#A26 INTEGERS 12 (2012)

VAN DER WAERDEN'S THEOREM ON HOMOTHETIC COPIES OF $\{1, 1+S, 1+S+T\}$[1]

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Received: 8/17/11, Revised: 2/2/12, Accepted: 3/12/12, Published: 3/23/12

Abstract

For all positive integers s and t, Brown et al. defined $f(s,t)$ to be the smallest positive integer N such that every 2-coloring of $[1,N]$ has a monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$. They proved that $f(s,t) \leq 4(s+t)+1$ for all s, t and that equality holds in the case where both $s/g \not\equiv 0 \pmod 4$ and $t/g \not\equiv 0 \pmod 4$ with $g = gcd(s,t)$ and in many other cases. They also proved that for all positive integers m, $f(4mt,t) = f(t,4mt) = 4(4mt+t) - t + 1$ or $4(4mt+t)+1$. In this paper, we show that $f(4mt,t) = f(t,4mt) = 4(4mt+t) - t + 1$ and that for all the other (s,t), $f(s,t) = 4(s+t)+1$.

1. Introduction

Van der Waerden's theorem on arithmetic progressions [5] states that for every positive integer k there is a smallest positive integer $w(k)$ such that every 2-coloring of $[1, w(k)] = \{1, 2, \ldots, w(k)\}$ has a monochromatic k-term arithmetic progression. There are lots of results on the estimation of $w(k)$ for large k (see [3], for example).

Let s, t, m be positive integers. A *homothetic* copy of $\{1, 1+s, 1+s+t\}$ is any set of the form $\{x, x+ys, x+ys+yt\}$ where x and y are positive integers. Regarding 3-term arithmetic progressions as homothetic copies of $\{1, 1+1, 1+1+1\}$, Brown et al. [2] considered van der Waerden's theorem on homothetic copies of $\{1, 1+s, 1+s+t\}$. They defined $f(s,t)$ to be the smallest positive integer N such that every 2-coloring

[1]This work is supported by the University of Incheon research grant in 2008-2009.
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of $[1,N]$ has a monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$. As they have noted, $f(s,t) = f(t,s)$ and hence we assume that $s \geq t$. They proved that $f(s,t) \leq 4(s+t)+1$ for all s, t and that the equality holds in the case where both $s/g \not\equiv 0 \pmod 4$ and $t/g \not\equiv 0 \pmod 4$ with $g = gcd(s,t)$ and in many other cases. Also they proved that $f(4mt,t) = 4(4mt+t)-t+1$ or $4(4mt+t)+1$. In this paper, we show that $f(4mt,t) = 4(4mt+t)-t+1$ and that for all the other (s,t), $f(s,t) = 4(s+t)+1$.

The following lemma is stated as Theorem 1 in [2].

Lemma 1. (Brown, Landman, Mishna) *Let s, t and c be positive integers. Then $f(cs,ct) = c(f(s,t)-1)+1$.*

The following lemma is from the Chinese Remainder Theorem.

Lemma 2. *Let s, t be positive integers such that $gcd(s,t) = 1$. Then any integer in $[1, 4s+4t]$ is equal to $1+is+jt$ for some unique $0 \leq j < s$ and $-jt/s \leq i < 4+(4-j)t/s$.*

Let s, t be positive integers and C be a 2-coloring of $[1,N]$. Due to Lemma 1 we may assume that $gcd(s,t) = 1$. As in Lemma 2, any integer in $[1,N]$ is equal to $1+is+jt$ for some $0 \leq j \leq \frac{N-is-1}{t}$. For those (i,j), construct Table 1 where each column consists of $\{C(1+is+jt)\}$ for fixed i over increasing j and each row consists of $\{C(1+is+jt)\}$ for fixed j over increasing i. Note that in Table 1, a triple $\{C(x), C(x+ys), C(x+ys+yt)\}$ composes an isosceles right triangle whose right angle is on the right top, which is recognized geometrically. For simplicity we call such a triangle an *IRT*. This geometric viewpoint is also helpful to solve the following problem which was raised by Bialostocki et al. [1]: Find the smallest positive integer N such that every 2-coloring of $[1,N]$ has a monochromatic triple $\{x, x+d, x+2d+b\}$. To solve this problem construct a table like Table 1 which contains $C(1+id+jb)$ instead of $C(1+is+jt)$. Then in that table, $\{C(x), C(x+d), C(x+2d+b)\}$ composes a triangle we get from an IRT by moving its lower vertex to the right for its length of base.

	$C(1)$	$C(1+s)$	$C(1+2s)$	$C(1+3s)$	$\dots$
$\dots$	$C(1+t)$	$C(1+s+t)$	$C(1+2s+t)$	$C(1+3s+t)$	$\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1

2. $f(4mt,t) = 4(4mt+t)-t+1$.

The following lemma is stated as Theorem 4 in [2].

Lemma 3. (Brown, Landman, Mishna) *We have:* $f(4mt, t) = 4(4mt + t) - t + 1$ *or* $4(4mt + t) + 1$ *for all positive integers* m *and* t.

To show that $f(4mt, t) = 4(4mt+t)-t+1$, it is enough to prove that $f(4m, 1) < 16m + 5$ by Lemma 1 and Lemma 3. Let $s = 4m$ and $t = 1$. We prove that $f(s, t) < 16m + 5$ in Theorem 6 by showing that Table 1 for $N = 16m + 4$ always contains an IRT which is composed of the same values. For simplicity we call such a triangle an *MIRT*. Each element of $[1, N]$ is $1 + is + jt$ for $0 \le i \le 4$ if $0 \le j \le 3$, and $0 \le i \le 3$ if $j \ge 4$ by Lemma 2. Therefore in Table 1, each of the first four rows which corresponds to some $0 \le j \le 3$ contains five elements and each of the other rows which corresponds to some $j \ge 4$ contains four elements. In particular the elements corresponding to $i = 4$ of Table 1 are $C(1 + 16m + j)$ for $0 \le j \le 3$.

From Table 1 we construct Table 2 where each row contains exactly four elements. For Table 2 to contain all $C(x)$ for all $x \in [1, 16m+4]$, we extend it to the $(4+4m)$-th row. Note that for $0 \le i \le 2$ and $4m + 1 \le j \le 4m + 4$, each $C(1 + is + jt)$ is the same as $C(1 + (i + 1)s + (j - 4m)t)$, i.e., it is located in two different places in Table 2.

$C(1)$	$C(1 + 4m)$	$C(1 + 8m)$	$C(1 + 12m)$
$C(2)$	$C(2 + 4m)$	$C(2 + 8m)$	$C(2 + 12m)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$C(4 + 4m)$	$C(4 + 8m)$	$C(4 + 12m)$	$C(4 + 16m)$

Table 2

The idea of the proof of Theorem 6 is as follows. Suppose Table 2 does not contain an MIRT. Then by (1) of Lemma 4, we show that no row which is not one of the last two rows and no column which is not one of the first two columns contains three consecutive elements which are of the same value. By the other parts of Lemma 4, we can show that the value of a row affects those of the following rows. This in turn shows that the table contains some consecutive rows as stated in Lemma 5. As a result, we obtain the rows of this table inductively. From these rows, we can find at least one element which has different value in its alternative location and we get a contradiction.

Throughout this section let N, s, t be positive integers and $C : [1, N] \to \{0, 1\}$ be a coloring without a monochromatic triple $\{x, x + sy, x + (s + t)y\}$ for $y = 1$, 2 or 3. Define $V(a) = (C(a), C(a + s), C(a + 2s), C(a + 3s))$ for $a \in [1, N - 3s]$. In the following lemmas, considering that $C(x) \in \{0, 1\}$ for each $x \in [1, N]$, we denote it by u or $1 - u$ for some $u \in \{0, 1\}$.

Lemma 4. *The following are true for each* $u \in \{0, 1\}$.
(1) *For* $1 \le a \le N - 2s - 2t$, *if* $C(a) = C(a + s) = u$, *then* $C(a + 2s) = 1 - u$.
For $2s + 1 \le a \le N - 2t$, *if* $C(a) = C(a + t) = u$, *then* $C(a + 2t) = 1 - u$.

(2) *For* $1 \le a \le N-3s-6t$*, if* $V(a) = (u,u,1-u,u)$*, then* $V(a+t) = (1-u,1-u,u,u)$*,* $V(a+2t) = (1-u,u,u,1-u)$ *and* $V(a+3t) = (u,u,1-u,1-u)$*.*
(3) *For* $1 \le a \le N-3s-5t$*, if* $V(a) = (u,1-u,u,u)$*, then* $V(a+2t) = (u,1-u,1-u,u)$ *and* $V(a+3t) = (1-u,u,u,1-u)$*.*
(4) *For* $1 \le a \le N-3s-4t$*, if* $V(a) = (u,1-u,1-u,u)$*, then either* $V(a+t) = (1-u,u,u,1-u)$*,* $V(a+2t) = (u,1-u,1-u,u)$ *or* $V(a+t) = (1-u,1-u,u,u)$*,* $V(a+2t) = (1-u,u,u,1-u)$*. Moreover in the latter case, if* $1 \le a \le N-3s-6t$*, then* $V(a+3t) = (u,u,1-u,1-u)$*.*

Proof. Throughout the proof, we may assume that $u = 0$. Arranging elements $C(x)$ for $x \in [1,N]$ as in Table 1 gives a geometrical viewpoint.
(1) Suppose $C(a+s+t) = 0$ also. Then each of $C(a+s+t), C(a+2s+t), C(a+2s+2t)$ composes an IRT together with a pair from $C(a), C(a+s), C(a+2s)$ and hence $C(a+s+t) = C(a+2s+t) = C(a+2s+2t) = 1$. Then $C(a+s+t), C(a+2s+t), C(a+2s+2t)$ compose an MIRT, a contradiction. Thus $C(a+s+t) = 1$. The other statement can be proved similarly.
(2) As $V(a) = (0,0,1,0)$, each of $C(a+s+t), C(a+3s+2t), C(a+3s+3t)$ composes an IRT together with a pair of $C(a), C(a+s), C(a+3s)$ and hence $C(a+s+t) = C(a+3s+2t) = C(a+3s+3t) = 1$. Then by (1), $C(a+3s+t) = C(a+3s+4t) = 0$. This implies $C(a+t) = 1$ as it composes an IRT together with $C(a+3s+t)$ and $C(a+3s+4t)$. Again by (1), $V(a+t) = (1,1,0,0)$. Similarly, $V(a+2t) = (1,0,0,1)$ and $V(a+3t) = (0,0,1,1)$. Statements (3) and (4) are proved similarly by using the fact that there is no MIRT and (1). □

From (4) of Lemma 4, we get the following lemma.

Lemma 5. *Let* $a \in [1, N-3s-4t]$*. Then the following are true for each* $u \in \{0,1\}$*.*
(1) *If* $V(a) = (u,1-u,1-u,u)$ *and* $V(a+t) = (1-u,1-u,u,u)$*, then*

$$V(a+\ell t) = \begin{cases} (u,1-u,1-u,u), & \ell \equiv 0 \pmod 4 \\ (1-u,1-u,u,u), & \ell \equiv 1 \pmod 4 \\ (1-u,u,u,1-u), & \ell \equiv 2 \pmod 4 \\ (u,u,1-u,1-u), & \ell \equiv 3 \pmod 4 \end{cases} \quad (1)$$

for all ℓ *such that* $0 \le \ell \le 2\lfloor \frac{N-a-3s-2t}{2t} \rfloor$*.*
(2) *If* $V(a) = (u,1-u,1-u,u)$ *and* $V(a+t) = (1-u,u,u,1-u)$*, then*

$$V(a+\ell t) = \begin{cases} (u,1-u,1-u,u), & \ell \equiv 0 \pmod 2 \\ (1-u,u,u,1-u), & \ell \equiv 1 \pmod 2 \end{cases} \quad (2)$$

for all ℓ *such that* $0 \le \ell \le \lfloor \frac{N-a-3s-2t}{t} \rfloor$*.*

Proof. Throughout the proof, we may assume that $u = 0$.

(1) Let k be the largest integer such that the equality (1) holds for all $0 \le \ell \le k$. Assume $k \equiv 0 \pmod 4$. Then $V(a+(k-2)t) = (1,0,0,1)$ and $V(a+(k-1)t) =$

$(0,0,1,1)$. Suppose $k < 2\lfloor\frac{N-a-3s-2t}{2t}\rfloor$. Then $a+(k-2)t \le N-3s-6t$ and hence by (4) of Lemma 4, $V(a+(k+1)t) = (1,1,0,0)$, which is a contradiction. So $k \ge 2\lfloor\frac{N-a-3s-2t}{2t}\rfloor$. Similarly, $k \ge 2\lfloor\frac{N-a-3s-2t}{2t}\rfloor$ if $k \equiv 1,2,3 \pmod 4$. Thus the equality (1) holds.

Statement (2) is proved similarly. □

Theorem 6. *For all m, $t \in Z^+$, $f(4m,1) < 16m+5$.*

Proof. Suppose not. Let $N = 16m+4$, $s = 4m$ and $t = 1$. Then there is a coloring C of [1, N] which has no monochromatic homothetic copy of $\{1, 4m+1, 4m+2\}$. We may assume that $C(1) = 0$. There are five cases to consider by (1) of Lemma 4. In each case we have a contradiction by using Lemma 4 and Lemma 5.

`Case 1`: $V(1) = (0,0,1,0)$. Then $V(2) = (1,1,0,0)$, $V(3) = (1,0,0,1)$ and $V(4) = (0,0,1,1)$ by (2) of Lemma 4. So for all $0 \le \ell \le 4m-2$,

$$V(3+\ell) = \begin{cases} (1,0,0,1), & \ell \equiv 0 \pmod 4 \\ (0,0,1,1), & \ell \equiv 1 \pmod 4 \\ (0,1,1,0), & \ell \equiv 2 \pmod 4 \\ (1,1,0,0), & \ell \equiv 3 \pmod 4 \end{cases} \tag{3}$$

by (1) of Lemma 5. In particular, $V(4m+1) = (0,1,1,0)$ and hence $C(12m+1) = 1$, which contradicts $V(1) = (0,0,1,0)$.

`Case 2`: $V(1) = (0,1,0,0)$. Then $V(3) = (0,1,1,0)$ and $V(4) = (1,0,0,1)$ by (3) of Lemma 4. By using (2) of Lemma 5, each of $V(3+\ell)$ for $0 \le \ell \le 4m-1$ is determined and by the same method as in Case 1, we get a contradiction.

`Case 3`: $V(1) = (0,1,1,0)$. Then $V(2)$ is $(1,1,0,0)$ or $(1,0,0,1)$ by (4) of Lemma 4. In each case, we can prove the theorem similarly as in Case 1.

`Case 4`: $V(1) = (0,0,1,1)$. Then $V(4m+1) = (0,1,1,0) \ne (0,0,1,1)$. So there is the minimum $2 \le k \le 4m+1$ such that

$$V(k) \ne \begin{cases} (0,0,1,1), & k \text{ is odd} \\ (1,1,0,0), & k \text{ is even.} \end{cases}$$

Consider the case where k is odd. As $k \ge 3$, $V(2) = (1,1,0,0)$ and hence $V(4m+2) = (1,0,0,1)$ by (1) of Lemma 4. As $V(k-1) = (1,1,0,0)$, $C(4m+k) = 0$ and $C(12m+k) = 1$. So $V(k)$ is $(1,0,1,1)$ or $(1,0,0,1)$ and $k \ne 4m+1$. Therefore $k \le 4m-1$.

First assume that $V(k) = (1,0,1,1)$. Then $V(k+2) = (1,0,0,1)$ and $V(k+3) = (0,1,1,0)$ by (3) of Lemma 4. So for all $0 \le \ell \le 4m-k-1$, $V(k+2+\ell)$ is the same as $V(a+\ell)$ in (2) of Lemma 5. In particular $V(4m+1)$ is $(1,0,0,1)$, which is a contradiction.

Second assume that $V(k) = (1,0,0,1)$. Then $V(k+1)$ is $(0,1,1,0)$ or $(0,0,1,1)$ by (4) of Lemma 4. In each case, we can proceed similarly as in Case 1.

When k is even, we get a contradiction similarly.

Case 5: $V(1) = (0,1,0,1)$. Thus $V(2)$ is $(0,0,1,0)$, $(0,0,1,1)$, $(0,1,0,1)$, $(1,0,0,1)$, $(1,0,1,0)$, $(1,0,1,1)$ or $(1,1,0,1)$.

If $V(2) = (0,0,1,0)$, $(0,0,1,1)$, $(1,0,0,1)$, $(1,0,1,1)$ or $(1,1,0,1)$, then we get a contradiction as in Case 1 - Case 4.

If $V(2) = (0,1,0,1)$, then by (1) of Lemma 4, $V(3) = (1,0,1,0)$. Since $C(4m+1) = 1$, $V(4m+1) \neq (0,1,0,1)$. So there is the minimum k such that

$$V(k) \neq \begin{cases} (0,1,0,1), & k \equiv 1,2 \pmod 4 \\ (1,0,1,0), & k \equiv 0,3 \pmod 4. \end{cases}$$

If $k \equiv 1 \pmod 4$, then $V(k-2) = V(k-1) = (1,0,1,0)$ and therefore $V(k) = (0,1,0,1)$, a contradiction. Similarly we get a contradiction if $k \equiv 3 \pmod 4$. Assume that $k \equiv 2 \pmod 4$. Then $k \leq 4m-2$. From $V(k-2) = (1,0,1,0)$ and $V(k-1) = (0,1,0,1)$, we get $C(8m+k) = 0$, $C(8m+k+1) = 1$, $C(12m+k) = 1$ and $C(12m+k+1) = 0$. Suppose $C(4m+k) = 1$. Then $C(12m+k+2) = 0$ and hence by (1) of Lemma 4, $C(12m+k+3) = 1$. Thus $C(k) = 0$, i.e., $V(k) = (0,1,0,1)$, a contradiction. Therefore $C(4m+k) = 0$ and hence $V(k) = (1,0,0,1)$ by (1) of Lemma 4. By (4) of Lemma 4, $V(k+1) = (0,1,1,0)$. So for all $0 \leq \ell \leq 4m-k+2$, by (2) of Lemma 5, $V(k+\ell)$ takes the same value as $V(a+\ell)$ as in Case 2. In particular, $V(4m+1) = (0,1,1,0)$, which contradicts $V(1) = (0,1,0,1)$.

If $V(2) = (1,0,1,0)$, then we get a contradiction similarly. □

Remark 7. Investigating the proof of Theorem 6, we see that only in Case 4 we may have a coloring C of $[1, 16m+3]$ which has no monochromatic homothetic copy of $\{1, 4m+1, 4m+2\}$ under the following conditions: Either (i) k is even, $k \leq 4m-2$, $V(k) = (0,1,1,0)$ and $V(k+1) = (1,1,0,0)$; or (ii) k is odd, $k \leq 4m-1$, $V(k) = (1,0,0,1)$ and $V(k+1) = (0,0,1,1)$. In fact it is only under the conditions stated in (ii) with another restriction of $k \equiv 3 \pmod 4$ that we have such a coloring which satisfies

$$V(\ell) = \begin{cases} (0,0,1,1), & \ell \equiv 1 \pmod 2, \ell < k \\ (1,1,0,0), & \ell \equiv 0 \pmod 2, \ell < k \\ (1,0,0,1), & \ell \equiv 3 \pmod 4, k \leq \ell \leq 4m \\ (0,0,1,1), & \ell \equiv 0 \pmod 4, k \leq \ell \leq 4m \\ (0,1,1,0), & \ell \equiv 1 \pmod 4, k \leq \ell \leq 4m \\ (1,1,0,0), & \ell \equiv 2 \pmod 4, k \leq \ell \leq 4m \end{cases} \tag{4}$$

$C(16m+1) = C(16m+2) = 0$ and $C(16m+3) = 1$.

We can dispose of the coloring as shown in Table 3 with $C(4mi+\ell)$ in the ℓ-th row and the $(i+1)$-th column. Note that for distinct values of k satisfying $1 \leq k \leq 4m$ and $k \equiv 3 \pmod 4$, the corresponding colorings C are distinct. Also for each of those C, $1-C$ has no monochromatic homothetic copy of $\{1, 4m+1, 4m+2\}$.

Thus we have altogether $2m$ colorings of $[1, 16m+3]$ that have no monochromatic homothetic copies of $\{1, 4m+1, 4m+2\}$.

$\ell = 1$	0 0 1 1
	1 1 0 0
	0 0 1 1
	1 1 0 0
	$\vdots$
	0 0 1 1
	1 1 0 0
$\ell = k$	1 0 0 1
	0 0 1 1
	0 1 1 0
	1 1 0 0
	$\vdots$
	1 0 0 1
$\ell = 4m$	0 0 1 1
$\ell = 4m+1$	0 1 1 0
$\ell = 4m+2$	1 0 0 0
$\ell = 4m+3$	0 1 1 1

Table 3

3. $f(s,t) = 4(s+t)+1$, if $s \neq 4mt$.

Let s, t, m be positive integers where $s \neq 4mt$ for any m. We prove that $f(s,t) = 4(s+t)+1$. The following two lemmas and one theorem are stated as Theorem 2, Theorem 3 and Theorem 5, respectively in [2]. In particular Lemma 9 implies that $f(s,t) = 4(s+t)+1$ if t divides s and $s/t \not\equiv 0 \pmod 4$.

Assume t does not divide s. By Theorem 10, it is sufficient to consider the case where $\lfloor s/t \rfloor$ and $\lfloor 2s/t \rfloor$ are both odd and $s/t \in (1.5, 2)$. By Lemma 1 and Lemma 9, we need to prove only the case when $s \equiv 0 \pmod 4$ or $t \equiv 0 \pmod 4$; this is accomplished in Theorem 11.

Lemma 8. (Brown, Landman, Mishna) *$f(s,t) \leq 4(s+t)+1$ for all positive integers s and t.*

Lemma 9. (Brown, Landman, Mishna) *Let s, t be positive integers with $g = gcd(s,t)$. If $s/g \not\equiv 0 \pmod 4$ and $t/g \not\equiv 0 \pmod 4$, then $f(s,t) = 4(s+t)+1$.*

Theorem 10. (Brown, Landman, Mishna) *Let s, t be positive integers such that $s > t > 1$ and t does not divide s. If $\lfloor s/t \rfloor$ is even or $\lfloor 2s/t \rfloor$ is even, then $f(s,t) = 4(s+t)+1$. If $\lfloor s/t \rfloor$ and $\lfloor 2s/t \rfloor$ are both odd, then $f(s,t) = 4(s+t)+1$ provided that s, t satisfy the additional condition $s/t \notin (1.5, 2)$.*

Theorem 11. *Let s, t be positive integers such that $s > t > 1$ and t does not divide s. Assume $\lfloor s/t \rfloor$ and $\lfloor 2s/t \rfloor$ are both odd and $s/t \in (1.5, 2)$. Also assume $s \equiv 0 \pmod 4$ or $t \equiv 0 \pmod 4$. Then $f(s,t) = 4(s+t)+1$.*

Proof. By Lemma 1, it is enough to consider the case where $gcd(s,t) = 1$. By Lemma 8, $f(s,t) \le 4(s+t)+1$ for all positive integers s and t. We will consider three cases according to the values of $s \pmod 4$ and $t \pmod 4$. In each case, we show that the equality holds by proposing a 2-coloring C of $[1, 4s+4t]$ that contains no monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$, which is obtained by using the three steps described below. By Lemma 2, for each $1+is+jt \in [1, 4s+4t]$, $0 \le j < s$ and $-jt/s \le i < 4 + (4-j)t/s$. Also, any homothetic copy of $\{1, 1+s, 1+s+t\}$ is a triple $\{1+is+jt, 1+(i+y)s+jt, 1+(i+y)s+(j+y)t\}$ where $y = 1$, 2 or 3.

In the first step, we assign a coloring C in such a way that

$$V(1+jt) = \begin{cases} (0,0,1,1), & j \equiv 0 \pmod 2 \\ (1,1,0,0), & j \equiv 1 \pmod 2 \end{cases} \tag{5}$$

or

$$V(1+jt) = \begin{cases} (0,1,1,0), & j \equiv 0 \pmod 2 \\ (1,0,0,1), & j \equiv 1 \pmod 2 \end{cases} \tag{6}$$

for $0 \le j \le s-1$ and $-jt/s \le i < 4 + (4-j)t/s$. If $j \le s-4$, then $j+y \le s-1$ and hence the above triple is not monochromatic as $C(1+(i+y)s+jt) \ne C(1+(i+y)s+(j+y)t)$ for $y = 1$, 3 and $C(1+is+jt) \ne C(1+(i+y)s+jt)$ for $y = 2$. Assume $s-3 \le j \le s-1$. If $j+y \ge s$, then $1+(i+y)s+(j+y)t = 1+(t+i+y)s+(j+y-s)t \in [1, 4s+4t]$. Note that the color of this integer is shown in the $(j+y+2-s)$-th row instead of the $(j+y+2)$-th row in Table 1 and we must make sure that $\{C(1+is+jt), C(1+(i+y)s+jt), C(1+(i+y)s+(j+y)t)\}$ does not compose an MIRT. To help recognize such an MIRT, we extend Table 1 to the $(s+3)$-th row by taking $C(1+(t+i+y)s+(j+y-s)t)$ which is in the $(j+y+2-s)$-th row for $C(1+(i+y)s+(j+y)t)$ in the $(j+y+2)$-th row. Certainly, C has no monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$ if the extended table contains no MIRT. However, it turned out that in some cases, with the coloring given by equation (5) or (6), the extended table contains MIRT's.

In the second step, to get rid of the MIRT's in the extended table, in each case we consider two subcases depending on whether $s/t \ge 5/3$ or not. In each subcase, an integer $1+(-t+7)s+(s-1)t$ or $1+(-t+3)s+(s-5)t$ is not in $[1, 4s+4t]$. We put a mark @ or @$'$ in the position of its color in the extended table. In the first subcase, any integer in $[1, 4s+4t]$ is less than $1+(-t+7)s+(s-1)t$ and has its color on the left of or above the @ mark in the extended table. Similarly in the second subcase, each entry is on the right of or below the @$'$ mark in the extended table.

In the third step, we change the values of some elements below or to the left side of the double lines. Of course for each of them, we change the value of the same element in the other location of the extended table too.

After going through these three steps, finally we obtain Table 4 - Table 8 as shown in the appendix, which contain no MIRT. In each of these tables, the smallest value of i in between two consecutive vertical lines is shown in the first row. Below we list cases and subcases.

`Case 1`: $s \equiv 0 \pmod 4$ and $t \equiv 3 \pmod 4$. Consider the following two subcases.

`Case 1a`: $s/t \geq 5/3$. Table 4 shows C without a monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$. Note that there is an @ mark instead of $C(1+(-t+7)s+(s-1)t)$ and each entry of the table is on the left of or above that @.

`Case 1b`: $s/t < 5/3$. Table 5 shows C without a monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$. Note that there is an @′ mark instead of $C(1+(-t+3)s+(s-5)t)$ and each entry of the table is on the right of or below that @′.

`Case 2`: $s \equiv 0 \pmod 4$ and $t \equiv 1 \pmod 4$. We consider two subcases where $s/t \geq 5/3$ and $s/t < 5/3$, respectively. Table 6 or Table 7 shows C without a monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$ in each subcase.

`Case 3`: $s \equiv 1, 3 \pmod 4$ and $t \equiv 0 \pmod 4$. We also consider two subcases depending on whether $s/t \geq 5/3$ or not. In each subcase, the coloring we obtain from the one shown in Table 8 by replacing @′ with $C(1+(-t+3)s+(s-5)t) = 1$, or @ with $C(1 + (-t + 7)s + (s - 1)t) = 1$ contains no monochromatic homothetic copy of $\{1, 1 + s, 1 + s + t\}$. □

From Lemma 9 and Theorem 11, we conclude the following.

Theorem 12. *For each pair of positive integers* (s, t)*,*

$$f(s,t) = \begin{cases} 4s + 3t + 1, & s \equiv 0 \pmod{4t} \text{ or } t \equiv 0 \pmod{4s} \\ 4s + 4t + 1, & \text{otherwise.} \end{cases}$$

Remark 13. For $r \geq 2$, Brown et al. introduced the r-color van der Waerden's number $f^{(r)}(s, t)$ on homothetic copies of $\{1, 1 + s, 1 + s + t\}$ as the minimum integer N such that every r-coloring of $[1, N]$ has a monochromatic homothetic copy of $\{1, 1+s, 1+s+t\}$ [2]. The values of $f^{(3)}(1, 1)$ and $f^{(4)}(1, 1)$ are known [4]. The problem of finding $f^{(r)}(s, t)$ for $r \geq 5$ is still open.

Acknowledgements We are grateful to the anonymous referee for useful remarks and comments.

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Appendix

	i=-t-1	i=-t+3	i=-t+7			i=0	i=4
j=0					 1 0	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	0 0 1 1 1 0 0 1
		A ⋮	⋮	…	⋮	⋮	
j=s-8		0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	…	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	
j=s-4		0 0 1 1	0 0 1 1		0 0 1 1	0 0 1 1	
j=s-3	0 1 **B** 1 0	0 1 1 0 1 0 0 1 0 1 1 0	1 1 0 0 0 0 1 1 @	…	1 1 0 0	1 1 0 0	
j=s	0 0 1 1 1 0 1 0 0 1	1 0 0 1 0 1 1 1 0 0					

Table 4. $s \equiv 0 \pmod 4$, $t \equiv 3 \pmod 4$ and $s/t \geq 5/3$.

	i=-t+1	i=-t+3	i=-t+7			i=0	i=4
j=0						0 1 0 1 0 0 1 1	1 0 1 0 1
j=2					0 1	0 1 1 0 1 0 0 1	1 0 0
		1 0 0 1 1 0 0 1	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	…	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	
		⋮	⋮	…	⋮	⋮	
j=s-8 j=s-5		1 0 0 0 1 1 1 0 @′ 0 0 1	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	…	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1		
	 1 0 0 1	1 0 1 1 0 1 0 0 1 0 0 1 0 0 1 0	0 1 1 0 1 0 0 1 0 1	…			
j=s	0 1 1	1 1 0 1 1 0 1 0 1 0					

Table 5. $s \equiv 0 \pmod 4$, $t \equiv 3 \pmod 4$ and $s/t < 5/3$.

	i=-t+1	i=-t+5	i=-t+9			i=0	i=4
j=0					 0 1	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	0 1 0 1 0 0 1 1
j=4				…	1 1 0 0 0 1 1 1 0 0	0 0 1 0 1 1 0 1 0 0 1 1 1 0	
		⋮	⋮	…	⋮	⋮	
j=s-8	 0 1 0 0	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	0 0 1 1 1 0 0 1	…			
j=s-4	1 1	0 0 1 1					
	1 1 0 0 0 1 0 0 1 1	0 1 0 0 1 0 1 0 1 @		…			
j=s	1 1 0 0 0 0 1 1 1 1 0 0	1 0 0 1					

Table 6. $s \equiv 0 \pmod 4$, $t \equiv 1 \pmod 4$ and $s/t \geq 5/3$.

	i=-t+1	i=-t+5	i=-t+9			i=0	i=4
j=0						0 1 0 1 1 0 1 0	1 0 0 0 1
j=2					0 1	0 1 1 0 1 0 0 1	1 0 0
				…	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	
		⋮	⋮	…	⋮	⋮	
j=s-8 j=s-5	 1 @′ 0	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	0 1 1 0 1 0 0 1 0 1 1 1 0	…			
j=s-4	1 1 1 0 0 0 1 1 1 1 0 0	0 0 1 0 1 1 0 1 0 0 1 1 1 0	0				
j=s	1 0 1 1 0 1 0 0 1 1 0 1	0 0 1 0					

Table 7. $s \equiv 0\,(\mathrm{mod}\ 4)$, $t \equiv 1\,(\mathrm{mod}\ 4)$ and $s/t < 5/3$.

	i=-t+1	i=-t+4	i=-t+8		i=0	i=4
j=0				 0 1	0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1	0 1 0 1 0 0 1 1
j=4			…	1 1 1 0 0 0 0 1 1 1 1 0 0	0 0 1 0 1 1 0 1 0 0 1 1 1 1 0 0	
		…	…	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	
		⋮	…	⋮	⋮	
j=s-5	@′ 0 1 1 0 0	0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0	…			
j=s-1	0 0 1	1 0 1 @				
j=s	1 1 0 0 0 1 1 1 0	0 1 0 1 0 0 1		…		

Table 8. $s \equiv 1, 3\,(\mathrm{mod}\ 4)$ and $t \equiv 0\,(\mathrm{mod}\ 4)$.