A PROBABILISTIC LOOK AT SERIES INVOLVING EULER'S TOTIENT FUNCTION

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Received: 2/8/10, Revised: 10/18/11, Accepted: 1/12/12, Published: 2/1/12

Abstract

We use a probabilistic method to evaluate the limit of $\sum_{x=1}^{n} \phi(x)\, x^{r-1}\, n^{-(r+1)}$, where $\phi(x)$ is the Euler totient function and r is a nonnegative integer. We extend the probabilistic method to evaluate two other generalized types of series that involve Euler's totient function. In addition to the probabilistic method, an analytic approach is presented to evaluate the series when the exponent parameter r is a positive real number.

1. Introduction

Let x be a positive integer. The *Euler totient function* $\phi(x)$ counts the number of positive integers not exceeding x and coprime to x. It is known that the series $\sum_{x=1}^{n} \frac{\phi(x)}{x} \frac{1}{n}$ converges to $6/\pi^2$ as $n \to \infty$, and the limiting value $6/\pi^2$ can be interpreted as the probability that two *randomly chosen* positive integers are coprime.

We need to be precise about *how* numbers are *randomly chosen*. Here, we consider two different versions of choosing two positive integers at random. The first version is to pick a random number X uniformly from the set $\{1, 2, \ldots n\}$, where n is a positive integer, and then to pick another random number Y uniformly from $\{1, 2, \ldots, X\}$. Because Y can not exceed X, the two randomly chosen numbers X and Y are not statistically independent. For $x \le n$, the value $\phi(x)/x$ is the conditional probability that X and Y are coprime given the occurrence of the event $\{X = x\}$. Let $(X, Y) = \gcd(X, Y)$ denote the greatest common divisor of X and Y, and let $\mathbb{P}_n$ be the joint probability measure of X and Y. Using the law of total probability and conditioning on random variable X, the probability that X and Y are coprime can be expressed as

$$\mathbb{P}_n\{(X,Y)=1\} = \sum_{x=1}^{n} \mathbb{P}_n\{(x,Y)=1 | X=x\} \frac{1}{n} = \sum_{x=1}^{n} \frac{\phi(x)}{x} \frac{1}{n}. \qquad (1)$$

The series in (1) is extensively studied in number theory. The discovery of its limiting value $6/\pi^2 = 1/\zeta(2)$ can be traced back to Dirichlet [1], and Euler's product formula for the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{\text{all primes } p} (1 - \frac{1}{p^s})^{-1}$$

is used with $s = 2$.

The second version of selecting two positive integers at random is to have two independent random numbers X and W chosen uniformly from $\{1, 2, \ldots n\}$, where n is a positive integer. There are n^2 equally likely sample points under this second sampling scheme, whereas there are $n(n+1)/2$ sample points under the first version. Again, let $\mathbb{P}_n$ be the joint probability measure of X and W. Partitioning the event $\{(X, W) = 1\}$ based on the order between X and W and using symmetry, we have

$$\begin{aligned}
&\mathbb{P}_n\{(X, W) = 1\} \\
&= \mathbb{P}_n\{(X, W) = 1, X \geq W\} + \mathbb{P}_n\{(X, W) = 1, X \leq W\} - \mathbb{P}_n\{(X, W) = 1, X = W\} \\
&= 2 \times \mathbb{P}_n\{(X, W) = 1, X \geq W\} - \mathbb{P}_n\{X = W = 1\} \\
&= 2 \times \sum_{x=1}^{n} \mathbb{P}_n\{(x, W) = 1, X = x, W \leq x\} - \frac{1}{n^2} \\
&= 2 \times \sum_{x=1}^{n} \frac{\phi(x)}{n^2} - \frac{1}{n^2}. \qquad (2)
\end{aligned}$$

The series in (1) and (2) are related, but different. Notice that with $1 \leq x \leq n$, the convergence of $\sum_{x=1}^{n} \frac{\phi(x)}{x} \frac{1}{n}$ implies the covergence of

$$\sum_{x=1}^{n} \frac{\phi(x)}{x} (\frac{x}{n})^r \frac{1}{n}, \qquad (3)$$

for any real number $r \geq 0$. If we set $r = 1$ in (3), the series becomes $\sum_{x=1}^{n} \frac{\phi(x)}{n^2}$, which appears in (2). Our main goal is to introduce a probabilistic method to evaluate the series in (3) for any positive integer r. For the special case $r = 1$, our evaluation will yield the result that $\sum_{x=1}^{n} \frac{\phi(x)}{n^2}$ converges to $3/\pi^2$ as $n \to \infty$. So the asymptotic probability that two randomly chosen integers are coprime is $6/\pi^2$ under either of the two sampling schemes. We then extend the probabilistic interpretation of the series in (3) to evaluate two other generalized types of series involving Euler's totient function.

The formula we are going to establish for the series in (3) is not new in the sense that it can be deduecd using the Euler summation formula. We will also present this analytic approach to evaluate the series in (3) for any positive real exponent parameter r. Without further explanation in this note, we will use $\mathbb{P}_n$ to denote the joint probability measure for the random variables assuming values in the set $\{1, 2, \ldots n\}$, where n is a positive integer.

2. Probabilistic Evaluation of Series Involving $\phi(x)$

First, we will evaluate the series in (3) using a probabilistic method. Under the first version of sampling two positive integers, we pick a random number X uniformly from the set $\{1, 2, \ldots, n\}$, and then we pick another random number Y uniformly from $\{1, 2, \ldots, X\}$. Additionally, statistically independent of X and Y, we generate another set of r random numbers $Z_1, Z_2, \ldots, Z_r$ that are independent and uniformly selected from $\{1, 2, \ldots, n\}$. There are $n^{r+1}(n+1)/2$ equally likely sample points in the sample space. Let $\{X = Z_k\}$ denote the set of all sample points representing the occurrence of the outcome $X = Z_k$ for some $k = 1, 2, \ldots, r$. Then we have $\mathbb{P}_n\{X = Z_k\} = 1/n$ for any $k = 1, 2, \ldots, r$. Thus the probability of any subset of $\{X = Z_k\}$ converges to zero at a rate no slower than $1/n$ as $n \to \infty$.

Now, let A_j be the event that X is a jth order statistic among $X, Z_1, \ldots, Z_r$, where $j = 1, 2, \ldots, r + 1$. That is, A_j is the event that exactly $(j - 1)$ of Z's are no larger than X and the rest $(r - j + 1)$ of Z's are no less than X. In particular, $A_{r+1} = \{Z_1 \le X, Z_2 \le X, \ldots, Z_r \le X\}$ is the event that X is a maximum among $X, Z_1, \ldots, Z_r$. Let Ω be the sample space, and let $B = \{(X, Y) = 1\}$. The events $A_1, A_2, \ldots, A_{r+1}$ are not mutually disjoint due to possible ties among $X, Z_1, \ldots, Z_r$. But every sample point belongs to at least one of those events. Thus $\Omega = \cup_{j=1}^{r+1} A_j$ and $\{(X, Y) = 1\} = B = \cup_{j=1}^{r+1}(A_j \cap B)$. Let us write $E_j = A_j \cap B$, $j = 1, 2, \ldots, r+1$. Applying the inclusion-exclusion principle on $E_1, E_2, \ldots, E_{r+1}$, we have

$$\begin{aligned}
\mathbb{P}_n\{(X, Y) = 1\} &= \mathbb{P}_n[\cup_{j=1}^{r+1}(A_j \cap B)] = \mathbb{P}_n[\cup_{j=1}^{r+1} E_j] \\
&= \sum_{j=1}^{r+1} \mathbb{P}_n[E_j] - \sum_{1 \le j_1 < j_2 \le r+1} \mathbb{P}_n[E_{j_1} \cap E_{j_2}] \\
&\quad + \sum_{1 \le j_1 < j_2 < j_3 \le r+1} \mathbb{P}_n[E_{j_1} \cap E_{j_2} \cap E_{j_3}] \\
&\quad - \cdots + (-1)^r \, \mathbb{P}_n[E_1 \cap E_2 \cap \ldots \cap E_{r+1}]. \qquad (4)
\end{aligned}$$

On the left hand of equation (4), we know $\mathbb{P}_n\{(X, Y) = 1\}$ converges to $6/\pi^2$ as $n \to \infty$ under the first version of selecting X and Y at random. On the right hand of equation (4), there are totally $(2^{r+1} - 1)$ probability terms with either a positive or a negative sign. Let us first look at $\mathbb{P}_n[E_j]$, $j = 1, 2, \ldots, r + 1$. Because $X, Z_1, \ldots, Z_r$ are independent and identically distributed (we may label X by Z_{r+1}), the events $A_1, A_2, \ldots, A_{r+1}$ have an equal probability of occurrence. By a similar argument, we can show that all the events $E_j = A_j \cap B$, $j = 1, 2, \ldots, r + 1$, have an equal probability of occurrence as well. For the other $(2^{r+1} - r - 2)$ probability terms, we want to prove that all of them converge to zero as $n \to \infty$. It is sufficient to show $\mathbb{P}_n[E_1 \cap E_2]$ converges to zero as $n \to \infty$. Clearly $E_1 \cap E_2$ is a subset of $A_1 \cap A_2$. The event $A_1 \cap A_2$ means X is both a minimum and a second order statistic among $X, Z_1, \ldots, Z_r$, which implies $X = Z_k$ for some $k = 1, 2, \ldots, r$. Because the probability of any subset of $\{X = Z_k\}$ converges to zero at a rate no slower than $1/n$ as $n \to \infty$, this establishes that $\mathbb{P}_n[E_1 \cap E_2]$ converges to zero as $n \to \infty$. Hence, the right-hand side of (4) converges to $(r+1)$ times the limiting value of $\mathbb{P}_n[E_{r+1}]$ as $n \to \infty$. It follows that $\mathbb{P}_n[E_{r+1}] = \mathbb{P}_n[A_{r+1} \cap B] = \mathbb{P}_n\{(X,Y) = 1, Z_1 \leq X, Z_2 \leq X, \ldots, Z_r \leq X\}$ converges to $1/(r+1)$ times $6/\pi^2$ as $n \to \infty$ in equation (4). We can express this probability as

$$\begin{aligned} & \mathbb{P}_n\{(X,Y) = 1, Z_1 \leq X, Z_2 \leq X, \ldots, Z_r \leq X\} \\ = & \sum_{x=1}^{n} \mathbb{P}_n\{(x,Y) = 1, Z_1 \leq x, Z_2 \leq x, \ldots, Z_r \leq x \mid X = x\} \frac{1}{n} \qquad (5) \\ = & \sum_{x=1}^{n} \frac{\phi(x)}{x} (\frac{x}{n})^r \frac{1}{n}. \qquad (6) \end{aligned}$$

This is the series in (3). Notice that from (5) to (6), we need the assumption of statistical independence among random numbers $Y, Z_1, Z_2, \ldots, Z_r$. We have proved the first series involving Euler's totient function, which is recapitulated below.

Formula 1. Let $\phi(x)$ be the Euler totient function. Let r be a positive integer. Then

$$\lim_{n\to\infty} \sum_{x=1}^{n} \frac{\phi(x)}{x} (\frac{x}{n})^r \frac{1}{n} = \frac{6}{(r+1)\pi^2}.$$

For $r = 1$, we have the special case that $\sum_{x=1}^{n} \frac{\phi(x)}{n^2}$ converges to $3/\pi^2$ as $n \to \infty$. This shows that the series in (2) also converges to $6/\pi^2$, and hence the limiting probability of choosing two coprime positive integers at random under both versions of sampling scheme is $6/\pi^2$.

We have just looked at the event $\{(X,Y) = 1, Z_1 \leq X, Z_2 \leq X, \ldots, Z_r \leq X\}$ probabilistically to derive Formula 1. While we take $n \to \infty$ in equation (4), the same argument can be used to show that the probability

$$\mathbb{P}_n[E_j] = \mathbb{P}_n\{(X,Y) = 1, X \text{ is a } j\text{th order statistic among } X, Z_1, Z_2, \ldots, Z_r\} \qquad (7)$$

has the same limiting value $(r+1)^{-1} \times 6/\pi^2$ for any $j = 1, 2, \ldots, r+1$. As we have explained, the event that X is a jth order statistic means exactly $j-1$ of $Z_1, Z_2, \ldots, Z_r$ are no larger than X and the rest $(r-j+1)$ of $Z_1, Z_2, \ldots, Z_r$ are no less than X. There are $\binom{r}{j-1}$ distinct subsets of size $j-1$ chosen from $Z_1, Z_2, \ldots, Z_r$. Using the law of total probability again, we can express the probability in (7) as follows:

$$\begin{aligned}
&\mathbb{P}_n\{(X,Y)=1, X \text{ is a } j\text{th order statistic among } X, Z_1, Z_2, \ldots, Z_r\} \\
=&\sum_{x=1}^{n} \frac{\mathbb{P}_n\{(x,Y)=1, j-1 \text{ of } Z_1,\ldots,Z_r \le x, (r-j+1) \text{ of } Z_1,\ldots,Z_r \ge x \mid X=x\}}{n} \\
=&\sum_{x=1}^{n} \frac{\phi(x)}{x}\binom{r}{j-1}(\frac{x}{n})^{j-1}(\frac{n-x+1}{n})^{r-j+1}\frac{1}{n}.
\end{aligned}$$

This proves the second series involving Euler's totient function, which generalizes Formula 1.

Formula 2. Let $\phi(x)$ be the Euler totient function. Let r be a positive integer. Then for any $j = 1, 2, \ldots, r+1$,

$$\lim_{n\to\infty}\sum_{x=1}^{n} \frac{\phi(x)}{x}\binom{r}{j-1}(\frac{x}{n})^{j-1}(\frac{n-x+1}{n})^{r-j+1}\frac{1}{n} = \frac{6}{(r+1)\pi^2}.$$

Next, we use a similar probabilistic argument to evaluate the limit of the series

$$\sum_{x=1}^{n}(\frac{\phi(x)}{x})^l\binom{r}{j-1}(\frac{x}{n})^{j-1}(\frac{n-x+1}{n})^{r-j+1}\frac{1}{n}, \tag{8}$$

where l and r are both positive integers, and $j = 1, 2, \ldots, r+1$. The convergence of the series in (8) follows from the convergence of $\sum_{x=1}^{n}\frac{\phi(x)}{x}\frac{1}{n}$ and the fact that $\phi(x)/x$, x/n, and $(n-x+1)/n$ are all between 0 and 1. We now give a probabilistic interpretation for the term $(\phi(x)/x)^l$. Start with a random integer X drawn uniformly from the set $\{1, 2, \ldots, n\}$. This time, we generate a set of l random integers $Y_1, Y_2, \ldots, Y_l$ that are independent and uniformly distributed over the set $\{1, 2, \ldots, X\}$. The quantity $(\phi(x)/x)^l$ is the conditional probability that $(X, Y_i) = 1$ for all $i = 1, 2, \ldots, l$, given the occurrence of the event $\{X = x\}$. In fact, we have

$$\begin{aligned}
&\mathbb{P}_n\{(X,Y_1)=1,(X,Y_2)=1,\ldots,(X,Y_l)=1\} \\
&\quad= \sum_{x=1}^{n}\mathbb{P}_n\{(x,Y_1)=1,(x,Y_2)=1,\ldots,(x,Y_l)=1 \mid X=x\}\frac{1}{n} \\
&\quad= \sum_{x=1}^{n}(\frac{\phi(x)}{x})^l\frac{1}{n}.
\end{aligned}$$

For any positive integer l, Kac ([3], pp. 57-58) justified the following formula due to I. Schur:

$$\lim_{n\to\infty}\sum_{x=1}^{n}(\frac{\phi(x)}{x})^l\frac{1}{n} = \prod_{\text{all primes } p}[1-\frac{1}{p}+\frac{1}{p}(1-\frac{1}{p})^l], \tag{9}$$

where the infinite product is taken over all prime numbers. Let us label the value in (9) by $C(l)$, which stands for the limiting probability that a random positive integer X is *coprime* to each of the l positive integers $Y_1, Y_2, \ldots, Y_l$ that are drawn independently and uniformly from $\{1, 2, \ldots, X\}$. There is no closed-form expression for $C(l)$ found in the literature except for the case $C(1) = 1/\zeta(2) = 6/\pi^2$.

As for the term $\binom{r}{j-1}(x/n)^{j-1}((n-x+1)/n)^{r-j+1}$ in the series of (8), we can use the same argument that establishes Formula 2. That is, we introduce another set of r independent integers $Z_1, Z_2, \ldots, Z_r$ selected uniformly from the set $\{1, 2, \ldots, n\}$, and they are independent of X, $Y_1, Y_2, \ldots, Y_l$. We consider events A_j that X is a jth order statistic among $X, Z_1, Z_2, \ldots, Z_r$, $j = 1, 2, \ldots, r+1$. Then we apply the principle of inclusion-exclusion and the argument of symmetry on events $A_j \cap \{(X, Y_1) = 1, (X, Y_2) = 1, \ldots, (X, Y_l) = 1\}$, $j = 1, 2, \ldots, r+1$, to establish the following result, which is the third series involving Euler's totient function.

Formula 3. Let $\phi(x)$ be the Euler totient function. Let l and r be two positive integers. Then for any $j = 1, 2, \ldots, r+1$,

$$\lim_{n\to\infty} \sum_{x=1}^{n} (\frac{\phi(x)}{x})^l \binom{r}{j-1} (\frac{x}{n})^{j-1} (\frac{n-x+1}{n})^{r-j+1} \frac{1}{n} = \frac{1}{r+1} C(l),$$

where $C(l)$ is the value given in (9).

3. Series Evaluation Using an Analytic Approach

For the three formulas presented in the previous section, Formula 2 generalizes Formula 1, and Formula 3 generalizes both Formula 1 and Formula 2. The probabilistic method we used to evaluate the three related series is based on introducing a finite number of independent and identically distributed random variables and the argument of symmetry. Nevertheless, the probabilistic interpretation can not be applied to the cases when any of l, r, and j is not an integer. The following Euler summation formula can be used to prove Formula 3 for any real numbers l, r and j with $l \geq 1$, $r > 0$, and $0 \leq j \leq r$.

Euler Summation Formula Let a_n be an arithmetic function with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} a_x = C.$$

For any real-valued differentiable function $f(t)$ defined in interval $[0, 1]$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} a_x f(x/n) = C \int_0^1 f(t)\, dt.$$

Proof. Let $A(n) = \sum_{x=1}^{n} a_x$, and let $\chi_A(t) = 1$ if $t \in A$ and zero elsewhere. The condition that the Cesaro sum of a_n converges to C is equivalent to the statement that $A(n) = Cn + o(n)$. We can rewrite $\sum_{x=1}^{n} a_x[f(x/n) - f(0)]$ as follows.

$$\begin{aligned}
\sum_{x=1}^{n} a_x[f(x/n) - f(0)] &= \sum_{x=1}^{n} a_x \int_0^{x/n} f'(t)\,dt \\
&= \int_0^1 \sum_{x=1}^{n} a_x \chi_{[0,x/n]}(t) f'(t)\,dt \\
&= \int_0^1 f'(t) \sum_{nt<x\le n} a_x\,dt \\
&= \int_0^1 f'(t)A(n)\,dt - \int_0^1 f'(t)A(nt)\,dt \\
&= A(n)[f(1) - f(0)] - \int_0^1 f'(t)A(nt)\,dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{x=1}^{n} a_x f(x/n) &= A(n)f(1) - \int_0^1 f'(t)A(nt)\,dt \\
&= [Cn + o(n)]f(1) - \int_0^1 f'(t)[Cnt + o(nt)]\,dt \\
&= Cn[f(1) - \int_0^1 f'(t)\,t\,dt] + o(n) \\
&= Cn \int_0^1 f(t)\,dt + o(n).
\end{aligned}$$

Now divide both sides of the above equation by n and let $n \to \infty$. □

Now, the three formulas presented in the previous section become corollaries to the Euler summation formula. For Formula 1, we can use $a_x = \phi(x)/x$ and $f(t) = t^r$. Formula 2 follows with $a_x = \phi(x)/x$ and $f(t) = \binom{r}{j-1} t^{j-1}(1-t)^{r-j+1}$. Formula 3, the most generalized one, is justified with $a_x = (\phi(x)/x)^l$ and $f(t) = \binom{r}{j-1} t^{j-1}(1-t)^{r-j+1}$.

4. Concluding Remarks

For the three formulas presented in this note, Formula 2 generalizes Formula 1, and Formula 3 generalizes both Formula 1 and Formula 2. Equipped with the value

$C(l)$ given in (9) and Formula 3, we can show that as $n \to \infty$, the limit of the series

$$\sum_{x=1}^{n} \phi(x)^l x^{j-l}(n-x+1)^{r-j} n^{-(r+1)}$$

is $\binom{r}{j}^{-1}(r+1)^{-1}C(l)$ for any positive integers l and r, and any $j = 0, 1, \ldots, r$. The Euler summation formula is employed to evaluate the series for any real numbers l, r and j with $l \geq 1$, $r > 0$.

We have computed the limiting probability that a random positive integer X chosen uniformly from $\{1, 2, \ldots, n\}$ is coprime to each of a set of l random positive integers Y_1, Y_2, $\ldots$ Y_l that are chosen independently and uniformly from $\{1, 2, \ldots, X\}$. Because none of the Y_i can exceed X, those l random positive integers are not independent of X. Would this limiting probability change if the set of l random positive integers is independent of X? That is, if X, $W_1, W_2, \ldots, W_l$ are generated independently and uniformly from $\{1, 2, \ldots, n\}$, would the limiting value of

$$\mathbb{P}_n\{(X, W_1) = 1, (X, W_2) = 1, \ldots, (X, W_l) = 1\} \tag{10}$$

be equal to $C(l)$ given in (9) as $n \to \infty$? For $l = 1$, we have shown the answer is $C(1) = 6/\pi^2$ under both versions of sampling scheme. For $l \geq 2$, we need a way to count the number of positive integers not exceeding n and coprime to x. This calls for the *Legendre totient function* $\phi(n, x)$ that denotes the number of positive integers that are less than or equal to n and coprime to x. The Euler totient function is a special case of the Legendre totient function with $\phi(x) = \phi(x, x)$. We can express the probability in (10) in terms of the Legendre totient function as shown below.

$$\begin{aligned}
&\mathbb{P}_n\{(X, W_1) = 1, (X, W_2) = 1, \ldots, (X, W_l) = 1\} \\
&\quad = \sum_{x=1}^{n} \mathbb{P}_n\{(x, W_1) = 1, (x, W_2) = 1, \ldots, (x, W_l) = 1 \mid X = x\}\frac{1}{n} \\
&\quad = \sum_{x=1}^{n} (\frac{\phi(n, x)}{n})^l \frac{1}{n}. \qquad (11)
\end{aligned}$$

In [2] and [4], the asymptotic behavior of $\phi(n, x)$ is investigated when x is large. Here we need to assess $\phi(n, x)$ when n is large. We conjecture that the series in (11) converges to $C(l)$ for any $l \geq 2$, and leave it as a future research problem.

The most commonly adopted assumption for *random numbers* means that they are chosen independently and uniformly from a given set of numbers. We like to conclude by remarking that the probability of choosing two coprime positive integers at random could depend upon how the two numbers are generated.

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