FIRST REMARK ON A $\zeta$-ANALOGUE OF THE STIRLING NUMBERS

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Received: 1/12/10, Revised: 10/9/10, Accepted: 12/9/10, Published: 1/31/11

Abstract

The so-called $\zeta$-analogues of the Stirling numbers of the first and second kind are considered. These numbers cover ordinary binomial and Gaussian coefficients, $p,q$-Stirling numbers and other combinatorial numbers studied with the help of object selection, Ferrers diagrams and rook theory.

Our generalization includes these and now also the $p,q$-binomial coefficients. This special subfamily of $F$-nomial coefficients encompasses among others, Fibonomial ones. The recurrence relations with generating functions of the $\zeta$-analogues are delivered here. A few examples of $\zeta$-analogues are presented.

1. Introduction

Let $w = \{w_i\}_{i \geq 1}$ be a vector of complex numbers $w_i$. The generalized Stirling numbers of the first kind $C^\zeta_k(w)$ and the second kind $S^\zeta_k(w)$ are defined as follows:

$$C^\zeta_k(w) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k},$$

$$S^\zeta_k(w) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k}.$$  \hspace{1cm} (1)

If the elements of the weight vector $w$ are positive integers then the coefficients are interpreted as a selection of $k$ objects from $k$ of $n$ boxes without and with box repetition allowed, respectively. In this case the number of distinct objects in the $s$-th box is designated by the $s$-th element of the weight vector $w$.

One shows that the numbers $C^\zeta_k(w)$ and $S^\zeta_k(w)$ cover among others, binomial coefficients, Gaussian coefficients and the Stirling numbers of the first and second kind, see for example Konvalina [6, 7]. Indeed, if we fix $w_i = 1$, we obtain ordinary
binomial coefficients:
\[ C_k^n(1) = \binom{n}{k}, \quad S_k^n(1) = \binom{n + k - 1}{k}. \]
Setting \( w_i = q^{i-1} \) gives us Gaussian coefficients:
\[ C_k^n(q) = q^{\binom{k}{2}} \binom{n}{k}, \quad S_k^n(q) = \binom{n + k - 1}{k}_q. \]
In this note, the ordinary Stirling numbers of the first kind are defined in the following way
\[ (1 - x)(1 - 2x) \cdots (1 - nx) = \sum_{k=0}^{n} (-1)^k \binom{n + 1}{n + 1 - k} x^k, \]
and the second kind
\[ \frac{1}{(1 - x)(1 - 2x) \cdots (1 - nx)} = \sum_{k=0}^{n} \frac{n + k}{n} x^k. \]
Letting \( i = (1, 2, 3, \ldots) \), i.e., \( w_i = i \), gives
\[ C_k^n(i) = \binom{n + 1}{n - k + 1}, \quad S_k^n(i) = \binom{n + k}{n}. \]
Furthermore, if \( i_{p,q} = \langle [1]_{p,q}, [2]_{p,q}, \ldots \rangle \) where \( [i]_{p,q} = \sum_{s=1}^{i} p^{i-s} q^{s-1} \), then we obtain \( p,q \)-Stirling numbers considered by Wachs and White [12]
\[ p^{\binom{\ell}{2}} S_k^n(i_{p,q}) = \binom{n + k}{n}_{p,q}, \]
which satisfy the following recursive relation
\[ \binom{n}{k}_{p,q} = p^{k-1} \binom{n-1}{k-1}_{p,q} + [k]_{p,q} \binom{n-1}{k}_{p,q}. \]
We refer the reader also to Wagner [13], Médics and Leroux [11].

Notice, that the weight vector \( w \) in the definition of the coefficients \( C_k^n(w) \) and \( S_k^n(w) \) is constant and independent of the number \( n \). The \( \zeta \)-analogue of the Stirling numbers introduced in the next section do not require this assumption. We define the weight vector \( w_n(\zeta) \) dependent on the number \( n \) and a complex number \( \zeta \).

We show that our approach covers the well-known combinatorial numbers mentioned above and contains, e.g., Fibonomial and more general \( p,q \)-binomial coefficients [2, 3, 4] relevant with \( cobweb \) posets’ partitions and hyper-boxes tilings considered by Kwaśniewski [9] and the present author [5].
2. A $\zeta$-analogue of the Stirling Numbers

Take a vector $\mathbf{w}_n(\zeta)$ of $n$ complex numbers $w_i\zeta^{n-i}$, where $i = 1, 2, \ldots, n$, i.e.,

$$ \mathbf{w}_n(\zeta) = \langle w_1\zeta^{n-1}, w_2\zeta^{n-2}, \ldots, w_{n-1}\zeta, w_n \rangle. \quad (2) $$

We write it as $\hat{\mathbf{w}}_n$ for short and denote the $i$-th element of $\hat{\mathbf{w}}_n$ by $\hat{w}_n(i)$ or just $\hat{w}_i$ for fixed $n \in \mathbb{N}$. We assume $\mathbf{w}_0(\zeta) = \emptyset$ and $\hat{w}_0 = 0$.

It is important to notice, that the $j$-th element of $\mathbf{w}_n(\zeta)$ is not equal to the $j$-th element of $\mathbf{w}_m(\zeta)$ while $n \neq m$ in general. Indeed, $w_j\zeta^{n-j} \neq w_j\zeta^{m-j}$.

**Definition 1.** For any $n, k \in \mathbb{N} \cup \{0\}$ the $\zeta$-analogues of the Stirling numbers of the first kind $\hat{C}^n_k(\hat{\mathbf{w}}_n)$ and the second kind $\hat{S}^n_k(\hat{\mathbf{w}}_n)$ are defined as follows:

$$ \hat{C}^n_k(\hat{\mathbf{w}}_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \hat{w}_{i_1}\hat{w}_{i_2}\cdots\hat{w}_{i_k}, \quad (3a) $$

$$ \hat{S}^n_k(\hat{\mathbf{w}}_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \hat{w}_{i_1}\hat{w}_{i_2}\cdots\hat{w}_{i_k}, \quad (3b) $$

with $\hat{C}^n_0(\hat{\mathbf{w}}_n) = \hat{S}^n_0(\hat{\mathbf{w}}_n) = 1$ due to the empty product.

2.1. Combinatorial Interpretation

If the $\hat{w}_i$ are positive integers, the coefficients $\hat{C}^n_k(\hat{\mathbf{w}}_n)$ and $\hat{S}^n_k(\hat{\mathbf{w}}_n)$ denote the number of ways to select $k$ objects from $n$ boxes without box repetition allowed and with box repetition allowed, respectively. In this case, the size of the $i$-th box is designated by the $i$-th element of $\mathbf{w}_n(\zeta)$ for $i = 1, 2, \ldots, n$. However, all the results in this note holds for any vector $\hat{\mathbf{w}}$ of complex numbers and can be proved algebraically.

**Theorem 2.** For any $n, k \in \mathbb{N}$ we have

$$ \hat{C}^n_k(\hat{\mathbf{w}}_n) = w_n\zeta^{n-1}\hat{C}^n_{k-1}(\hat{\mathbf{w}}_{n-1}) + \zeta^k\hat{C}^n_{k-1}(\hat{\mathbf{w}}_{n-1}), \quad (4a) $$

$$ \hat{S}^n_k(\hat{\mathbf{w}}_n) = w_n\hat{S}^n_{k-1}(\hat{\mathbf{w}}_{n}) + \zeta^k\hat{S}^n_{k-1}(\hat{\mathbf{w}}_{n-1}), \quad (4b) $$

where $\hat{C}^n_0(\hat{\mathbf{w}}_n) = \hat{S}^n_0(\hat{\mathbf{w}}_n) = 1$ and $\hat{C}^n_0(\hat{\mathbf{w}}_n) = 0$ for $s > n$, $\hat{S}^n_0(\hat{\mathbf{w}}_n) = 0$ for $k > 0$.

*Proof.* The proof uses terms of the combinatorial interpretation of these coefficients, but still holds for any vector $\hat{\mathbf{w}}_n$ of complex numbers. In point of fact, we consider the sums (3a), (3b) and only play with its summations.

Fix a natural number $n$ and take the weight vector $\hat{\mathbf{w}}_n = (w_1\zeta^{n-1}, \ldots, w_{n-1}\zeta, w_n)$. (a) Consider a $k$-selection with the last $n$-th box being selected ($i_k = n$) and not
selected ($i_k < n$), respectively (repetition of boxes is not allowed)

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k < n} \prod_{j=1}^{k} w_{i_j} \zeta^{n-i_j} + \sum_{1 \leq i_1 < \cdots < i_k < n} \prod_{j=1}^{k} w_{i_j} \zeta^{n-i_j}. \tag{5a}$$

Observe that we can rewrite the right-hand side of the above as follows:

$$w_n \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n-1} \prod_{j=1}^{k-1} w_{i_j} \zeta^{n-1-i_j} + \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \prod_{j=1}^{k} w_{i_j} \zeta^{n-i_j}. \tag{5b}$$

As we have already noticed, the vector $\hat{\mathbf{w}}_n \equiv \mathbf{w}_n(\zeta)$ is dependent on $n$, and the $j$-th element of $\mathbf{w}_n(\zeta)$ is $\zeta$ times as large as the $j$-th element of $\mathbf{w}_{n-1}(\zeta)$ for $j = 1, 2, \ldots, n-1$. Hence

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = w_n \zeta^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n-1} \prod_{j=1}^{k-1} w_{i_j} \zeta^{n-1-i_j} +$$

$$+ \zeta^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} \prod_{j=1}^{k} w_{i_j} \zeta^{n-1-i_j} = w_n \zeta^{k-1} \hat{C}_{k-1}^{n-1}(\hat{\mathbf{w}}_{n-1}) + \zeta^k \hat{C}_k^{n-1}(\hat{\mathbf{w}}_{n-1}). \tag{5b}$$

(b) In the same way we prove the case with box repetition allowed. \hfill \square

**Notation 3.** Let $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. Denote by $\hat{\mathbf{w}}_n^{(m)}$ the vector

$$\hat{\mathbf{w}}_n^{(m)} = \langle w_{m+1} \zeta^{n-1}, w_{m+2} \zeta^{n-2}, \ldots, w_{m+n-1} \zeta, w_{m+n} \rangle.$$

For $m = 0$ we have $\hat{\mathbf{w}}_n^{(0)} \equiv \hat{\mathbf{w}}_n$.

**Proposition 4.** For any $n, m, k \in \mathbb{N}$ we have

$$\hat{C}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{j=0}^{k} \zeta^j \hat{C}_j^n(\hat{\mathbf{w}}_n) \hat{C}_k^{m-j}(\hat{\mathbf{w}}_m), \tag{5a}$$

$$\hat{S}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{j=0}^{k} \zeta^j \hat{S}_j^n(\hat{\mathbf{w}}_n) \hat{S}_k^{m-j}(\hat{\mathbf{w}}_m). \tag{5b}$$

**Proof.** (a) We prove the first equation (5a). Consider the left-hand side, i.e., the sum

$$\hat{C}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n+m} w_{i_1} \zeta^{n+m-i_1} \cdots w_{i_k} \zeta^{n+m-i_k}.$$

Take $j \in \{0, 1, \ldots, k\}$. We only need to show that the above summation might be separated into $(k+1)$ disjoint sums where in the $j$-th one the first $j$ variables
i_1, i_2, \ldots, i_j$ take on values from the set $\{1, 2, \ldots, n\}$ and the remaining $(k - j)$ variables from $\{n + 1, \ldots, n + m\}$, i.e.,

$$
\hat{C}^{n+m}_{k} (\hat{w}_{n+m}) = \sum_{j=0}^{k} \sum_{1 \leq i_1 < \cdots < i_j \leq n} w_{i_1} \zeta^{n+m-i_1} \cdots w_{i_j} \zeta^{n+m-i_j}
$$

$$
\hat{C}^{n+m}_{k+1} (\hat{w}_{n+m}) = \sum_{n+1 \leq i_{j+1} < \cdots < i_k \leq n+m} w_{i_{j+1}} \zeta^{n+m-i_{j+1}} \cdots w_{i_k} \zeta^{n+m-i_k}.
$$

Finally, we need to correct the powers of $\zeta$’s as follows:

$$
\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} w_{i_1} \zeta^{n+m-i_1} \cdots w_{i_j} \zeta^{n+m-i_j} = \zeta^{j-m} \hat{C}^{n}_{j} (\hat{w}_{n}).
$$

(b) The same proof remains valid for the coefficients $\hat{S}^{n+m}_{k} (\hat{w}_{n+m})$. \qed

This result provides a more general form of the recurrence relation (4a) for the coefficients $\hat{C}^{n}_{k} (\hat{w}_{n})$. Indeed, letting $n = n' - 1$ and $m = 1$ in the equation (5a) gives (4a).

\textbf{Proposition 5.} For any $n, k \in \mathbb{N}$ we have

$$
\hat{C}^{n}_{k} (\hat{w}_{n}) = \sum_{j=k}^{n} w_{j} \zeta^{k(n-j+1)-1} \hat{C}^{j-1}_{k-1} (\hat{w}_{j-1}),
$$

(6a)

$$
\hat{S}^{n}_{k} (\hat{w}_{n}) = \sum_{j=1}^{n} w_{j} \zeta^{k(n-j)} \hat{S}^{j}_{k-1} (\hat{w}_{j}).
$$

(6b)

\textbf{Proof.} (a) Consider the sum (3a) from the definition of the coefficient $\hat{C}^{n}_{k} (\hat{w}_{n})$ and separate it into $(n - k + 1)$ sums where in the $j$-th one the last variable $i_k$ is equal to $(k+j)$ for $j = 0, 1, \ldots, n - k$, i.e.,

$$
\hat{C}^{n}_{k} (\hat{w}_{n}) = \sum_{j=0}^{n-k} \sum_{1 \leq i_1 < \cdots < i_k = k+j} w_{i_1} \zeta^{n-i_1} \cdots w_{i_k} \zeta^{n-i_k}
$$

(7)

$$
= \sum_{j=k}^{n} w_{j} \zeta^{n-j} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq j-1} w_{i_1} \zeta^{n-i_1} \cdots w_{i_{k-1}} \zeta^{n-i_{k-1}}.
$$

(8)

Taking out the common factor $\zeta^{(n-j+1)}$ from $(k - 1)$ factors $(w_{i} \zeta^{n-i})$ gives

$$
\hat{C}^{n}_{k} (\hat{w}_{n}) = \sum_{j=k}^{n} w_{j} \zeta^{k(n-j+1)-1} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq j-1} w_{i_1} \zeta^{j-1-i_1} \cdots w_{i_{k-1}} \zeta^{j-1-i_{k-1}}
$$

$$
= \sum_{j=k}^{n} w_{j} \zeta^{k(n-j+1)-1} \hat{C}^{j-1}_{k-1} (\hat{w}_{j-1}).
$$
(b) The second equation (6a) might be handled in much the same way. Observe only that the variable \( j \) takes on values from the set \( \{1, 2, \ldots, n\} \).

Proposition 6. For any \( n, k \in \mathbb{N} \) we have

\[
\begin{align*}
\hat{C}_{k}^{n+1}(\hat{w}_{n+1}) &= \sum_{j=0}^{k} \hat{C}_{k-j}^{n-1}(\hat{w}_{n-j}) \zeta^{(j+1)(k-j)+(j)} \prod_{i=0}^{j-1} w_{n+1-i}, \tag{9a} \\
\hat{S}_{k}^{n+1}(\hat{w}_{n+1}) &= \sum_{j=0}^{k} \hat{S}_{k-j}^{n}(\hat{w}_{n}) \zeta^{(k-j)w_{n+1}j}. \tag{9b}
\end{align*}
\]

Proof. (a) Consider the sum (3a) of \( \hat{C}_{k}^{n+1}(\hat{w}_{n+1}) \) and observe that it may be separated into \( (k + 1) \) sums where in the \( j \)-th one \( (j = 0, 1, 2, \ldots, k) \) we have

\[
1 \leq i_1 < \cdots < i_{k-j} \leq n - j; \quad i_{k-j+1} = n + 1 - j + 1, \ldots, i_k = n + 1.
\]

(b) In the case of the coefficient \( \hat{S}_{k}^{n+1}(\hat{w}_{n+1}) \) we may separate the sum (3b) into \( (k + 1) \) sums where in the \( j \)-th one \( (j = 0, 1, 2, \ldots, k) \) we have

\[
1 \leq i_1 \leq \cdots \leq i_{k-j} \leq n; \quad i_{k-j+1} = i_{k-j+2} = \cdots = i_k = n + 1.
\]

The rest of the proof is straightforward and goes in much the same way as the proofs of Proposition 4 and Proposition 5.

\[\null\]

3. Generating Functions

Let \( n \geq 0 \) and define two generating functions:

\[
\begin{align*}
\mathcal{A}_n(x, y) &= \sum_{k \geq 0} (-1)^k \hat{C}_k^n(\hat{w}_n)x^ky^{n-k}, \tag{10a} \\
\mathcal{B}_n(x) &= \sum_{k \geq 0} \hat{S}_k^n(\hat{w}_n)x^k. \tag{10b}
\end{align*}
\]

Theorem 7. For \( n \geq 1 \) we have

\[
\begin{align*}
\mathcal{A}_n(x, y) &= \prod_{i=1}^{n} \left(y - w_i \zeta^{n-i}x\right), \tag{11a} \\
\mathcal{B}_n(x) &= \prod_{i=1}^{n} \left(1 - w_i \zeta^{n-i}x\right), \tag{11b}
\end{align*}
\]

with \( \mathcal{A}_0(x, y) = 1 \) and \( \mathcal{B}_0(x) = 1 \).
Proof. Applying recurrences (4a) and (4b), respectively, shows that $A_n(x, y)$ and $B_n(x)$ satisfy

$$A_n(x, y) = (y - w_n x) A_{n-1}(\zeta x, y) \quad \text{with} \quad A_0(x, y) = 1,$$

$$B_n(x) = \frac{1}{(1 - w_n x)} B_{n-1}(\zeta x) \quad \text{with} \quad B_0(x) = 1.$$  

Solving these recurrence relations proves (11a) and (11b).

Corollary 8. For any $n, j \in \mathbb{N}$ the coefficients $\hat{C}_k^n(\hat{w}_n)$ and $\hat{S}_k^n(\hat{w}_n)$ satisfy the following relations

$$\sum_{k=0}^{j} (-1)^k \hat{C}_k^n(\hat{w}) \hat{S}_{j-k}^n(\hat{w}) = 0, \quad (12a)$$

$$\sum_{k=0}^{j} \hat{S}_k^n(\hat{w})(-1)^{j-k} \hat{C}_{j-k}^n(\hat{w}) = 0. \quad (12b)$$

Proof. Indeed, notice that $A_n(x, 1) B_n(x) = B_n(x) A_n(x, 1) = 1$ for any $n \in \mathbb{N}$. Using the Cauchy product of power series with (11a) and (11b) finishes the proof.

Let $f(x)$ be a series in powers of $x$. Then by the symbol $[x^n] f(x)$ we will mean the coefficient of $x^n$ in the series $f(x)$. For example we have

$$\hat{S}_k^n(\hat{w}_n) = [x^k] B_n(x) = [x^k] \prod_{i=1}^{n} \frac{1}{(1 - w_i \zeta^{n-i} x)}.$$  

Proposition 9. Let $\hat{w} = (\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_n)$ be the vector $w_n(\zeta)$, such that $\hat{w}_i \neq \hat{w}_j$ for any $i \neq j$. Then for $n, k \in \mathbb{N}$ we have

$$\hat{S}_k^n(\hat{w}_n) = \sum_{i=1}^{n} (-1)^{n-i} \frac{\hat{w}_i^{(n+k-1)}}{\prod_{j=1}^{i-1}(\hat{w}_i - \hat{w}_j) \prod_{j=i+1}^{n}(\hat{w}_j - \hat{w}_i)}, \quad (13)$$

where $\hat{w}_i = w_i \zeta^{n-i}$ for $i = 1, 2, \ldots, n$.

Proof. Let us consider the generating function (11b). From the partial fraction decomposition we get

$$\hat{S}_k^n(\hat{w}_n) = [x^k] \prod_{i=1}^{n} \frac{1}{(1 - w_i \zeta^{n-i} x)} = [x^k] \sum_{i=1}^{n} \frac{a_i}{(1 - w_i \zeta^{n-i} x)} = \sum_{i=1}^{n} a_i (w_i \zeta^{n-i})^k.$$  

What is left is to find the coefficients $a_1, a_2, \ldots, a_n$. First, we multiply the above by the denominator of (11b), i.e., by $\prod_{j=1}^{n} (1 - w_j \zeta^{n-j} x)$ to get

$$1 \equiv \sum_{i=1}^{n} a_i \prod_{j=1}^{n} (1 - w_j \zeta^{n-j} x).$$
Observe that if we evaluate the above with \( x = (w_i \zeta^{n-i})^{-1} \), all summands except the \( i \)-th one vanish. Thus we obtain \( a_i \), i.e.,

\[
1 = a_i \prod_{j=1 \atop j \neq i}^n \left(1 - \frac{w_j}{w_i} \zeta^{i-j}\right) \Rightarrow a_i = \prod_{j=1 \atop j \neq i}^n \frac{1}{1 - \frac{w_j}{w_i} \zeta^{i-j}} = \prod_{j=1 \atop j \neq i}^n \frac{(w_i \zeta^{n-i})^{n-1}}{(w_i \zeta^{n-i} - w_j \zeta^{n-j})}.
\]

Replacing \( w_i \zeta^{n-i} \) by \( \hat{w}_i \) for each \( i = 1, 2, \ldots, n \), we can rewrite the above as

\[
a_i = \frac{\hat{w}_i^{n-1}}{\prod_{j=1 \atop j \neq i}^n (\hat{w}_i - \hat{w}_j)} = \frac{\hat{w}_i^{n-1}}{\prod_{j=1}^{i-1} (\hat{w}_i - \hat{w}_j) \prod_{j=i+1}^n (\hat{w}_i - \hat{w}_j)} = (-1)^{n-i} \frac{\hat{w}_i^{n-1}}{\prod_{j=1}^{i-1} (\hat{w}_i - \hat{w}_j) \prod_{j=i+1}^n (\hat{w}_j - \hat{w}_i)}.
\]

\( \square \)

**Example 10.** Let \( \mathbf{i} \) be the vector \( (1, 2, 3, \ldots) \), i.e., \( \hat{w}_i = i \) for \( i \in \mathbb{N} \). Then by Proposition 9 we obtain the well-known identity for the Stirling numbers of the second kind:

\[
\hat{S}_n^k(i) = \binom{n}{k} = \sum_{i=1}^k (-1)^{k-i} i^n \frac{i!}{i!(k-i)!}.
\]

4. Remarks and Examples

It is clear that \( \hat{C}_k^n(\mathbf{w}_n) \) and \( \hat{S}_k^n(\mathbf{w}_n) \) generalize the Stirling numbers of the first kind \( C_k^n(\mathbf{w}) \) and the second kind \( S_k^n(\mathbf{w}) \) if \( \zeta = 1 \) i.e., \( \mathbf{w} = (w_1, w_2, \ldots, w_k, \ldots) \) and

\[
\hat{C}_k^n(\mathbf{w}_n(1)) \equiv C_k^n(\mathbf{w}), \quad \hat{S}_k^n(\mathbf{w}_n(1)) \equiv S_k^n(\mathbf{w}).
\]

Fix \( p, q \in \mathbb{C} \). A sequence \( \{n_{p,q}\}_{n>0} \) of the elements \( n_{p,q} = \sum_{i=1}^n p^{n-i} q^{i-1} \) is called a \((p, q)\)-sequence. In the literature, the elements of \((p, q)\)-sequences are called \((p, q)\)-analogues and are denoted by \( n_{p,q} \equiv \lfloor n \rfloor_{p,q} \) (see Briggs and Remmel [1]).

**Example 11.** \((p, q\)-binomial coefficients)

The \((p, q\)-binomial coefficients generalize binomial, Gaussian and Fibonacci coefficients [2, 3, 4] and are defined as

\[
\binom{n}{k}_{p,q} = \frac{n_{p,q}!}{k_{p,q}!(n-k)_{p,q}} = \frac{n_{p,q}(n-1)_{p,q}\cdots(n-k+1)_{p,q}}{k_{p,q}(k-1)_{p,q}\cdots1_{p,q}},
\]

where \( n_{p,q}! = n_{p,q}(n-1)_{p,q}\cdots1_{p,q} \) and \( 0_{p,q} = 1 \).
Therefore, if the weight vector \( \mathbf{w}_n(p) \) takes the form \( (p^{n-1}, q^{n-2}, \ldots, q^{n-2}p, q^{n-1}) \), one covers the family of \( p, q \)-\textit{binomial coefficients} [2, 3, 4], i.e.,

\[
\hat{C}_k^n(p, q) = \binom{n}{k} p^{k(n-k)} q^{k(n-k)} , \quad \hat{S}_k^n(p, q) = \binom{n+k-1}{k}_{p,q} . \tag{15}
\]

Thus for any \((p, q)\)-sequence with \( p, q \in \mathbb{N} \), we have at least three different combinatorial interpretations of its \( p, q \)-\textit{binomial coefficients}: expressed in the language of cobweb posets partitions [9], tilings of hyper-boxes [4] and now as an object selection from weighted boxes.

\textbf{Example 12.} (Fibonomial coefficients)

It is easy to show that the Fibonacci numbers define a \((\varphi, \rho)\)-sequence where \( \varphi = (1 + \sqrt{5})/2 \) and \( \rho = (1 - \sqrt{5})/2 \). Therefore, from the previous example, the \( \zeta \)-analogue also generalize the Fibonomial coefficients, i.e.,

\[
\hat{C}_k^n(\varphi_n) = (-1)^k \binom{n}{k}_{Fib} , \quad \hat{S}_k^n(\varphi_n) = \binom{n+k-1}{k}_{Fib} , \tag{16}
\]

with the weight vector \( \varphi_n = (\varphi^{n-1}, \rho \varphi^{n-2}, \ldots, \rho^{n-2} \varphi, \rho^{n-1}) \). However, the combinatorial interpretation in terms of object selection cannot be applied in this case vector \( \varphi_n \) does not consist of only nonnegative integers. Fixing \( s, n \in \mathbb{N} \), from Corollary 8 we have also

\[
\sum_{k=0}^s (-1)^{k+1} \binom{n}{k}_{Fib} \binom{n+k-1}{s-k}_{Fib} = 0.
\]

\textbf{Example 13.} (\( p, q \)-\textit{Stirling numbers})

The \( \zeta \)-analogue generalizes the \( p, q \)-\textit{Stirling numbers} [12]. Indeed, let us consider the vector \( i_n(\zeta) = (\lfloor i \rfloor_{p,q} \zeta^{n-1}, [2]_{p,q} \zeta^{n-2}, \ldots, [n]_{p,q}) \), where \( [i]_{p,q} = \sum_{s=1}^i p^{s-1} q^{s-1} \) for \( i \in \mathbb{N} \) and \( \zeta = 1 \). Then we have

\[
\hat{S}_k^n(i_n(1)) = \binom{n+k}{n}_{p,q} , \quad \hat{S}_{n-k}(i_n(1)) = \binom{n}{k}_{p,q} . \tag{17}
\]

Finally, by Theorem 2 we have that the \( \zeta \)-analogue of \( p, q \)-\textit{Stirling numbers} satisfy

\[
\binom{n}{k}_{\zeta} = p^{k-1} \zeta^{n-k} \binom{n-1}{k-1}_{\zeta} + [k]_{p,q} \binom{n-1}{k}_{\zeta} . \tag{18}
\]

\section{Final Remarks}

The form of the weight vector \( \mathbf{w}_n(\zeta) \) given by (2) is one possible choice and we expect that there might be many other useful forms that can be applied here, e.g.
\[ \hat{w}_{i,n} = w_i^{n-i}, \] etc. We leave it for further investigation. Our choice is caused by unifying \( p, q \)-binomial coefficients and generalized Stirling numbers.

Acknowledgements This note was inspired by the paper of Professor Kwaśniewski [8] where he asked about relations between \( F \)-nomial and Konvalina coefficients. The author thanks the referee for useful suggestions and corrections of this note.

References


