THE ROBIN INEQUALITY FOR 7-FREE INTEGERS

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Abstract

Recall that an integer is t-free if and only if it is not divisible by p^t for some prime p. We give a method to check Robin's inequality, $\sigma(n) < e^{\gamma} n \log\log n$, for t-free integers n and apply it for $t = 6, 7$. We introduce Ψ_t, a generalization of the Dedekind Ψ function defined for any integer $t \geq 2$, by

$$\Psi_t(n) := n \prod_{p|n} \left(1 + 1/p + \cdots + 1/p^{t-1}\right).$$

If n is t-free then the sum of divisor function $\sigma(n)$ is $\leq \Psi_t(n)$. We characterize the champions for $x \mapsto \Psi_t(x)/x$, as primorial numbers. Define the ratio $R_t(n) := \frac{\Psi_t(n)}{n \log\log n}$. We prove that, for all t, there exists an integer $n_1(t)$, such that we have $R_t(N_n) < e^{\gamma}$ for $n \geq n_1$, where $N_n = \prod_{k=1}^{n} p_k$. Further, by combinatorial arguments, this can be extended to $R_t(N) \leq e^{\gamma}$ for all $N \geq N_n$, such that $n \geq n_1(t)$. This yields Robin's inequality for $t = 6$, 7. For t varying slowly with N, we also derive $R_t(N) < e^{\gamma}$.

1. Introduction

The Riemann Hypothesis (RH), which describes the non trivial zeroes of the Riemann ζ function has been deemed the Holy Grail of Mathematics by several authors [1, 7]. There exist many equivalent formulations in the literature [5]. The one of concern here is that of Robin [13], which is given in terms of the sum of divisor function $\sigma(n)$ as

$$\sigma(n) < e^{\gamma} n \log\log n,$$

for $n \geq 5041$. Recall that an integer is t-free if and only if it is not divisible by p^t for some prime p. The above inequality was checked for many infinite families of integers in [3], for instance 5-free integers. In the present work we introduce a method to check the inequality for t-free integers for larger values of t and apply it to $t = 6, 7$. The idea of our method is to introduce the generalized Dedekind Ψ function defined for any integer $t \geq 2$, by

$$\Psi_t(n) := n \prod_{p|n} \left(1 + 1/p + \cdots + 1/p^{t-1}\right).$$

If $t = 2$, this is just the classical Dedekind function which occurs in the theory of modular forms [4], in physics [10, 11], and in analytic number theory [9]. By construction, if n is t-free then the sum of divisors function $\sigma(n)$ is $\leq \Psi_t(n)$. To see this, note that the multiplicative function σ satisfies for any integer a in the range $t > a \geq 2$

$$\sigma(p^a) = 1 + p + \cdots + p^a,$$

when the multiplicative function Ψ_t satisfies

$$\Psi_t(p^a) = p^a + \cdots + 1 + \cdots + 1/p^{t-1-a}.$$

It turns out that the structure of champion numbers for the arithmetic function $x \mapsto \Psi_t(x)/x$ is much easier to understand than that of $x \mapsto \sigma(x)/x$, the super abundant numbers. They are exactly the so-called primorial numbers (product of first consecutive primes). We prove that, in order to maximize the ratio R_t it is enough to consider its value at primorial integers. Once this reduction is made, bounding above unconditionally R_t is easy by using classical lemmas on partial Eulerian products. We conclude the article by some results on t-free integers $N \geq N_n$, valid for t varying slowly with N.

2. Reduction to Primorial Numbers

Define the primorial number N_n of index n as the product of the first n primes

$$N_n = \prod_{k=1}^{n} p_k,$$

so that $N_0 = 1, N_1 = 2, N_2 = 6, \cdots$ and so on. The primorial numbers (OEIS sequence $A002110$ [12]) play the role of superabundant numbers in [13] or primorials in [8]. They are champion numbers (i.e., left to right maxima) of the function $x \mapsto \Psi_t(x)/x$:

$$\frac{\Psi_t(m)}{m} < \frac{\Psi_t(n)}{n} \text{ for any } m < n. \tag{1}$$

We give a rigorous proof of this fact.

Proposition 1. *The primorial numbers and their multiples are exactly the champion numbers of the function* $x \mapsto \Psi_t(x)/x$.

Proof. The proof is by induction on n. The induction hypothesis H_n is that the statement is true up to N_n. The Sloane sequence $A002110$ begins $1, 2, 4, 6\dots$ so that H_2 is true. Assume H_n true. Let $N_n \leq m < N_{n+1}$ denote a generic integer. The prime divisors of m are $\leq p_n$. Therefore $\Psi_t(m)/m \leq \Psi_t(N_n)/N_n$ with equality if and only if m is a multiple of N_n. Further $\Psi_t(N_n)/N_n < \Psi_t(N_{n+1})/N_{n+1}$. The proof of H_{n+1} follows. □

In this section, we reduce the maximization of $R_t(n)$ over all integers n to the maximization over primorials.

Proposition 2. *Let n be an integer ≥ 2. For any m in the range $N_n \leq m < N_{n+1}$ one has $R_t(m) < R_t(N_n)$.*

Proof. As in the preceding proof we have

$$\Psi_t(m)/m \leq \Psi_t(N_n)/N_n,$$

and, since $0 < \log\log N_n \leq \log\log m$, the result follows. □

3. Ψ_t at Primorial Numbers

We begin with an easy application of Mertens formula.

Proposition 3. *For n going to ∞ we have*

$$\lim R_t(N_n) = \frac{e^\gamma}{\zeta(t)}.$$

Proof. Writing $1 + 1/p = (1 - 1/p^2)/(1 - 1/p)$ in the definition of $\Psi(n)$, we can combine the Eulerian product for $\zeta(t)$ with Mertens formula

$$\prod_{p \leq x} (1 - 1/p)^{-1} \sim e^\gamma \log x$$

to obtain

$$\Psi(N_n) \sim \frac{e^\gamma}{\zeta(t)} \log p_n.$$

Now the Prime Number Theorem [6, Th. 6, Th. 420] shows that $x \sim \theta(x)$ for x large, where $\theta(x)$ stands for Chebyshev's first summatory function:

$$\theta(x) = \sum_{p \le x} \log p.$$

This shows that, taking $x = p_n$ we have

$$p_n \sim \theta(p_n) = \log N_n.$$

The result follows. $\square$

This motivates the search for explicit upper bounds on $R_t(N_n)$ of the form $\frac{e^\gamma}{\zeta(t)}(1 + o(1))$. In that direction we have the following bound.

Proposition 4. *For n large enough to have $p_n \ge 20000$, we have*

$$\frac{\Psi_t(N_n)}{N_n} \le \frac{\exp(\gamma + 2/p_n)}{\zeta(t)} \left(\log\log N_n + \frac{1.1253}{\log p_n} \right).$$

We prepare for the proof of the preceding Proposition by some lemmas. First an upper bound on a partial Eulerian product from [14, (3.30) p.70].

Lemma 5. *For $x \ge 2$, we have*

$$\prod_{p \le x} (1 - 1/p)^{-1} \le e^\gamma \left(\log x + \frac{1}{\log x} \right).$$

Next an upper bound on the tail of the Eulerian product for $\zeta(t)$.

Lemma 6. *For $n \ge 2$ we have*

$$\prod_{p > p_n} \left(1 - 1/p^t\right)^{-1} \le \exp\left(2/p_n\right).$$

Proof. Use Lemma 6.4 in [3] with $x = p_n$. Bound $\frac{t}{t-1} x^{1-t}$ above by $2/x$. $\square$

Lemma 7. *For $n \ge 2263$, we have*

$$\log p_n < \log\log N_n + \frac{0.1253}{\log p_n}.$$

Proof. If $n \ge 2263$, then $p_n \ge 20000$. By [14], we know then that

$$\log N_n > p_n \left(1 - \frac{1}{8 p_n} \right).$$

By taking logs we obtain

$$\log\log N_n > \log p_n - \frac{0.1253}{p_n},$$

where we used

$$\log\left(1 - \frac{x}{8}\right) > -0.1253\ x,$$

for x small enough. In particular, it is enough to assume $x < 1/20000$. $\square$

We are now ready for the proof of Proposition 4.

Proof. Write

$$\frac{\Psi_t(N_n)}{N_n} = \prod_{k=1}^{n} \frac{1 - 1/{p_k}^t}{1 - 1/p_k} = \frac{\prod_{p>p_n}(1 - 1/p^t)^{-1}}{\zeta(t)} \prod_{p \le p_n} (1 - 1/p)^{-1}$$

and use both lemmas to derive

$$\frac{\Psi_t(N_n)}{N_n} \le \frac{\exp(\gamma + 2/p_n)}{\zeta(t)} \left(\log p_n + \frac{1}{\log p_n}\right).$$

Now we get rid of the first log in the right hand sise by Lemma 3 and the result follows. $\square$

So, armed with this powerful tool, we derive the following significant corollaries. For convenience let

$$f(n) = 1 + \frac{1.1253}{\log p_n \log\log N_n}.$$

Corollary 8. *Let* $n_0 = 2263$. *Let* $n_1(t)$ *denote the least* $n \ge n_0$ *such that* $e^{2/p_n} f(n) < \zeta(t)$. *For* $n \ge n_1(t)$ *we have* $R_t(N_n) < e^\gamma$.

Proof. Let $n \ge n_0$. We need to check that

$$\exp(2/p_n)\left(1 + \frac{1.1253}{\log p_n \log\log N_n}\right) \le \zeta(t),$$

which, for fixed t, holds for n large enough. Indeed $\zeta(t) > 1$ and the left hand side goes monotonically to 1^+ for n large. $\square$

A numerical illustration of Corollary 1 is found in Table 1.

We can extend this Corollary to all integers $\ge n_0$ by using the reduction of preceding section.

Corollary 9. *For all* $N \ge N_n$ *such that* $n \ge n_1(t)$ *we have* $R_t(N) < e^\gamma$.

Proof. Combine Corollary 8 with Proposition 2 . $\square$

t	$n_1(t)$	$N_{n_1(t)}$
3	10	6.5×10^{9}
4	24	2.4×10^{34}
5	79	4.1×10^{163}
6	509	5.8×10^{1551}
7	10 596	2.5×10^{48337}

Table 1: The numbers in Corollary 8 .

We are now in a position to derive the main result of this note.

Theorem 10. *If N is a 7-free integer, then $\sigma(N) < Ne^{\gamma}\log\log N$.*

Proof. If N is $\geq N_n$ with $n \geq n_1(7)$, then the above upper bound holds for $\Psi_7(N)$ by Corollary 2 , hence for $\sigma(N)$ by the remark in the Introduction. Note that by [13, Proposition 1] it is enough to check Robin's inequality for colossally abundant numbers. If we denote by $R(n)$ the ratio $\sigma(n)/n\log\log n$, the cited result says that if m is an integer between two successive CA numbers $N < N'$, then $R(m) \leq \max(R(N), R(N'))$. Therefore, the results of [2, p.253, left column, line 12] imply that Robin's inequality holds for $5040 < N \leq 10^{10^{10}}$. The result follows then upon observing that all 7-free integers are > 5040. $\square$

The case of 6-free integers follows either in the same way, or by noticing that they are in particular 7-free.

4. Varying t

We begin with an easy lemma.

Lemma 11. *Let t be a real variable. For t large, we have $\zeta(t) = 1 + \frac{1}{2^t} + o(\frac{1}{2^t})$.*

Proof. By definition, for $t > 1$ we may write

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t}$$

so that

$$\zeta(t) \geq 1 + \frac{1}{2^t}.$$

In the other direction, we write

$$\zeta(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + \sum_{n=4}^{\infty} \frac{1}{n^t},$$

and compare the remainder of the series expansion of the ζ function with an integral:

$$\sum_{n=4}^{\infty} \frac{1}{n^t} < \int_3^{\infty} \frac{du}{u^t} = \frac{3}{(t-1)3^t} = O(\frac{1}{3^t}).$$

The result follows. $\square$

We can derive a result when t grows slowly with n.

Theorem 12. *Let S_n be a sequence of integers such that $S_n \geq N_n$ for n large, and such that S_n is t-free with $t = o(\log\log n)$. For n large enough, Robin's inequality holds for S_n.*

Proof. For Corollary 2 to hold we need

$$e^{2/p_n} f(n) < \zeta(t)$$

or, taking logs, the exact bound

$$2/p_n + \log f(n) < \log \zeta(t),$$

so that, up to $o(1)$ terms

$$2/p_n + \frac{1.1253}{\log p_n \log\log N_n} \leq \log \zeta(t).$$

In the left hand side, the dominant term is of order $1/(\log p_n)^2$, since, as in the proof of Proposition 3 , we may write $p_n \sim \log N_n$. Now $p_n \sim n \log n$ by [6, Th. 8], entailing $\log p_n \sim \log n$ and $(\log p_n)^2 \sim (\log n)^2$. In the right hand side, with the hypothesis made on t we have, by Lemma 4 , the estimate $\log \zeta(t) \sim \frac{1}{2^t}$. The result follows after comparing logarithms of both sides. $\square$

5. Conclusion

In this article we have proposed a technique to check Robin's inequality for t-free integers for some values of t. The main idea has been to investigate the complex structure of the divisor function σ though the sequence of Dedekind psi functions ψ_t. The latter are simpler to study for the following reasons

- $\Psi_t(n)$ solely depends on the prime divisors of n and not on their multiplicity
- the champions of Ψ_t are the primorials instead of the colossally abundant numbers
- Ψ_t is easier to bound for n large because of connections with Eulerian products

Further, $\sigma(n) \leq \Psi_t(n)$ for t-free integers n. We checked Robin's inequality for t-free integers for $t = 6, 7$ and $t = o(\log \log n)$. It is an interesting and difficult open problem to apply Theorem 2 to superabundant numbers or colossally abundant numbers for instance. We do not believe it is possible. New ideas are required to prove Robin's inequality in full generality.

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