\le l-1} s_2(n+k) + O(1)\\
&= \sum_{1\le n<N}\ \sum_{1\le m\le l-1} s_2(m) - \sum_{0\le k\le l-1}\ \sum_{l<n<N} s_2(n+k) + O(1)\\
&= N \sum_{1\le m\le l-1} s_2(m) - \sum_{0\le k\le l-1}\ \sum_{l<n<N} s_2(n+k) + O(1).
\end{aligned}$$

But

$$\begin{aligned}
\sum_{0\le k\le l-1}\ \sum_{l<n<N} s_2(n+k) &= \sum_{0\le k\le l-1}\ \sum_{1\le n<N} s_2(n+k) + O(1)\\
&= \sum_{0\le k\le l-1}\ \sum_{k+1\le n<N+k} s_2(j) + O(1)\\
&= \sum_{0\le k\le l-1}\ \sum_{1\le n<N+k} s_2(j) + O(1)\\
&= \sum_{0\le k\le l-1} \left(\sum_{1\le j<N} s_2(j) + \sum_{N\le j<N+k} s_2(j) \right) + O(1)\\
&= l \sum_{1\le j<N} s_2(j) + O(\log N)\\
&\quad \text{(since for any } m\ge 1,\ s_2(m)\le 1+\log m/\log 2).
\end{aligned}$$

So, finally,

$$\sum_{n<N} W_n = N \sum_{1\le m\le l-1} s_2(m) - l \sum_{1\le j<N} s_2(j) + O(\log N)$$

and

$$\sum_{1\le n<N}\ \sum_{1\le m\le n} f_l(m) = \frac{3lN(N-1)}{2} + N \sum_{1\le m\le l-1} s_2(m) - l \sum_{1\le j<N} s_2(j) + O(\log N).$$

Using Theorem 9 we get

$$\begin{aligned}
\sum_{1\le n<N}\ \sum_{1\le m\le n} f_l(m) = \frac{3lN(N-1)}{2} &+ N \sum_{1\le m\le l-1} s_2(m)\\
&- \frac{l}{2\log 2} N\log N - lNF\left(\frac{\log N}{\log 2}\right) + O(\log N).
\end{aligned}$$

The result holds by using the Fourier expansion of Delange's function F. □

Remark 11. It is worth noting that, while the term $O(\log N)$ depends on l, the function F (Delange's function) does *not* depend on l.

4. Conclusion

The method used in [15] to study the asymptotics of the double summatory function $\sum_{1\le n<N}\sum_{1\le m\le n} f_l(m) = \sum_{1\le k<n} f_l(k)(n-k)$ is based on the philosophy of [12] and involves the study of the Dirichlet series $\sum \frac{f_l(n)}{n^s}$. It happens that these Dirichlet series can be computed as infinite linear combinations of shifts of the zeta function.

It is worth noting that the series $\sum \frac{f_l(n)}{n^s}$ belong to a class of Dirichlet series that have the following properties: they satisfy infinite functional equations, being equal to infinite linear combinations of their shifts; they can be continued to meromorphic functions on the whole plane; their poles (if any) are located on a finite number of left half-lattices (see [1, Theorem 3 and Remark 4]). Namely the sequence $(f_l(n))_{n\ge 1}$ is 2-regular (see [2, 3, 4] for a definition), which is an immediate consequence of the 2-regularity of the sequence $(s_2(n))_n$ and of the stability properties of 2-regular sequences.

It is also worth noting that the method of [12], although giving asymptotic expansions of summatory functions of fairly general "digit-related sequences", does not give the (non-)differentiability properties of the oscillatory term. We have mentioned the result of Delange [11]. Several other examples can be found in the literature: a list of references and a unified treatment can be found in [10] and in [18].

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