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FIRST REMARK ON A ζ-ANALOGUE OF THE STIRLING NUMBERS

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Abstract

The so-called ζ-analogues of the Stirling numbers of the first and second kind are considered. These numbers cover ordinary binomial and Gaussian coefficients, p, q-Stirling numbers and other combinatorial numbers studied with the help of object selection, Ferrers diagrams and rook theory.

Our generalization includes these and now also the p, q-binomial coefficients. This special subfamily of F-nomial coefficients encompasses among others, Fibonomial ones. The recurrence relations with generating functions of the ζ-analogues are delivered here. A few examples of ζ-analogues are presented.

1. Introduction

Let $\mathbf{w} = \{w_i\}_{i\geq 1}$ be a vector of complex numbers w_i. The generalized Stirling numbers of the first kind $C_k^n(\mathbf{w})$ and the second kind $S_k^n(\mathbf{w})$ are defined as follows:

$$\begin{aligned} C_k^n(\mathbf{w}) &= \sum_{1\leq i_1 < i_2 < \cdots < i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k}, \\ S_k^n(\mathbf{w}) &= \sum_{1\leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k}. \end{aligned} \tag{1}$$

If the elements of the weight vector $\mathbf{w}$ are positive integers then the coefficients are interpreted as a selection of k objects from k of n boxes without and with box repetition allowed, respectively. In this case the number of distinct objects in the s-th box is designated by the s-th element of the weight vector $\mathbf{w}$.

One shows that the numbers $C_k^n(\mathbf{w})$ and $S_k^n(\mathbf{w})$ cover among others, binomial coefficients, Gaussian coefficients and the Stirling numbers of the first and second kind, see for example Konvalina [6, 7]. Indeed, if we fix $w_i = 1$, we obtain ordinary binomial coefficients:

$$C_k^n(\mathbf{1}) = \binom{n}{k}, \qquad S_k^n(\mathbf{1}) = \binom{n+k-1}{k}.$$

Setting $w_i = q^{i-1}$ gives us Gaussian coefficients:

$$C_k^n(\mathbf{q}) = q^{\binom{k}{2}}\binom{n}{k}_q, \qquad S_k^n(\mathbf{q}) = \binom{n+k-1}{k}_q.$$

In this note, the ordinary Stirling numbers of the first kind are defined in the following way

$$(1-x)(1-2x)\cdots(1-nx) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n+1 \\ n+1-k \end{bmatrix} x^k,$$

and the second kind

$$\frac{1}{(1-x)(1-2x)\cdots(1-nx)} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+k \\ n \end{matrix} \right\} x^k.$$

Letting $\mathbf{i} = \langle 1, 2, 3, \ldots \rangle$, i.e., $w_i = i$, gives

$$C_k^n(\mathbf{i}) = \begin{bmatrix} n+1 \\ n-k+1 \end{bmatrix}, \qquad S_k^n(\mathbf{i}) = \left\{ \begin{matrix} n+k \\ n \end{matrix} \right\}.$$

Furthermore, if $\mathbf{i}_{p,q} = \langle [1]_{p,q}, [2]_{p,q}, \ldots \rangle$ where $[i]_{p,q} = \sum_{s=1}^{i} p^{i-s}q^{s-1}$, then we obtain p,q-Stirling numbers considered by Wachs and White [12]

$$p^{\binom{n}{2}} S_k^n(\mathbf{i}_{p,q}) = \left\{ \begin{matrix} n+k \\ n \end{matrix} \right\}_{p,q},$$

which satisfy the following recursive relation

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{p,q} = p^{k-1} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_{p,q} + [k]_{p,q} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{p,q}.$$

We refer the reader also to Wagner [13], Médicis and Leroux [11].

Notice, that the weight vector $\mathbf{w}$ in the definition of the coefficients $C_k^n(\mathbf{w})$ and $S_k^n(\mathbf{w})$ is constant and independent of the number n. The ζ-*analogue* of the Stirling numbers introduced in the next section do not require this assumption. We define the weight vector $\mathbf{w}_n(\zeta)$ dependent on the number n and a complex number ζ.

We show that our approach covers the well-known combinatorial numbers mentioned above and contains, e.g., Fibonomial and more general p,q-*binomial* coefficients [2, 3, 4] relevant with *cobweb* posets' partitions and hyper-boxes tilings considered by Kwaśniewski [9] and the present author [5].

2. A ζ-analogue of the Stirling Numbers

Take a vector $\mathbf{w}_n(\zeta)$ of n complex numbers $w_i\zeta^{n-i}$, where $i = 1, 2, ..., n$, i.e.,

$$\mathbf{w}_n(\zeta) = \left\langle w_1\zeta^{n-1},\, w_2\zeta^{n-2},\, \ldots,\, w_{n-1}\zeta,\, w_n\right\rangle. \tag{2}$$

We write it as $\hat{\mathbf{w}}_n$ for short and denote the i-th element of $\hat{\mathbf{w}}_n$ by $\hat{w}_{n,i}$ or just $\hat{w}_i$ for fixed $n \in \mathbb{N}$. We assume $\mathbf{w}_0(\zeta) = \emptyset$ and $\hat{w}_0 = 0$.

It is important to notice, that the j-th element of $\mathbf{w}_n(\zeta)$ is not equal to the j-th element of $\mathbf{w}_m(\zeta)$ while $n \neq m$ in general. Indeed, $w_j\zeta^{n-j} \neq w_j\zeta^{m-j}$.

Definition 1. For any $n, k \in \mathbb{N} \cup \{0\}$ the ζ-*analogues* of the Stirling numbers of the first kind $\hat{C}^n_k(\hat{\mathbf{w}}_n)$ and the second kind $\hat{S}^n_k(\hat{\mathbf{w}}_n)$ are defined as follows:

$$\hat{C}^n_k(\hat{\mathbf{w}}_n) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \hat{w}_{i_1}\hat{w}_{i_2}\cdots\hat{w}_{i_k}, \tag{3a}$$

$$\hat{S}^n_k(\hat{\mathbf{w}}_n) = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} \hat{w}_{i_1}\hat{w}_{i_2}\cdots\hat{w}_{i_k}, \tag{3b}$$

with $\hat{C}^n_0(\hat{\mathbf{w}}_n) = \hat{S}^n_0(\hat{\mathbf{w}}_n) = 1$ due to the empty product.

2.1. Combinatorial Interpretation

If the $\hat{w}_i$ are positive integers, the coefficients $\hat{C}^n_k(\hat{\mathbf{w}}_n)$ and $\hat{S}^n_k(\hat{\mathbf{w}}_n)$ denote the number of ways to select k objects from k of n boxes without box repetition allowed and with box repetition allowed, respectively. In this case, the size of the i-th box is designated by the i-th element of $\mathbf{w}_n(\zeta)$ for $i = 1, 2, ..., n$. However, all the results in this note holds for any vector $\hat{\mathbf{w}}$ of complex numbers and can be proved algebraically.

Theorem 2. *For any $n, k \in \mathbb{N}$ we have*

$$\hat{C}^n_k(\hat{\mathbf{w}}_n) = w_n\zeta^{k-1}\hat{C}^{n-1}_{k-1}(\hat{\mathbf{w}}_{n-1}) \;+\; \zeta^k\hat{C}^{n-1}_k(\hat{\mathbf{w}}_{n-1}), \tag{4a}$$

$$\hat{S}^n_k(\hat{\mathbf{w}}_n) = w_n\hat{S}^n_{k-1}(\hat{\mathbf{w}}_n) \;+\; \zeta^k\hat{S}^{n-1}_k(\hat{\mathbf{w}}_{n-1}), \tag{4b}$$

where $\hat{C}^n_0(\hat{\mathbf{w}}_n) = \hat{S}^n_0(\hat{\mathbf{w}}_n) = 1$ *and* $\hat{C}^n_s(\hat{\mathbf{w}}_n) = 0$ *for* $s > n$, $\hat{S}^0_k(\hat{\mathbf{w}}_0) = 0$ *for* $k > 0$.

Proof. The proof uses terms of the combinatorial interpretation of these coefficients, but still holds for any vector $\hat{\mathbf{w}}_n$ of complex numbers. In point of fact, we consider the sums (3a), (3b) and only play with its summations.

Fix a natural number n and take the weight vector $\hat{\mathbf{w}}_n = \langle w_1\zeta^{n-1}, \ldots, w_{n-1}\zeta, w_n\rangle$.
(a) Consider a k-selection with the last n-th box being selected ($i_k = n$) and not

selected ($i_k < n$), respectively (repetition of boxes is not allowed)

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = \sum_{1\le i_1<i_2<\cdots<i_k=n} \prod_{j=1}^{k} w_{i_j}\zeta^{n-i_j} + \sum_{1\le i_1<i_2<\cdots<i_k<n} \prod_{j=1}^{k} w_{i_j}\zeta^{n-i_j}.$$

Observe that we can rewrite the right-hand side of the above as follows:

$$w_n \sum_{1\le i_1<\cdots<i_{k-1}\le n-1} \prod_{j=1}^{k-1} w_{i_j}\zeta^{n-i_j} + \sum_{1\le i_1<\cdots<i_k\le n-1} \prod_{j=1}^{k} w_{i_j}\zeta^{n-i_j}.$$

As we have already noticed, the vector $\hat{\mathbf{w}}_n \equiv \mathbf{w}_n(\zeta)$ is dependent on n, and the j-th element of $\mathbf{w}_n(\zeta)$ is ζ times as large as the j-th element of $\mathbf{w}_{n-1}(\zeta)$ for $j = 1, 2, \ldots, n-1$. Hence

$$\begin{aligned}\hat{C}_k^n(\hat{\mathbf{w}}_n) &= w_n\zeta^{k-1} \sum_{1\le i_1<i_2<\cdots<i_{k-1}\le n-1} \prod_{j=1}^{k-1} w_{i_j}\zeta^{n-1-i_j} + \\ &\quad + \zeta^k \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} \prod_{j=1}^{k} w_{i_j}\zeta^{n-1-i_j} \\ &= w_n\zeta^{k-1}\hat{C}_{k-1}^{n-1}(\hat{\mathbf{w}}_{n-1}) + \zeta^k\hat{C}_k^{n-1}(\hat{\mathbf{w}}_{n-1}).\end{aligned}$$

(b) In the same way we prove the case with box repetition allowed. □

Notation 3. Let $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. Denote by $\hat{\mathbf{w}}_n^{(m)}$ the vector

$$\hat{\mathbf{w}}_n^{(m)} = \langle w_{m+1}\zeta^{n-1}, w_{m+2}\zeta^{n-2}, \ldots, w_{m+n-1}\zeta, w_{m+n}\rangle.$$

For $m = 0$ we have $\hat{\mathbf{w}}_n^{(0)} \equiv \hat{\mathbf{w}}_n$.

Proposition 4. *For any $n, m, k \in \mathbb{N}$ we have*

$$\hat{C}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{j=0}^{k} \zeta^{j\cdot m}\hat{C}_j^n(\hat{\mathbf{w}}_n)\hat{C}_{k-j}^m(\hat{\mathbf{w}}_m^{(n)}), \tag{5a}$$

$$\hat{S}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{j=0}^{k} \zeta^{j\cdot m}\hat{S}_j^n(\hat{\mathbf{w}}_n)\hat{S}_{k-j}^m(\hat{\mathbf{w}}_m^{(n)}). \tag{5b}$$

Proof. (a) We prove the first equation (5a). Consider the left-hand side, i.e., the sum

$$\hat{C}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{1\le i_1<i_2<\cdots<i_k\le n+m} w_{i_1}\zeta^{n+m-i_1}\cdots w_{i_k}\zeta^{n+m-i_k}.$$

Take $j \in \{0, 1, \ldots, k\}$. We only need to show that the above summation might be separated into $(k+1)$ disjoint sums where in the j-th one the first j variables

$i_1, i_2, \ldots, i_j$ take on values from the set $\{1, 2, \ldots, n\}$ and the remaining $(k-j)$ variables from $\{n+1, \ldots, n+m\}$, i.e.,

$$\hat{C}_k^{n+m}(\hat{\mathbf{w}}_{n+m}) = \sum_{j=0}^{k} \sum_{1 \le i_1 < \cdots < i_j \le n} w_{i_1}\zeta^{n+m-i_1} \cdots w_{i_j}\zeta^{n+m-i_j} \cdot \sum_{n+1 \le i_{j+1} < \cdots < i_k \le n+m} w_{i_{j+1}}\zeta^{n+m-i_{j+1}} \cdots w_{i_k}\zeta^{n+m-i_k}.$$

Finally, we need to correct the powers of ζ's as follows:

$$\sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} w_{i_1}\zeta^{n+m-i_1} \cdots w_{i_j}\zeta^{n+m-i_j} = \zeta^{j \cdot m}\hat{C}_j^n(\hat{\mathbf{w}}_n).$$

(b) The same proof remains valid for the coefficients $\hat{S}_k^{n+m}(\hat{\mathbf{w}}_{n+m})$. □

This result provides a more general form of the recurrence relation (4a) for the coefficients $\hat{C}_k^n(\hat{\mathbf{w}}_n)$. Indeed, letting $n = n' - 1$ and $m = 1$ in the equation (5a) gives (4a).

Proposition 5. *For any $n, k \in \mathbb{N}$ we have*

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = \sum_{j=k}^{n} w_j \zeta^{k(n-j+1)-1}\hat{C}_{k-1}^{j-1}(\hat{\mathbf{w}}_{j-1}), \tag{6a}$$

$$\hat{S}_k^n(\hat{\mathbf{w}}_n) = \sum_{j=1}^{n} w_j \zeta^{k(n-j)}\hat{S}_{k-1}^{j}(\hat{\mathbf{w}}_j). \tag{6b}$$

Proof. (a) Consider the sum (3a) from the definition of the coefficient $\hat{C}_k^n(\hat{\mathbf{w}}_n)$ and separate it into $(n-k+1)$ sums where in the j-th one the last variable i_k is equal to $(k+j)$ for $j = 0, 1, \ldots, n-k$, i.e.,

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = \sum_{j=0}^{n-k} \sum_{1 \le i_1 < \cdots < i_{k-1} < i_k = k+j} w_{i_1}\zeta^{n-i_1} \cdots w_{i_k}\zeta^{n-i_k} \tag{7}$$

$$= \sum_{j=k}^{n} w_j \zeta^{n-j} \sum_{1 \le i_1 < \cdots < i_{k-1} \le j-1} w_{i_1}\zeta^{n-i_1} \cdots w_{i_{k-1}}\zeta^{n-i_{k-1}}. \tag{8}$$

Taking out the common factor $\zeta^{(n-j+1)}$ from $(k-1)$ factors $(w_i\zeta^{n-i})$ gives

$$\hat{C}_k^n(\hat{\mathbf{w}}_n) = \sum_{j=k}^{n} w_j \zeta^{k(n-j+1)-1} \sum_{1 \le i_1 < \cdots < i_{k-1} \le j-1} w_{i_1}\zeta^{j-1-i_1} \cdots w_{i_{k-1}}\zeta^{j-1-i_{k-1}}$$

$$= \sum_{j=k}^{n} w_j \zeta^{k(n-j+1)-1}\hat{C}_{k-1}^{j-1}(\hat{\mathbf{w}}_{j-1}).$$

(b) The second equation (6a) might be handled in much the same way. Observe only that the variable j takes on values from the set $\{1, 2 \ldots, n\}$. $\square$

Proposition 6. *For any $n, k \in \mathbb{N}$ we have*

$$\hat{C}_k^{n+1}(\hat{\mathbf{w}}_{n+1}) = \sum_{j=0}^{k} \hat{C}_{k-j}^{n-j}(\hat{\mathbf{w}}_{n-j})\zeta^{(j+1)(k-j)+\binom{j}{2}} \prod_{i=0}^{j-1} w_{n+1-i}, \tag{9a}$$

$$\hat{S}_k^{n+1}(\hat{\mathbf{w}}_{n+1}) = \sum_{j=0}^{k} \hat{S}_{k-j}^{n}(\hat{\mathbf{w}}_n)\zeta^{(k-j)} w_{n+1}^{j}. \tag{9b}$$

Proof. (a) Consider the sum (3a) of $\hat{C}_k^{n+1}(\hat{\mathbf{w}}_{n+1})$ and observe that it may be separated into $(k+1)$ sums where in the j-th one $(j = 0, 1, 2, \ldots, k)$ we have

$$1 \leq i_1 < \cdots < i_{k-j} \leq n - j; \quad i_{k-j+1} = n + 1 - j + 1, \ldots, i_k = n + 1.$$

(b) In the case of the coefficient $\hat{S}_k^{n+1}(\hat{\mathbf{w}}_{n+1})$ we may separate the sum (3b) into $(k+1)$ sums where in the j-th one $(j = 0, 1, 2, \ldots, k)$ we have

$$1 \leq i_1 \leq \cdots \leq i_{k-j} \leq n; \quad i_{k-j+1} = i_{k-j+2} = \cdots = i_k = n + 1.$$

The rest of the proof is straightforward and goes in much the same way as the proofs of Proposition 4 and Proposition 5. $\square$

3. Generating Functions

Let $n \geq 0$ and define two generating functions:

$$\mathcal{A}_n(x, y) = \sum_{k \geq 0} (-1)^k \hat{C}_k^n(\hat{\mathbf{w}}_n) x^k y^{n-k}, \tag{10a}$$

$$\mathcal{B}_n(x) = \sum_{k \geq 0} \hat{S}_k^n(\hat{\mathbf{w}}_n) x^k. \tag{10b}$$

Theorem 7. *For $n \geq 1$ we have*

$$\mathcal{A}_n(x, y) = \prod_{i=1}^{n} \left(y - w_i \zeta^{n-i} x\right), \tag{11a}$$

$$\mathcal{B}_n(x) = \prod_{i=1}^{n} \frac{1}{(1 - w_i \zeta^{n-i} x)}, \tag{11b}$$

with $\mathcal{A}_0(x, y) = 1$ and $\mathcal{B}_0(x) = 1$.

Proof. Applying recurrences (4a) and (4b), respectively, shows that $\mathcal{A}_n(x,y)$ and $\mathcal{B}_n(x)$ satisfy

$$\mathcal{A}_n(x,y) = (y - w_n\, x)\, \mathcal{A}_{n-1}(\zeta x, y) \quad \text{with} \quad \mathcal{A}_0(x,y) = 1,$$
$$\mathcal{B}_n(x) = \frac{1}{(1 - w_n\, x)} \mathcal{B}_{n-1}(\zeta x) \quad \text{with} \quad \mathcal{B}_0(x) = 1.$$

Solving these recurrence relations proves (11a) and (11b). □

Corollary 8. *For any $n, j \in \mathbb{N}$ the coefficients $\hat{C}_k^n(\hat{\mathbf{w}}_n)$ and $\hat{S}_k^n(\hat{\mathbf{w}}_n)$ satisfy the following relations*

$$\sum_{k=0}^{j} (-1)^k \hat{C}_k^n(\hat{\mathbf{w}}) \hat{S}_{j-k}^n(\hat{\mathbf{w}}) = 0, \tag{12a}$$

$$\sum_{k=0}^{j} \hat{S}_k^n(\hat{\mathbf{w}}) (-1)^{j-k} \hat{C}_{j-k}^n(\hat{\mathbf{w}}) = 0. \tag{12b}$$

Proof. Indeed, notice that $\mathcal{A}_n(x,1)\mathcal{B}_n(x) = \mathcal{B}_n(x)\mathcal{A}_n(x,1) = 1$ for any $n \in \mathbb{N}$. Using the Cauchy product of power series with (11a) and (11b) finishes the proof. □

Let $f(x)$ be a series in powers of x. Then by the symbol $[x^n]f(x)$ we will mean the coefficient of x^n in the series $f(x)$. For example we have

$$\hat{S}_k^n(\hat{\mathbf{w}}_n) = [x^k]\mathcal{B}_n(x) = [x^k] \prod_{i=1}^{n} \frac{1}{(1 - w_i \zeta^{n-i} x)}.$$

Proposition 9. *Let $\hat{\mathbf{w}} = \langle \hat{w}_1, \hat{w}_2, \ldots, \hat{w}_n \rangle$ be the vector $\mathbf{w}_n(\zeta)$, such that $\hat{w}_i \neq \hat{w}_j$ for any $i \neq j$. Then for $n, k \in \mathbb{N}$ we have*

$$\hat{S}_k^n(\hat{\mathbf{w}}_n) = \sum_{i=1}^{n} (-1)^{n-i} \frac{\hat{w}_i^{(n+k-1)}}{\prod_{j=1}^{i-1} (\hat{w}_i - \hat{w}_j) \prod_{j=i+1}^{n} (\hat{w}_j - \hat{w}_i)}, \tag{13}$$

where $\hat{w}_i = w_i \zeta^{n-i}$ for $i = 1, 2, \ldots, n$.

Proof. Let us consider the generating function (11b). From the partial fraction decomposition we get

$$\hat{S}_k^n(\hat{\mathbf{w}}_n) = [x^k] \prod_{i=1}^{n} \frac{1}{(1 - w_i \zeta^{n-i} x)} = [x^k] \sum_{i=1}^{n} \frac{a_i}{(1 - w_i \zeta^{n-i} x)} = \sum_{i=1}^{n} a_i \left(w_i \zeta^{n-i} \right)^k.$$

What is left is to find the coefficients $a_1, a_2, \ldots, a_n$. First, we multiply the above by the denominator of (11b), i.e., by $\prod_{j=1}^{n} \left(1 - w_j \zeta^{n-j} x\right)$ to get

$$1 \equiv \sum_{i=1}^{n} a_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(1 - w_j \zeta^{n-j} x\right).$$

Observe that if we evaluate the above with $x = (w_i\zeta^{n-i})^{-1}$, all summands except the i-th one vanish. Thus we obtain a_i, i.e.,

$$1 = a_i \prod_{\substack{j=1\\ j\neq i}} \left(1 - \frac{w_j}{w_i}\zeta^{i-j}\right) \Rightarrow a_i = \prod_{\substack{j=1\\ j\neq i}} \frac{1}{\left(1-\frac{w_j}{w_i}\zeta^{i-j}\right)} = \frac{\left(w_i\zeta^{n-i}\right)^{n-1}}{\prod_{\substack{j=1\\ j\neq i}}^{n}\left(w_i\zeta^{n-i} - w_j\zeta^{n-j}\right)}.$$

Replacing $w_i\zeta^{n-i}$ by $\hat{w}_i$ for each $i = 1, 2, \ldots, n$, we can rewrite the above as

$$\begin{aligned} a_i &= \frac{\hat{w}_i^{n-1}}{\prod_{\substack{j=1\\ j\neq i}}^{n}(\hat{w}_i - \hat{w}_j)} = \frac{\hat{w}_i^{n-1}}{\prod_{j=1}^{i-1}(\hat{w}_i-\hat{w}_j)\prod_{j=i+1}^{n}(\hat{w}_i-\hat{w}_j)} \\ &= (-1)^{n-i}\frac{\hat{w}_i^{n-1}}{\prod_{j=1}^{i-1}(\hat{w}_i-\hat{w}_j)\prod_{j=i+1}^{n}(\hat{w}_j-\hat{w}_i)}. \end{aligned}$$

□

Example 10. Let $\mathbf{i}$ be the vector $\langle 1, 2, 3, \ldots\rangle$, i.e., $\hat{w}_i = i$ for $i \in \mathbb{N}$. Then by Proposition 9 we obtain the well-known identity for the Stirling numbers of the second kind:

$$\hat{S}^k_{n-k}(\mathbf{i}) = \left\{ n \atop k \right\} = \sum_{i=1}^{k} (-1)^{k-i} \frac{i^n}{i!(k-i)!}.$$

4. Remarks and Examples

It is clear that $\hat{C}^n_k(\hat{\mathbf{w}}_n)$ and $\hat{S}^n_k(\hat{\mathbf{w}}_n)$ generalize the Stirling numbers of the first kind $C^n_k(\mathbf{w})$ and the second kind $S^n_k(\mathbf{w})$ if $\zeta = 1$, i.e., $\mathbf{w} = (w_1, w_2, \ldots, w_k, \ldots)$ and

$$\hat{C}^n_k(\mathbf{w}_n(1)) \equiv C^n_k(\mathbf{w}), \qquad \hat{S}^n_k(\mathbf{w}_n(1)) \equiv S^n_k(\mathbf{w}). \tag{14}$$

Fix $p, q \in \mathbb{C}$. A sequence $\{n_{p,q}\}_{n\geq 0}$ of the elements $n_{p,q} = \sum_{i=1}^{n} p^{n-i}q^{i-1}$ is called a (p, q)-*sequence*. In the literature, the elements of (p, q)-sequences are called (p, q)-*analogues* and are denoted by $n_{p,q} \equiv [n]_{p,q}$ (see Briggs and Remmel [1]).

Example 11. (p, q-*binomial* coefficients)
The p, q-binomial coefficients generalize binomial, Gaussian and Fibonomial coefficients [2, 3, 4] and are defined as

$$\binom{n}{k}_{p,q} = \frac{n_{p,q}!}{k_{p,q}!(n-k)_{p,q}!} = \frac{n_{p,q}(n-1)_{p,q}\cdots(n-k+1)_{p,q}}{k_{p,q}(k-1)_{p,q}\cdots 1_{p,q}},$$

where $n_{p,q}! = n_{p,q}(n-1)_{p,q}\cdots 1_{p,q}$ and $0_{p,q} = 1$.

Therefore, if the weight vector $\mathbf{w}_n(p)$ takes the form $\langle p^{n-1}, qp^{n-2}, \ldots, q^{n-2}p, q^{n-1}\rangle$, one covers the family of p, q-*binomial* coefficients [2, 3, 4], i.e.,

$$\hat{C}_k^n(\mathbf{w}_n(p)) = p^{\binom{k}{2}} q^{\binom{k}{2}} \binom{n}{k}_{p,q}, \qquad \hat{S}_k^n(\mathbf{w}_n(p)) = \binom{n+k-1}{k}_{p,q}. \tag{15}$$

Thus for any (p, q)-sequence with $p, q \in \mathbb{N}$, we have at least three different combinatorial interpretations of its p, q-*binomial* coefficients: expressed in the language of cobweb posets partitions [9], tilings of hyper-boxes [4] and now as an object selection from weighted boxes.

Example 12. (Fibonomial coefficients)
It is easy to show that the Fibonacci numbers define a (φ, ρ)-sequence where $\varphi = (1+\sqrt{5})/2$ and $\rho = (1-\sqrt{5})/2$. Therefore, from the previous example, the ζ-analogue also generalize the Fibonomial coefficients, i.e.,

$$\hat{C}_k^n(\boldsymbol{\varphi}_n) = (-1)^{\binom{k}{2}} \binom{n}{k}_{Fib}, \qquad \hat{S}_k^n(\boldsymbol{\varphi}_n) = \binom{n+k-1}{k}_{Fib}, \tag{16}$$

with the weight vector $\boldsymbol{\varphi}_n = \langle \varphi^{n-1}, \rho\varphi^{n-2}, \ldots, \rho^{n-2}\varphi, \rho^{n-1}\rangle$. However, the combinatorial interpretation in terms of object selection cannot be applied in this case vector $\boldsymbol{\varphi}_n$ does not consist of only nonnegative integers. Fixing $s, n \in \mathbb{N}$, from Corollary 8 we have also

$$\sum_{k=0}^{s} (-1)^{\binom{k+1}{2}} \binom{n}{k}_{Fib} \binom{n+s-k-1}{s-k}_{Fib} = 0.$$

Example 13. (p, q-Stirling numbers)
The ζ-analogue generalizes the p, q-Stirling numbers [12]. Indeed, let us consider the vector $\mathbf{i}_n(\zeta) = \langle [1]_{p,q}\zeta^{n-1}, [2]_{p,q}\zeta^{n-2}, \ldots, [n]_{p,q}\rangle$, where $[i]_{p,q} = \sum_{s=1}^{i} p^{i-s}q^{s-1}$ for $i \in \mathbb{N}$ and $\zeta = 1$. Then we have

$$\hat{S}_k^n(\mathbf{i}_n(1)) = \left\{ {n+k \atop n} \right\}_{p,q}, \qquad \hat{S}_{n-k}^k(\mathbf{i}_n(1)) = \left\{ {n \atop k} \right\}_{p,q}. \tag{17}$$

Finally, by Theorem 2 we have that the ζ-analogues of p, q-Stirling numbers satisfy

$$\left\{ {n \atop k} \right\}_{\zeta} = p^{k-1}\zeta^{n-k} \left\{ {n-1 \atop k-1} \right\}_{\zeta} + [k]_{p,q} \left\{ {n-1 \atop k} \right\}_{\zeta}. \tag{18}$$

5. Final Remarks

The form of the weight vector $\mathbf{w}_n(\zeta)$ given by (2) is one possible choice and we expect that there might be many other useful forms that can be applied here, e.g.

$\hat{w}_{i,n} = w_i^{n-i}$, etc. We leave it for further investigation. Our choice is caused by unifying p,q-*binomial* coefficients and generalized Stirling numbers.

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