COMMUNAL PARTITIONS OF INTEGERS

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Abstract

There is a well-known formula due to Andrews that counts the number of incongruent triangles with integer sides and a fixed perimeter. In this note, we consider the analagous question counting the number of k-tuples of nonnegative integers none of which is more than $\frac{1}{k-1}$ of the sum of all the integers. We give an explicit function for the generating function which counts these k-tuples in the case where they are ordered, unordered, or partially ordered. Finally, we discuss the application to algebraic geometry which motivated this question.

1. Introduction and Notation

In this note, we wish to count the number of k-tuples of nonnegative integers which sum to a fixed integer n so that no entry in the k-tuple is "too big". The notion of "too big" that we will use is that no entry can exceed $\frac{n}{k-1}$. When $k = 3$, this problem corresponds to counting triples so that no entry is more than half of the sum of all three entries, which is the same as the question of counting how many incongruent triangles there are with integer-lengthed sides and a fixed perimeter, a problem considered by George Andrews in [1]. Depending on the application, we may want to consider our k-tuples to be unordered sets of numbers (typically referred to as 'partitions') or ordered lists of numbers (typically called 'compositions'). We will also consider intermediate cases, which we call semiordered k-tuples. These are defined precisely below.

Definition 1. A *communal partition* of a number n into k parts is a set $\{a_1, a_2, \ldots, a_k\}$ of integers so that $0 \le a_i \le \frac{n}{k-1}$ for all i and $\sum a_i = n$. Without loss of generality, we typically assume that $a_1 \ge a_2 \ge \ldots \ge a_k$.

Definition 2. A *communal composition* of a number n into k parts is a k-tuple $(a_1, a_2, \ldots, a_k)$ so that $0 \le a_i \le \frac{n}{k-1}$ for all i and $\sum a_i = n$.

Definition 3. A *j-semiordered communal k-tuple* specifies (in order) the j largest elements but gives no ordering on the other elements. In particular, a j-semiordered communal k-tuple is given by $[a_1, \ldots, a_j; a_{j+1}, \ldots, a_k]$ where $\sum a_i = n$, $\frac{n}{k-1} \geq a_1 \geq a_2 \geq \ldots \geq a_j$, and $a_j \geq a_\ell \geq 0$ for all $\ell > j$.

We note that 0-semiordered k-tuples correspond to compositions and k-semiordered k-tuples correspond to partitions. While there are other ways that one could interpolate between compositions and partitions, this approach was particularly interesting to the author due to an application to algebraic geometry which is discussed in Section 5. To give a better feel for semiordered k-tuples, we illustrate with an example:

Example 4. For this example, we will let $n = 20$ and $k = 5$. In particular, we will be looking at the ways to break up 20 as the sum of 5 nonnegative integers, none of which are greater than $\frac{20}{5-1} = 5$.

- The partitions (aka 5-semiordered) are $\{5,5,5,5,0\}$, $\{5,5,5,4,1\}$, $\{5,5,5,3,2\}$, $\{5,5,4,4,2\}$, $\{5,5,4,3,3\}$, $\{5,4,4,4,3\}$, $\{4,4,4,4,4\}$.

- The 4-semiordered k-tuples are $[5,5,5,5;0]$, $[5,5,5,4;1]$, $[5,5,5,3;2]$, $[5,5,4,4;2]$, $[5,5,4,3;3]$, $[5,4,4,4;3]$, $[4,4,4,4;4]$

- The 3-semiordered k-tuples are $[5,5,5;5,0]$, $[5,5,5;0,5]$, $[5,5,5;4,1][5,5,5;1,4]$, $[5,5,5;3,2]$, $[5,5,5;2,3]$, $[5,5,4;4,2]$, $[5,5,4;2,4]$, $[5,5,4;3,3]$, $[5,4,4;4,3]$, $[5,4,4;3,4]$, $[4,4,4;4,4]$.

- The 2-semiordered k-tuples are $[5,5;5,5,0]$, $[5,5;5,0,5]$, $[5,5;0,5,5]$, $[5,5;5,4,1]$, $[5,5;5,1,4]$, $[5,5;4,5,1]$, $[5,5;4,1,5]$, $[5,5;1,4,5]$, $[5,5;1,5,4]$, $[5,5;5,3,2]$, $[5,5;5,2,3]$, $[5,5;2,3,5]$, $[5,5;2,5,3]$, $[5,5;3,5,2]$, $[5,5;3,2,5]$, $[5,5;4,4,2]$, $[5,5;4,2,4]$, $[5,5;2,4,4]$, $[5,5;4,3,3]$, $[5,5;3,4,3]$, $[5,5;3,3,4]$, $[5,4;4,4,3]$, $[5,4;4,3,4]$, $[5,4;3,4,4]$, $[4,4;4,4,4]$.

- The 1-semiordered k-tuples are $[5;5,5,5,0]$, $[5;5,5,0,5]$, $[5;5,0,5,5]$, $[5;0,5,5,5]$, $[5;5,5,4,1]$, $[5;5,5,1,4]$, $[5;5,4,5,1]$, $[5;5,1,5,4]$, $[5;5,4,1,5]$, $[5;5,1,4,5]$, $[5;1,5,5,4]$, $[5;4,5,5,1]$, $[5;1,5,4,5]$, $[5;4,5,1,5]$, $[5;4,1,5,5]$, $[5;1,4,5,5]$, $[5;5,5,3,2]$, $[5;5,5,2,3]$, $[5;5,3,5,2]$, $[5;5,2,5,3]$, $[5;5,3,2,5]$, $[5;5,2,3,5]$, $[5;2,5,5,3]$, $[5;3,5,5,2]$, $[5;2,5,3,5]$, $[5;3,5,2,5]$, $[5;3,2,5,5]$, $[5;2,3,5,5]$, $[5;4,4,5,2]$, $[5;4,4,2,5]$, $[5;4,5,4,2]$, $[5;4,2,4,5]$, $[5;4,5,2,4]$, $[5;4,2,5,4]$, $[5;2,4,4,5]$, $[5;5,4,4,2]$, $[5;2,4,5,4]$, $[5;5,4,2,4]$, $[5;5,2,4,4]$, $[5;2,5,4,4]$, $[5;3,3,4,5]$, $[5;3,3,5,4]$, $[5;3,4,3,5]$, $[5;3,5,3,4]$, $[5;3,4,5,3]$, $[5;3,5,4,3]$, $[5;5,3,3,4]$, $[5;4,3,3,5]$, $[5;5,3,4,3]$, $[5;4,3,5,3]$, $[5;4,5,3,3]$, $[5;5,4,3,3]$, $[5;4,4,4,3]$, $[5;4,4,3,4]$, $[5;4,3,4,4]$, $[5;3,4,4,4]$, $[4;4,4,4,4]$.

- The 0-semiordered (aka compositions) k-tuples are the 5 rearrangements of $(5,5,5,5,0)$, the 20 rearrangements each of of $(5,5,5,4,1), (5,5,5,3,2)$, and $(5,4,4,4,3)$, the 30 rearrangements each of $(5,5,4,4,2)$ and $(5,5,4,3,3)$, and the single element $(4,4,4,4,4)$.

	n=16	17	18	19	20
j=0	70	35	15	5	126
1	35	20	10	4	57
2	16	10	6	3	25
3	8	5	3	2	12
4	5	3	2	1	7
5	5	3	2	1	7

Table 1: The number of j-semiordered 5-tuples whose entries sum to n

More generally, Table 1 gives the number of j-semiordered communal partitions of n into 5 parts for various choices of n and j. In the remainder of this note, we will calculate the generating function for the number of j-semiordered communal partitions into k parts. In particular, Section 2 looks at the case of partitions and Section 3 looks at the case of compositions. In each case, we construct a bijection between the set of k-tuples whose entries sum to n and a more straightforward set of integers that will be easy to count using generating functions. In particular, we will make use of the following standard theorem in the study of generating functions (see [10] or [2]).

Theorem 5. *Fix positive integers $r_1, \ldots, r_k$. Let $f(n)$ be the number of ways to decompose n as a sum $n = x_1 + x_2 + \cdots + x_k$, where $r_i | x_i$ for each i. Then the generating function associated with $f(n)$ is given by*

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n = \frac{1}{(1 - z^{r_1})(1 - z^{r_2}) \ldots (1 - z^{r_k})}.$$

We go on in Section 4 to consider the case of j-semiordered k-tuples, and in particular we prove a recursive relationship between their generating functions which allows us to give explicit formulas. As mentioned above, the author became interested in this problem due to an application involving counting the irreducible components of certain Hurwitz spaces, and we discuss this application in Section 5.

We point out that there are other approaches to counting communal k-tuples than the one presented in this note. In particular, the results of Section 3 could be derived from results appearing in [4]. Additionally, in [8] related problems are approached using MacMahon's Partition Analysis and the Omega Transform developed in [3]. The author would like to thank Andrew Maturo and Rachel Pries for useful conversations.

2. Partitions

In this section, we wish to count the number of ways that a given integer n can be expressed as $n = \sum_{i=1}^{k} a_i$ so that $\frac{n}{k-1} \geq a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$. As an abuse of notation, we will write such a partition as $(a_1, \ldots, a_k)$ and we will talk about adding two k-tuples by adding them componentwise. For $1 \leq j \leq k-1$, we define $\mathbf{x}_j$ be the k-tuple which has $k - j$ as its first j coordinates and $k - j - 1$ in the remaining coordinates. In particular, $\mathbf{x}_1 = (k-1, k-2, \ldots, k-2, k-2), \ldots, \mathbf{x}_{k-1} = (1, 1, \ldots, 1, 1, 0)$. Additionally, we define $\mathbf{y} = (1, 1, \ldots, 1, 1)$. In order to compute the generating function, we will make use of the following observation:

Lemma 6. *Any communal unordered partition of n into k parts can be decomposed in a unique way as $b_1\mathbf{x}_1 + \ldots b_{k-1}\mathbf{x}_{k-1} + c\mathbf{y}$ where $b_i, c \in \mathbb{N}$.*

Proof. Given a communal partition $(a_1, \ldots, a_k)$ one can decompose it in a unique was as a sum of the $\mathbf{x}_i$ and $\mathbf{y}$ by noting that the only one of these partitions in which the first two coordinates differ is $\mathbf{x}_1$ and therefore we must set $b_1 = a_1 - a_2$. Similarly, we must set $b_j = a_j - a_{j+1}$ for all $1 \leq j \leq k-1$. Note that by assumption $a_j \geq a_{j+1}$ so the $b_i \in \mathbb{N}$. Finally, we let $c = \sum a_i - (k-1)a_1$. Our original partition was communal, so $a_1 \leq (\sum a_i)/(k-1)$ and therefore $c \in \mathbb{N}$. One can check that this choice of b_i and c gives $(a_1, \ldots, a_k) = \sum b_i\mathbf{x}_i + c\mathbf{y}$ as desired. □

Additionally, we note that if $\mathfrak{p}$ is a communal k-tuple whose elements sum to n then $\mathfrak{p} + \mathbf{x}_j$ and $\mathfrak{p} + \mathbf{y}$ will be also be communal. In particular, the entries of $\mathfrak{p} + \mathbf{x}_j$ will still be nonnegative integers in descending order and they will sum to $n' = n + j(k-j) + (k-j-1)(k-j) = n + (k-j)(k-1)$. Moreover, the largest entry will be $a_1 + k - j \leq n/(k-1) + (k-j) = n'/(k-1)$. The case of $\mathfrak{p} + \mathbf{y}$ is similar. In particular, any nonnegative linear combination of the $\mathbf{x}_i$ and $\mathbf{y}$ will be communal, which implies that there is actually a bijection between communal partitions and nonnegative integral linear combinations of these vectors.

Theorem 7. *The number of communal partitions whose entries sum to n can be described by the generating function*

$$P_k(z) = \frac{1}{(1 - z^k)(1 - z^{k-1})(1 - z^{2(k-1)}) \ldots (1 - z^{(k-1)^2})}$$

Proof. By the above, the number of communal partitions whose entries sum to n is the same as the number of k-tuples $(b_1, \ldots, b_{k-1}, c)$ of nonnegative integers with $(k-1)^2 b_1 + (k-2)(k-1)b_2 + \ldots + (k-1)b_{k-1} + kc = n$. In other words, the number of ways to write n as the sum of a multiple of $(k-1)^2$, a multiple of $(k-1)(k-2)$, and so forth down to a multiple of $(k-1)$ and a multiple of k. The result follows from Theorem 5. □

As an example of the above formula, we note that when $k = 5$ then

$$
\begin{aligned}
P_5(z) &= \frac{1}{(1 - z^5)(1 - z^4)(1 - z^8)(1 - z^{12})(1 - z^{16})} \\
&= 1 + z^4 + z^5 + 2z^8 + z^9 + z^{10} + 3z^{12} + 2z^{13} + z^{14} + z^{15} + 5z^{16} + 3z^{17} \\
&\quad + 2z^{18} + z^{19} + 7z^{20} + \cdots.
\end{aligned}
$$

In particular, note that these coefficients agree with the values in Table 1.

3. Compositions

In this section, we will be counting the number of ways that an integer n can be expressed as $n = a_1 + \ldots + a_k$ where each a_i is an integer between 0 and $n/(k-1)$. As in the previous section, our argument will rely on showing that such an ordered k-tuple $(a_1, \ldots, a_k)$ can be broken down in a unique way as the sum of k-tuples lying in a specific set. Specifically, for the duration of this section we set $\mathbf{x}_i$ to be the k-tuple which has a 0 in the ith coordinate and a 1 in each of the other coordinates and $\mathbf{y}$ to be the k-tuple with a 1 in each coordinate.

Lemma 8. *Let $\mathfrak{p}$ be a communal composition whose entries sum to n. Then $\mathfrak{p} + \mathbf{x}_i$ is a communal composition whose entries sum to $n + k - 1$.*

Proof. The fact that the entries of $\mathfrak{p} + \mathbf{x}_i$ sum to $n + k - 1$ is trivial. Moreover, if all entries of $\mathfrak{p}$ are at most $\frac{n}{k-1}$ then the entries of $\mathfrak{p} + \mathbf{x}_i$ are at most $\frac{n}{k-1} + 1 = \frac{n+k-1}{k-1}$ and therefore the new composition is also communal. $\qquad \square$

After noting that the k-tuple $\epsilon\mathbf{y} = (\epsilon, \epsilon, \ldots, \epsilon)$ is communal for any number ϵ, the first half of the following theorem is an immediate consequence of the Lemma:

Theorem 9. *Let ϵ be an integer with $0 \leq \epsilon \leq k - 2$. Then the k-tuple $\epsilon\mathbf{y} + \sum b_i\mathbf{x}_i$ is communal for every choice of nonnegative integers $b_1, \ldots, b_k$. Moreover, any communal k-tuple can be expressed uniquely in this form.*

Proof. Let $\mathfrak{p} = (a_1, \ldots, a_k)$ be a communal composition whose entries sum to n. It suffices to show that there exists a unique $0 \leq \epsilon \leq k - 2$ and $b_i \in \mathbb{N}$ so that $\mathfrak{p} = \epsilon\mathbf{y} + b_1\mathbf{x}_1 + \ldots b_k\mathbf{x}_k$. We note that for any nonnegative integers b_i, the sum of the entries of $\epsilon\mathbf{y} + \sum b_i\mathbf{x}_i$ will be congruent to $\epsilon \mod k - 1$.

We begin by considering the case that $k - 1$ divides n. In particular, if $n = (k-1)n'$ then it follows from above that $\epsilon = 0$ and that $\sum b_i = n'$; moreover, the entry in the ith component will be equal to $n' - b_i$, so we need to set $b_i = n' - a_i$, which will be a nonnegative integer, as $\mathfrak{p}$ is communal. It is straightforward to show that this choice of b_i gives $\mathfrak{p} = \sum b_i\mathbf{x}_i$. Note that an alternative proof of uniqueness

in this case is to show that the matrix $\mathbf{1} - I$ is nonsingular, where $\mathbf{1}$ is the $k \times k$ matrix all of whose entries are 1 and I is the $k \times k$ identity matrix.

If we wish to construct a k-tuple whose entries sum to n where $(k-1) \nmid n$ then the $\mathbf{x}_i$ do not suffice to generate $\mathfrak{p}$. To decompose $\mathfrak{p}$ we set ϵ to be the least residue of $n \bmod k - 1$ and consider the k-tuple $\mathfrak{p} - \epsilon \mathbf{y}$. The sum of the entries of this k-tuple is a multiple of n and therefore by the argument in the preceding paragraph is representable in a unique way as the sum of the $\mathbf{x}_i$. The theorem follows. $\square$

Theorem 10. *The number of communal compositions whose entries sum to n can be described by the generating function*

$$C_k(z) = \frac{1 - z^{k(k-1)}}{(1 - z^k)(1 - z^{k-1})^k}.$$

Proof. It follows from the above theorem that the communal compositions whose entries sum to n are in a bijective correspondence with k-tuples $(b_1, \ldots, b_k)$ of natural numbers so that $(k-1) \sum b_i = n - \epsilon k$ for some $0 \le \epsilon \le k - 2$. It follows from Theorem 5 that the corresponding generating function is given by $C_k(z) = \frac{1 + z^k + z^{2k} + \ldots z^{(k-2)k}}{(1 - z^{k-1})^k} = \frac{1 - z^{k(k-1)}}{(1 - z^k)(1 - z^{k-1})^k}.$ $\square$

As an example of the above formula, we compute that when $k = 5$ then

$$\begin{aligned}
C_5(z) &= \frac{1 - z^{20}}{(1 - z^5)(1 - z^4)^5} \\
&= 1 + 5z^4 + z^5 + 15z^8 + 5z^9 + z^{10} + 35z^{12} + 15z^{13} + 5z^{14} + z^{15} + 70z^{16} \\
&\quad + 35z^{17} + 15z^{18} + 5z^{19} + 126z^{20} + \ldots
\end{aligned}$$

We note that these coefficients again agree with the values in the Table 1.

4. Semiordered

In this section, we wish to compute the generating function for the number of communal j-semiordered k-tuples whose entries sum to n. While it is possible to write down an explicit basis for the set of j-semiordered k-tuples in the manner of the earlier sections, we follow a different approach. In particular, we wish to compare the number of j-semiordered k-tuples with the number of $(j-1)$-semiordered k-tuples. Let $\mathfrak{p} = [a_1, \ldots, a_{j-1}; a_j, \ldots, a_k]$ be a $(j-1)$-semiordered k-tuple. Recall that this means that $\frac{\sum a_i}{k-1} \ge a_1 \ge a_2 \ldots \ge a_{j-1}$ and $a_\ell \ge a_{j-1}$ for all $j \le \ell \le k$. For each $\mathfrak{p}$, define t to be the smallest integer so that $t \ge j$ and $a_t \ge a_i$ for all $i \ge j$. In particular, a_t is the first occurrence of the largest value in the unordered part of $\mathfrak{p}$. Let $\tau = t - j$ and define $\phi(\mathfrak{p})$ as follows:

$$\phi(\mathfrak{p}) = [a_1 - \tau, \ldots, a_{j-1} - \tau, a_t - \tau; a_j - \tau + 1, \ldots, a_{t-1} - \tau + 1, a_{t+1} - \tau, \ldots, a_k - \tau].$$

One can easily check the following facts:

- If $1 \leq i \leq j - 2$ then $a_i - \tau \geq a_{i+1} - \tau$.

- $a_{j-1} - \tau \geq a_t - \tau$.

- For all $j \leq i \leq t - 1$, $a_i < a_t$. It follows that $a_t - \tau \geq a_i - \tau + 1$.

- If $t + 1 \leq i \leq k$ then $a_t - \tau \geq a_i - \tau$.

- If the sum of the entries of $\mathfrak{p}$ is n then the entries of $\phi(\mathfrak{p})$ sum to $n' = n - (k-1)\tau$.

- $a_1 \leq \frac{n}{k-1}$ implies $a_1 - \tau \leq \frac{n'}{k-1}$.

Note that if $j \leq \ell \leq t - 1$ then $a_\ell \leq a_1 - 1$; otherwise $a_\ell \leq a_1$. One can show that if $a_\ell < \tau$ for any $\ell \geq t$ then $\sum a_i \leq a_1(j + k - t) + (a_1 - 1)(t - j) + a_\ell < a_1(k-1)$, contradicting the assumption that $\mathfrak{p}$ is communal. Similarly, $a_\ell \geq \tau - 1$ for $j \leq \ell \leq t-1$. This shows that all entries of $\phi(\mathfrak{p})$ are nonnegative, and concludes the proof that $\phi(\mathfrak{p})$ is a communal j-semiordered k-tuple. Moreover, this process is invertible. In particular, given a j-semiordered k-tuple $\mathfrak{q} = [b_1, \ldots, b_j; b_{j+1}, \ldots, b_k]$ whose entries sum to $n - (k-1)s$ for some $0 \leq s \leq k - j$, we define the following:

$$\psi(\mathfrak{q}) = [b_1 + s, \ldots, b_{j-1} + s; b_{j+1} + s - 1, \ldots, b_{j+s} + s - 1, b_j + s, b_{j+s+1} + s, \ldots, b_k + s]$$

One can easily check that $\psi(\mathfrak{q})$ is a $(j-1)$-semiordered k-tuple whose entries sum to $(n - (k-1)s) + ks - s = n$. Moreover, $\psi(\phi(\mathfrak{p})) = \mathfrak{p}$ and $\phi(\psi(\mathfrak{q})) = \mathfrak{q}$.

We have therefore shown that the number of $(j-1)$-semiordered k-tuples whose entries sum to n is equal to the number of j-semiordered k-tuples whose entries sum to $n - s(k-1)$ for some $0 \leq s \leq k - j$. This immediately proves the following.

Lemma 11. *Let $SO_k^{(j)}(z)$ be the generating function describing the number of j-semiordered k-tuples. (ie, the coefficient of z^n gives the number of such k-tuples whose entries sum to n). Then we have the following relationship:*

$$SO_k^{(j-1)}(z) = (1 + z^{k-1} + \ldots z^{(k-1)(k-j)})SO_k^{(j)}(z) = \frac{1 - z^{(k-1)(k-j+1)}}{1 - z^{k-1}} SO_k^{(j)}(z).$$

Using the fact that $SO_k^{(k)}(z) = UO_k(z)$ and the results of Section 2, we now obtain the following formula.

Theorem 12. *For $0 \leq j \leq k$ we have the following description of the generating function describing j-semiordered k-tuples.*

$$SO_k^{(j)}(z) = \frac{1 - z^{k(k-1)}}{(1 - z^k)(1 - z^{k-1})^{k-j} \prod_{i=k-j+1}^{k}(1 - z^{i(k-1)})}$$

Note, in particular, that we get the same formula for $SO_k^{(0)}(z)$ as in Corollary 10.

As a final example, we again compute the following generating functions in the case $k = 5$ and note that the relevant coefficients agree with the values in Table 1:

$$
\begin{aligned}
SO_5^{(4)}(z) &= \frac{1}{(1-z^5)(1-z^4)(1-z^8)(1-z^{12})(1-z^{16})} \\
&= 1 + z^4 + z^5 + 2z^8 + z^9 + z^{10} + 3z^{12} + 2z^{13} + z^{14} + z^{15} + 5z^{16} + 3z^{17} \\
&\quad + 2z^{18} + z^{19} + 7z^{20} + \cdots ;
\end{aligned}
$$

$$
\begin{aligned}
SO_5^{(3)}(z) &= \frac{1}{(1-z^5)(1-z^4)^2(1-z^{12})(1-z^{16})} \\
&= 1 + 2z^4 + z^5 + 3z^8 + 2z^9 + z^{10} + 5z^{12} + 3z^{13} + 2z^{14} + z^{15} + 8z^{16} \\
&\quad + 5z^{17} + 3z^{18} + 2z^{19} + 12z^{20} + \cdots ;
\end{aligned}
$$

$$
\begin{aligned}
SO_5^{(2)}(z) &= \frac{1}{(1-z^5)(1-z^4)^3(1-z^{16})} \\
&= 1 + 3z^4 + z^5 + 6z^8 + 3z^9 + z^{10} + 10z^{12} + 6z^{13} + 3z^{14} + z^{15} \\
&\quad + 16z^{16} + 10z^{17} + 6z^{18} + 3z^{19} + 25z^{20} + \cdots ;
\end{aligned}
$$

$$
\begin{aligned}
SO_5^{(1)}(z) &= \frac{1}{(1-z^5)(1-z^4)^4} \\
&= 1 + 4z^4 + z^5 + 10z^8 + 4z^9 + z^{10} + 20z^{12} + 10z^{13} + 4z^{14} + z^{15} \\
&\quad + 35z^{16} + 20z^{17} + 10z^{18} + 4z^{19} + 57z^{20} + \cdots .
\end{aligned}
$$

5. Application to Hurwitz Spaces

Our interest in counting communal partitions came about from the study of the moduli space of covers of the projective line with a fixed Galois group, often referred to in the literature as Hurwitz Spaces. In particular, in [7] the author and Rachel Pries look at the moduli space of $(\mathbb{Z}/2\mathbb{Z})^2$-covers of the projective line. That work can be generalized to look at curves X of genus g_X admitting an action by a group G isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ so that the quotient X/G is isomorphic to $\mathbb{P}^1$. To see this, we first recall a few ideas from group theory. In particular, note that the elements of the group $G = (\mathbb{Z}/p\mathbb{Z})^2$ can be written as ordered pairs (a, b) with $a, b \in \mathbb{Z}/p\mathbb{Z}$. Moreover, we note that this group has $p + 1$ subgroups of index p, which are each generated by a single element in the group. More specifically, this set of subgroups is given by $\{\langle (1,0) \rangle, \langle (0,1) \rangle, \langle (1,1) \rangle, \ldots, \langle (p-1,1) \rangle\}$. It is easy to see that the automorphism group of G acts doubly transitively on this set of subgroups, as one can send the generator $(1,0)$ to any nontrivial element x in the group and then can send the generator $(0,1)$ to any element which is not a multiple of x. However, once one has made these two choices the automorphism is uniquely determined, so $Aut(G)$ is not triply transitive on the set of subgroups.

Let X be a curve which has an action by G so that the quotient X/G has genus zero. Consider the actions on X by the $p+1$ subgroups of G of index p. In particular, for $1 \leq i \leq p+1$ we can consider the curve X_i which is the quotients of X by one such subgroup. It follows from the Riemann-Hurwitz formula that each quotient will have genus $g_i \leq \frac{g_X+p-1}{p}$. Moreover, it follows from work of Kani and Rosen in [9] that the sum of the g_i will be g_X. In particular, setting $k = p+1$, for any such curve we get a k-tuple whose entries sum to g_X each of whom is at most $\frac{g_X+p-1}{k-1}$. Moreover, as we saw above the action of $Aut(G)$ is doubly transitive but not triply transitive on the elements of a given k-tuple, so without loss of generality we can specify that $g_1 \geq g_2 \geq g_i$ for all $i > 2$ but we can not induce a specific ordering on the remaining $p-1$ entries. In particular, for any such curve we can obtain a 2-semiordered k-tuple which is either communal or has entries $\frac{g_X}{p} < g_i \leq \frac{g_X+p-1}{p}$. This is the analogue of the $\mathcal{K}$-type discussed in [5], and we will use that terminology here as well. As long as the characteristic of the base field is different from p, one can use the methods of [6] to show that, given two curves X_1 and X_2 of the same $\mathcal{K}$-type, one can deform the branch points of the covering $X_1 \to \mathbb{P}^1$ to obtain the cover $X_2 \to \mathbb{P}^1$. In particular, one sees that these $\mathcal{K}$-types correspond to the irreducible components of the moduli space of G-covers of $\mathbb{P}^1$. We note that this condition is sufficient but not necessary, as curves may admit two different G actions leading to different $\mathcal{K}$-types. In particular, there may be intersections of the irreducible components of the Hurwitz Space, leading to a smaller number of connected components. See [7] for more details when $p = 2$.

Continuing our analysis, we note that every branch point of the $(\mathbb{Z}/p\mathbb{Z})^2$-cover $X \to \mathbb{P}^1$ will have cyclic inertia group by Abhayankar's Lemma, and in particular has inertia group $\mathbb{Z}/p\mathbb{Z}$. In other words, over a generic point of $\mathbb{P}^1$ there will lie p^2 points of X, but over branch points there will lie exactly p points. One can then use the Riemann-Hurwitz formula to compute that the genus of X is given by $g_X = 1 - p^2 + p'B$, where $p' = \frac{p-1}{2}$ and B is the number of branch points of this cover. It further follows that for any branch point $b \in \mathbb{P}^1$ there will be exactly one quotient cover X_i so that X_i has p points lying over b – more than one such quotient would give p^2 points of X lying over b, while no such covers would give a unique point, contradicting Abhayankar's Lemma. In particular, each branch point of this cover is also a branch point of exactly p of the quotient covers $X_i \to \mathbb{P}^1$.

Letting B_i be the number of branch points of the cover $X_i \to \mathbb{P}^1$ we then obtain a communal 2-semiordered $(p+1)$-tuple $[B_1, B_2; B_3, \ldots B_{p+1}]$ whose entries sum to pB. Moreover, each such $(p+1)$-tuple can be obtained with an appropriate choice of overlapping branch points. It follows from arguments similar to those in Sections 2 and 3 that $[B_1, B_2; B_3, \ldots, B_{p+1}]$ can be generated as a nonnegative linear combination of the vectors $\mathbf{x}_i$ which contain a 0 in the ith component and a 1 in all other components for $3 \leq i \leq p+1$ (corresponding to a point which is branched in each cover except X_i), the vector $\mathbf{y} = [p, p; p, \ldots, p]$ (corresponding

to $p+1$ points, each branched in all but one of the X_i for a different i) and $\mathbf{z} = [p, p-1; p-1, \ldots, p-1]$ (corresponding to p points, each branched in all but one of the X_i for a different $i \neq 1$). Setting $g_i = p'(B_i - 2)$, one can show that the 2-semiordered (but not quite communal) $(p+1)$-tuple $[g_1, g_2; g_3, \ldots g_{p+1}]$ can be obtained by beginning with one of the 2-semiordered $(p+1)$-tuples in the set $\mathcal{S}$ described below and then adding a nonnegative linear combination of $p'\mathbf{x}_i, p'\mathbf{y}, p'\mathbf{z}$ as defined above.

$$\mathcal{S} = \begin{cases} [(p-2)p', (p-2)p'; (p-2)p', \ldots, (p-2)p'] \\ [(p-2)p', (p-3)p'; (p-3)p', \ldots, (p-3)p'] \\ [a_1, a_2; a_3, \ldots, a_{p+1}], a_i = 0 \text{ for exactly three } i \geq 3, a_i = p' \text{ otherwise} \\ [a_1, a_2; a_3, \ldots, a_{p+1}], a_i = (p-1)p' \text{ for exactly two } i \geq 3, a_i = pp' \text{ otherwise} \end{cases}$$

In particular, we are left with the following result.

Theorem 13. *Given an odd prime p and a genus g, the number of $\mathcal{K}$-types of $(\mathbb{Z}/p\mathbb{Z})^2$-covers of the projective line is given by the coefficient of z^g in the generating function*

$$F(z) = \frac{z^{(p^2-2p-2)p'} + z^{(p^2-p-2)p'} + \binom{p-1}{3} z^{(p-2)p'} + \binom{p-1}{2} z^{(p^2+p-2)p'}}{(1 - z^p p')^{p-1}(1 - z^{pp'(p+1)})(1 - z^{p^2 p'})}.$$

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