A RECURRENCE RELATED TO THE BELL NUMBERS

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Abstract

In this paper, we solve a general, four-parameter recurrence by both algebraic and combinatorial methods. The Bell numbers and some closely related sequences are solutions to the recurrence corresponding to particular choices of the parameters.

1. Introduction

It is well-known that there is no general procedure for solving recurrence relations, which is why some might say it is an art; see, for example, [4, 8, 9, 17, 18]. In this paper, we solve a seemingly new recurrence related to the Bell numbers involving a single index and four parameters given by

$$C_n(a,b,c,d) = abC_{n-1}(a,b,c,d) + cC_{n-1}(a+d,b,c,d), \qquad n \geq 1, \tag{1}$$

where $C_0(a,b,c,d) = 1$. Taking specific values of the parameters will yield the Bell numbers and several related sequences. We provide both algebraic and combinatorial arguments. In addition, we supply a new proof of an explicit formula for a well-known q-generalization of the Stirling numbers by combining one of its combinatorial interpretations with the ideas used in our proofs of (1). By modifying our arguments further, we also obtain a combinatorial proof of a related recurrence in [6], which was established there by an algebraic method.

Let $[n] = \{1, 2, \ldots, n\}$ if $n \geq 1$, with $[0] = \varnothing$. By a *partition of* $[n]$, we will mean any collection of non-empty, pairwise disjoint subsets, called *blocks*, whose union is $[n]$. (If $n = 0$, then there is a single empty partition which has no blocks.) The set of all partitions of $[n]$ will be denoted by P_n and has cardinality given by the Bell number B_n (see, e.g., p. 33 of [14]). Let $S_{n,k}$ be the Stirling number of the second kind (see, e.g., p. 103 of [1]) which counts the set of partitions of $[n]$ having exactly k blocks, denoted by $P_{n,k}$. Throughout, we will express $\pi \in P_{n,k}$ in the *standard form* as $\pi = E_1/E_2/\cdots/E_k$, where $\min E_1 < \min E_2 < \cdots < \min E_k$. Recall that the Bell numbers are given by the recurrence

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \qquad n \geq 0, \tag{2}$$

with $B_0 = 1$; see, e.g., p. 373 of [7].

2. Algebraic Solution

In order to solve (1), let $F(x;a) = \sum_{n\geq 0} C_n(a,b,c,d)x^n$ be the generating function for the sequence $\{C_n(a,b,c,d)\}_{n\geq 0}$, where b, c, and d are fixed constants. Multiplying (1) by x^n and summing over $n \geq 0$, we obtain

$$F(x;a) = 1 + abxF(x;a) + cxF(x;a+d),$$

which implies

$$F(x;a) = \frac{1}{1-abx} + \frac{cx}{1-abx}F(x;a+d).$$

Applying this relation an infinite number of times yields

$$F(x;a) = \sum_{i\geq 0} \frac{c^i x^i}{(1-bxy_0)(1-bxy_1)\cdots(1-bxy_i)},$$

where $y_i = a + id$. By partial fractions, we may write

$$\frac{x^i}{(1-bxy_0)(1-bxy_1)\cdots(1-bxy_i)} = \sum_{j=0}^{i} \frac{a_{i,j}}{1-bxy_j},$$

where $a_{i,j} = \frac{(-1)^{i-j}}{(bd)^i i!}\binom{i}{j}$. Hence, the coefficient of x^n in the generating function $F(x;a)$ is given by

$$\begin{aligned} C_n(a,b,c,d) = [x^n](F(x;a)) &= [x^n]\left(\sum_{i\geq 0}\sum_{j=0}^{i} \frac{a_{i,j}c^i}{1-bxy_j}\right) \\ &= \sum_{i\geq 0}\sum_{j=0}^{i} \frac{(-1)^{i-j}c^i}{(bd)^i i!}\binom{i}{j} b^n (a+jd)^n. \end{aligned}$$

Now we convert the ordinary generating function $F(x;a)$ to the exponential $G(x;a)$ and find an explicit formula for it:

$$\begin{aligned} G(x;a) &= \sum_{n\geq 0} C_n(a,b,c,d)\frac{x^n}{n!} = \sum_{i\geq 0}\sum_{j=0}^{i} \frac{(-1)^{i-j}c^i}{(bd)^i i!}\binom{i}{j}\sum_{n\geq 0}\frac{(a+jd)^n b^n}{n!}x^n \\ &= \sum_{i\geq 0}\sum_{j=0}^{i} \frac{(-1)^{i-j}c^i}{(bd)^i i!}\binom{i}{j} e^{(a+jd)bx} = e^{abx}\sum_{i\geq 0}\frac{c^i}{(bd)^i i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}e^{jbdx} \\ &= e^{abx}\sum_{i\geq 0}\frac{c^i}{(bd)^i i!}(e^{bdx}-1)^i, \end{aligned}$$

which implies the following result.

Theorem 1. *The exponential generating function for the sequence $\{C_n(a,b,c,d)\}$ defined in* (1) *is given by*

$$G(x;a) = \sum_{n\geq 0} C_n(a,b,c,d)\frac{x^n}{n!} = e^{abx+\frac{c}{bd}(e^{bdx}-1)}.$$

Recall the Bell polynomials $B_n(y)$ given by $B_n(y) = \sum_{k=0}^{n} S(n,k)y^k$, which reduce to the ordinary Bell numbers when $y=1$ (see, e.g., p. 5 of [5]). They have exponential generating function (see, e.g., 7.54 on p. 351 of [7]) given by

$$\sum_{n\geq 0} B_n(y)\frac{x^n}{n!} = e^{y(e^x-1)}.$$

Writing

$$G(x;a) = e^{abx}e^{\frac{c}{bd}(e^{bdx}-1)},$$

and collecting the coefficient of $\frac{x^n}{n!}$ in the convolution, yields the following explicit formula for the polynomials $C_n(a,b,c,d)$ and hence solution to recurrence (1).

Corollary 2. *If $n \geq 0$, then*

$$C_n(a,b,c,d) = b^n\sum_{j=0}^{n} a^{n-j}d^j\binom{n}{j}B_j(c/bd). \tag{3}$$

Remark 3. In particular, we have $C_n(1,1,1,1) = B_{n+1}$, by (2), and $C_n(0,1,1,1) = B_n$ for all $n \geq 0$.

3. Combinatorial Solution

We will assume that the blocks of a partition are arranged in increasing order according to the size of their minimal (i.e., smallest) elements. We define four statistics on P_{n+1} as follows. Given $\pi \in P_{n+1}$, let $s_1(\pi)$ be the number of elements $r > 1$ in the block of π containing 1, let $s_2(\pi)$ be the number of non-minimal elements of π (which is also $n+1-k$, where k is the number of blocks of π), let $s_3(\pi)$ be the number of minimal elements of π greater than 1 (i.e., $k-1$), and let $s_4(\pi)$ be the number of non-minimal elements of π in all the blocks, excluding the first one. From the definitions, note that $s_2(\pi)+s_3(\pi)=n$ and $s_1(\pi)+s_4(\pi)=n+1-k$ for all $\pi \in P_{n+1}$, where k denotes the number of blocks.

Define the distribution polynomial $D_n(a,b,c,d)$ on P_{n+1} as

$$D_n(a,b,c,d) = \sum_{\pi \in P_{n+1}} a^{s_1(\pi)} b^{s_2(\pi)} c^{s_3(\pi)} d^{s_4(\pi)}, \qquad n \geq 0. \tag{4}$$

The following proposition provides a combinatorial interpretation of $C_n(a,b,c,d)$.

Proposition 4. *If $n \geq 0$, then $C_n(a,b,c,d) = D_n(a,b,c,d)$.*

Proof. Note first that $C_n(1,1,1,1) = D_n(1,1,1,1) = B_{n+1}$. Expanding formula (3), we may write

$$C_n(a,b,c,d) = \sum_{j=0}^{n} a^{n-j} \binom{n}{j} \sum_{i=0}^{j} b^{n-i} c^i d^{j-i} S_{j,i}, \qquad n \geq 0. \tag{5}$$

Note that j corresponds to the number of elements of $\{2,3,\ldots,n+1\}$ not lying in the block with 1 within a partition of $[n+1]$ and i corresponds to the number of blocks in addition to the block containing 1. The result now follows from the definition of $D_n(a,b,c,d)$ and comparison with (5). $\square$

Note that (5) implies $C_n(0,b,c,d) = (bd)^n B_n(c/bd)$, and $C_n(a,b,c,0) = (ab+c)^n$. In particular, we have $C_n(0,1,1,1) = B_n$ and $C_n(1,1,1,0) = 2^n$. These formulas may also be realized using the interpretation for $C_n(a,b,c,d)$ in (4). On the other hand, $C_n(-1,1,1,1)$ is the number of partitions of $[n]$ having no singleton blocks (see sequence A000296 of [13]). To show this using our combinatorial interpretation, we construct a sign-reversing involution on a certain set of partitions. Consider those numbers within a member of P_{n+1} which are in the block with 1 or in a singleton block. If the largest of these numbers is in a singleton, then put it in the block containing 1, and if it is in the block containing 1, then put it in a singleton. Note that this operation defines an involution which changes the parity of s_1. It is not defined on those partitions having the form $\{1\} \cup \pi'$, where π' is a partition of $\{2,3,\ldots,n+1\}$ containing no singleton blocks. Note that all such partitions have zero, hence even, s_1 value.

Choosing other specific values for the parameters in $C_n(a,b,c,d)$ yields additional previously studied sequences. For example, the specific cases $C_n(1,1,2,1)$ and $2^n C_n(1/2,1,1,1)$ correspond to sequences A035009 and A126390, respectively, in [13], which are certain transforms involving the Stirling and Bell numbers. The sequence

$$2^n C_n(1,1,1/2,1/2) = \sum_{j=0}^{n} 2^{n-j}\binom{n}{j} B_j = B_{n+2} - B_{n+1},$$

which occurs as A005493, counts, among other things, the number of blocks in all the members of P_{n+1}, the number of elements of P_{n+2} in which 1 and 2 belong to different blocks, and the number of Boolean sublattices of the Boolean lattice of subsets of $[n]$. See also sequence A005494, which is $3^n C_n(1,1,1/3,1/3)$.

We now give a combinatorial proof showing that $C_n(a,b,c,d)$ given by (5) satisfies recurrence (1) using the interpretation given it in (4).

Combinatorial proof of (1). It is enough to show recurrence (1) in the case when a, b, c and d are positive integers since both sides are polynomials in these variables. Let (S_1, S_2, S_3, S_4) denote disjoint sets of colors having cardinalities a, b, c and d, respectively. Given $\pi \in P_{n+1}$ and $1 \le i \le 4$, we assign, in an independent fashion, each of the elements of π counted by the statistic s_i a color from the set S_i. For example, each element $r > 1$ in the block containing 1 is assigned one of the a colors from S_1. Note that elements of π counted by two of the statistics s_i will receive two colors (this occurs with the elements counted by the statistics s_1 and s_2 and by s_2 and s_4). We will call partitions of $[n]$ whose elements are assigned colors as described *colored partitions*. By Proposition 4, the left side of (1) counts all of the colored partitions of $[n+1]$.

We now argue that the right side of (1) also achieves this. Note that the first term on the right side counts all colored partitions of $[n+1]$ in which the elements 1 and 2 belong to the same block. For one may first form a colored partition of $\{2, 3, \ldots, n+1\}$ and then add the element 1 to the block containing 2, noting that we then must assign the element 2 a color from S_1 (since it now belongs to the block containing 1) as well as a color from S_2 (since it is now a non-minimal element in its block). Thus, there are $abC_n(a,b,c,d)$ possibilities in this case.

To complete the proof, we must show that $cC_n(a+d,b,c,d)$ counts all the colored partitions of $[n+1]$ in which 1 and 2 belong to different blocks. First, we create colored partitions λ of $\{2, 3, \ldots, n+1\}$ using the sets $(S_1 \cup S_4, S_2, S_3, S_4)$. Given any λ, we then take any elements in the block containing 2 and marked with a color from S_1 and form a separate block together with 1. Any elements in the block marked with a color from S_4 remain in the block with 2. (In either case, the original elements in the block of λ with 2 retain the color that they were assigned from S_2.) Furthermore, the element 2 must now be given a color from S_3 since it is a minimal element, but no longer the smallest; hence, there are $cC_n(a+d,b,c,d)$ colored partitions in this case, which completes the proof. $\square$

One can also consider the comparable version of (1) in two indices corresponding to partitions of $[n+1]$ having a fixed number k of blocks. We take $b = c = 1$ in what follows since the s_2 and s_3 statistics are constant on $P_{n+1,k}$ for fixed n and k. Define the distribution polynomial $C_{n,k}(a,d)$ by

$$C_{n,k}(a,d)=\sum_{\pi\in P_{n+1,k}} a^{s_1(\pi)}d^{s_4(\pi)}, \qquad n,k\geq 0.$$

Then a similar combinatorial argument to the one above shows

$$C_{n,k}(a,d)=aC_{n-1,k}(a,d)+C_{n-1,k-1}(a+d,d), \qquad n,k\geq 1, \tag{6}$$

where $C_{0,k}(a,d)=\delta_{k,1}$ for integers $k\geq 0$.

Using either of the methods above, one can then show

$$C_{n,k}(a,d)=\sum_{j=k-1}^{n} a^{n-j}d^{j-k+1}\binom{n}{j}S_{j,k-1}, \qquad n,k\geq 1. \tag{7}$$

From (7), we get $C_{n,k}(0,1)=S_{n,k-1}$ and $C_{n,k}(1,1)=S_{n+1,k}$, the latter by a well-known recurrence for the Stirling numbers.

4. Applications

4.1. New Proof of a q-Stirling Number Formula

We first give a bit of notation that we use below. The letter q denotes an indeterminate, with $0_q:=0$, $n_q:=1+q+\cdots+q^{n-1}$ if $n\geq 1$, $0_q!$:=1, $n_q!:=1_q2_q\cdots n_q$ if $n\geq 1$, and $\binom{n}{k}_q:=\frac{n_q!}{k_q!(n-k)_q!}$ if $n\geq 0$ and $0\leq k\leq n$. The binomial coefficient $\binom{n}{k}$ is equal to zero if k is a negative integer or if $0\leq n<k$.

In this section, we supply a new proof of an explicit formula for a well-known q-generalization of $S_{n,k}$ by combining one of its combinatorial interpretations with the reasoning used in our proofs of (1). Perhaps the ideas could be extended to finding comparable explicit formulas for other q-generalizations of the Stirling and Bell numbers.

Let $\tilde{S}_{n,k}(q)$ denote the sequence of numbers determined by the relation

$$(x_q)^n=\sum_{k=0}^{n}\tilde{S}_{n,k}(q)x_q(x-1)_q\cdots(x-k+1)_q, \qquad n\geq 0,$$

where $x_q:=(q^x-1)/(q-1)$, or, equivalently, by the recurrence

$$\tilde{S}_{n,k}(q)=\tilde{S}_{n-1,k-1}(q)+k_q\tilde{S}_{n-1,k}(q), \qquad n,k\geq 1,$$

with $\tilde{S}_{0,k}(q)=\delta_{k,0}$ for $k\geq 0$. Note that $\tilde{S}_{n,k}(q)=S_{n,k}$ when $q=1$. Starting with the two relations above, Carlitz [2] derived the following beautiful explicit formula for $\tilde{S}_{n,k}(q)$.

Theorem 5. *If $n, k \geq 0$, then*

$$\tilde{S}_{n,k}(q) = \frac{1}{q^{\binom{k}{2}} k_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [(k-j)_q]^n. \tag{8}$$

To do so, he used a sequence of difference operators defined, recursively, by

$$\Delta_{q,1} f(x) = \Delta f(x) = f(x+1) - f(x)$$

and

$$\Delta_{q,k+1} f(x) = \Delta_{q,k} f(x+1) - q^k \Delta_{q,k} f(x),$$

where q is regarded as an arbitrary parameter.

Carlitz [3] later gave $\tilde{S}_{n,k}(q)$ a combinatorial interpretation as the distribution polynomial on $P_{n,k}$ for a statistic, denoted $\tilde{w}$, and defined as $\tilde{w}(\pi) = \sum_{i=1}^{k}(i-1)(|E_i|-1)$, where $\pi = E_1/E_2/\cdots/E_k \in P_{n,k}$ is expressed in standard form. The $\tilde{S}_{n,k}(q)$ were later given other combinatorial interpretations by Wachs and White [15] and by Sagan [11] in terms of placement of rooks on Ferrers boards as well as in terms of certain statistics on set partitions concerning the relative order of the elements and the blocks (see also the related paper by Milne [10]). In [16] and [12], algebraic and combinatorial proofs, respectively, are given which establish the value of $\tilde{S}_{n,k}(q)$ when $q = -1$, along with three other closely related q-analogues.

We now recall a partition statistic considered in [10] and [11]. An *inversion* of $\pi = E_1/E_2/\cdots/E_k$
$\in P_{n,k}$, expressed in standard form, is a pair (a, E_j), where $a \in E_i$, $i < j$, and $a > \min E_j$. Let $inv(\pi)$ denote the number of inversions of π. For example, the partition

$$\pi = 14/27/359/68 = E_1/E_2/E_3/E_4$$

has inversions $(4, E_2)$, $(4, E_3)$, $(7, E_3)$, $(7, E_4)$ and $(9, E_4)$, so $inv(\pi) = 5$. The distribution of the inv statistic on $P_{n,k}$ is $\tilde{S}_{n,k}(q)$ since it is seen to satisfy the same recurrence.

Let $\tilde{B}_n(q)$ given by $\sum_{k=0}^{n} \tilde{S}_{n,k}(q)$ denote the corresponding Bell numbers. We consider a slight refinement of $\tilde{B}_n(q)$. Given $\pi \in P_{n+1}$, let $s_1(\pi)$ once again denote the number of elements $r > 1$ in the block containing 1 and let $\nu(\pi)$ denote the number of blocks of π. Define the polynomials $E_n(q, t, u)$ by

$$E_n(q, t, u) = \sum_{\pi \in P_{n+1}} q^{inv(\pi)} t^{\nu(\pi)} u^{s_1(\pi)}, \qquad n \geq 0.$$

If $n = 2$, for example, then $P_3 = \{123, 12/3, 13/2, 1/23, 1/2/3\}$ and thus $E_2(q, t, u) = tu^2 + t^2(u + qu + 1) + t^3$. Note the $E_n(q, 1, 1) = \tilde{B}_{n+1}(q)$.

The $E_n(q, t, u)$ have a two-term recurrence analogous to (1) above.

Proposition 6. *The polynomials $E_n(q,t,u)$ satisfy*

$$E_n(q,t,u) = uE_{n-1}(q,t,u) + tE_{n-1}(q,t,qu+1), \qquad n \geq 1, \tag{9}$$

with $E_0(q,t,u) = t$.

Proof. We reason as in the combinatorial proof of (1) given in the prior section. The first term on the right side of (9) accounts for the case when the elements 1 and 2 belong to the same block, while the second term accounts for the case when they do not. In the second case, note that the elements $r > 2$ going in the block with 2 which are subsequently moved to the block containing 1 all pick up an extra inversion and also account for the s_1 statistic value; hence, one must replace u with $qu+1$ in the third argument of the second term. The factor t accounts for the block which is added containing 1. □

We now supply an alternate proof of the explicit formula for $\tilde{S}_{n,k}(q)$ in Theorem 5 by solving the recurrence in Proposition 6.

Proof of (8). In order to solve recurrence (9), we define the generating function

$$F(x;q,t,u) = \sum_{n\geq 0} E_n(q,t,u)x^n.$$

Multiplying (9) by x^n and summing over $n \geq 1$ yields

$$F(x;q,t,u) - t = uxF(x;q,t,u) + txF(x;q,t,qu+1),$$

which we rewrite as

$$F(x;q,t,u) = \frac{t}{1-ux} + \frac{tx}{1-ux}F(x;q,t,qu+1).$$

Iterating this relation yields

$$F(x;q,t,u) = \sum_{k\geq 0} t^{k+1}x^k \prod_{j=0}^{k} \frac{1}{1-x[(u-1)q^j + (j+1)_q]}. \tag{10}$$

We seek the coefficient of $t^{k+1}x^n$ in $F(x;q,t,1)$. By partial fractions, we may write

$$\prod_{j=0}^{k} \frac{1}{1-x(j+1)_q} = \sum_{j=0}^{k} \frac{a_{k,j}}{1-x(j+1)_q},$$

where

$$a_{k,j} = \frac{(-1)^{k-j}q^{\binom{k-j}{2}-\binom{k+1}{2}}\binom{k}{j}_q[(j+1)_q]^k}{k_q!}, \qquad 0 \leq j \leq k.$$

Substituting this into (10) implies

$$
\begin{aligned}
[t^{k+1}x^n](F(x;q,t,1)) &= [t^{k+1}x^n]\left(\sum_{k\geq 0} t^{k+1}x^k \sum_{j=0}^{k} \frac{a_{k,j}}{1-x(j+1)_q}\right)\\
&= \sum_{j=0}^{k} a_{k,j}[(j+1)_q]^{n-k}\\
&= \frac{1}{q^{\binom{k+1}{2}}k_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q[(j+1)_q]^n\\
&= \frac{1}{q^{\binom{k+1}{2}}(k+1)_q!}\sum_{j=0}^{k+1}(-1)^{j}q^{\binom{j}{2}}\binom{k+1}{j}_q[(k+1-j)_q]^{n+1}.
\end{aligned}
$$

Replacing $n+1$ with n and $k+1$ with k in the last expression yields (8). □

4.2. Combinatorial Proof of a Recurrence

Let G_n be the sequence defined by the recurrence

$$G_{n+1} = \sum_{j=0}^{n}(-1)^j b^{n-j}\binom{n}{j}G_j, \qquad n \geq 0, \tag{11}$$

with $G_0 = 1$, where b is a fixed constant. Note that replacing -1 by a in (11) and then taking $a = b = 1$ yields the Bell numbers, by (2). In [6], it was shown algebraically that the G_n satisfy the two-term recurrence

$$G_n = bG_{n-1} - G_{n-2}, \qquad n \geq 2, \tag{12}$$

with $G_0 = G_1 = 1$. Here, we provide a combinatorial proof that the sequence G_n defined by (11) satisfies (12) by modifying the argument given in the third section.

To do so, given $\pi = E_1/E_2/\cdots$, expressed in standard form, let us first define $s(\pi)$ by

$$s(\pi) = \sum_{i \text{ even}} |E_i|.$$

If $\pi \in P_{n,k}$, then let the (signed) weight $w(\pi)$ be given by $w(\pi) = (-1)^{s(\pi)}b^{n-k}$. Note that the exponent $n-k$ gives the number of non-minimal elements within a member of $P_{n,k}$. If $n \geq 0$, then let $H_n(b) = \sum_{\pi\in P_n} w(\pi)$, which we will denote by H_n.

Proposition 7. *If $n \geq 0$, then $G_n = H_n$.*

Proof. To show that H_n satisfies

$$H_{n+1} = \sum_{j=0}^{n} (-1)^j b^{n-j} \binom{n}{j} H_j, \qquad n \geq 0,$$

we condition on the number of additional elements, $n-j$, occupying the block with 1 within a member of P_{n+1}. There are $\binom{n}{n-j} = \binom{n}{j}$ choices for these elements, which contributes b^{n-j} towards the weight w of a partition. Then $(-1)^j H_j$ accounts for the contribution towards w of the remaining j elements of $[n+1]$ arranged in the blocks $E_2, E_3, \ldots$. We multiply H_j by $(-1)^j$ since we wish for members lying in blocks E_i of even index to contribute -1 towards the sign and those in odd-indexed E_i to contribute $+1$. Since j is the number of elements in all of the blocks past the first one, multiplying by $(-1)^j$ changes the contribution of each element towards the sign, as desired. □

Using the combinatorial interpretation for G_n in Proposition 7, we now show that it satisfies recurrence (12).

Combinatorial proof of (12). To show that the right side of (12) gives the total w-weight of all the members of P_n where $n \geq 2$, we consider whether or not the elements 1 and 2 in a partition of $[n]$ belong to the same block. Note that the first term, bG_{n-1}, gives the weight of those partitions where 1 and 2 belong to the same block, for adding the 1 to the block containing 2 in a partition of $\{2, 3, \ldots, , n\}$ in standard form contributes only a factor of b and does not change the sign.

To show that the second term, $-G_{n-2}$, gives the total weight of all of the members of P_n in which the blocks containing 1 and 2 are distinct, suppose first that at least one of these blocks is not a singleton. Let $M > 2$ denote the largest element in the first two blocks. If M lies in the first block, then move it to the second, and vice-versa. This defines an involution $x \mapsto x'$ in which x and x' have opposite weight. It is not defined on those partitions of $[n]$ in which the elements 1 and 2 both belong to singleton blocks. The weight of such partitions is seen to be $-G_{n-2}$ since no factor of b is introduced by the blocks $\{1\}$ and $\{2\}$ coming before the blocks $E_3, E_4, \ldots$ and since the sign must be changed as there is a single element lying in the second block. □

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