A NOTE ON PARTITIONS OF NATURAL NUMBERS AND THEIR REPRESENTATION FUNCTIONS

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Abstract

Let $\mathbb{N}$ be the set of nonnegative integers. For a given set $A \subset \mathbb{N}$ the representation functions $R_2(A,n), R_3(A,n)$ are defined as the number of solutions of the equation $n = a + a', a, a' \in A$ with condition $a < a', a \le a'$, respectively. For $i = 2, 3$, are there subsets $A, B \subset \mathbb{N}$ with $R_i(A,n) = R_i(B,n)$ for all large enough integers n such that $\mathbb{N} = A \cup B$ and $A \cap B \neq \emptyset$? In this paper, we obtain some slightly weaker results in this direction.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For $A \subset \mathbb{N}$, let $A(n)$ be the counting function of set A and let $R_1(A,n)$, $R_2(A,n)$, $R_3(A,n)$ denote the number of solutions of

$$\begin{array}{rcllll} a + a' &=& n, & a,\ a' \in A, & \\ a + a' &=& n, & a,\ a' \in A, & a < a', \\ a + a' &=& n, & a,\ a' \in A, & a \le a', \end{array}$$

respectively. Sárközy asked ever whether there exist two sets A and B of nonnegative integers with infinite symmetric difference, i.e.

$$|(A \cup B) \setminus (A \cap B)| = \infty$$

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and

$$R_i(A,n) = R_i(B,n), \quad n \ge n_0$$

for $i = 1,2,3$. As Dombi [4] has shown, the answer is negative for $i = 1$ by the simple observation that $R_1(A,n)$ is odd if and only if $n = 2a$ for some $a \in A$, and positive for $i = 2$. For $i = 3$, Chen and Wang (see [3]) presented a partition of the set of all positive integers into two disjoint subsets A and B such that $R_3(A,n) = R_3(B,n)$ for all $n \ge n_0$.

Using generating functions, Lev [5] and independently Sándor [6] gave a simple common proof to the results of Dombi and of Chen and Wang. Sándor actually established the following two stronger results(which are also implicit in Lev's paper):

Theorem A. Let N be a positive integer. The equality $R_2(A,n) = R_2(\mathbb{N} \setminus A, n)$ holds for $n \ge 2N-1$ if and only if $A(2N-1) = N$ and $2m \in A \Leftrightarrow m \in A, 2m+1 \in A \Leftrightarrow m \notin A$ for $m \ge N$.

Theorem B. Let N be a positive integer. The equality $R_3(A,n) = R_3(\mathbb{N} \setminus A, n)$ holds for $n \ge 2N-1$ if and only if $A(2N-1) = N$ and $2m \in A \Leftrightarrow m \notin A, 2m+1 \in A \Leftrightarrow m \in A$ for $m \ge N$.

The second author of this paper [7] gave a more natural proof of Sándor's results. For the other related results, the reader is referred to see ([1],[2]).

It is natural to ask: for $i = 2,3$, are there subsets $A, B \subset \mathbb{N}$ with $R_i(A,n) = R_i(B,n)$ for all large enough integers n such that $\mathbb{N} = A \cup B$ and $A \cap B \ne \emptyset$? Noting that if $A \cap B = \{2k\}$, $A \cup B = \mathbb{N}$ and $R_2(A,n) = R_2(B,n)$ for $n \ge n_0$, then we have $R_2(A, 2k-1) = A(2k-1) - k$ and $R_2(B, 2k-1) = B(2k-1) - k$. Hence $A(2k-1) = B(2k-1)$ for $k \ge \frac{n_0+1}{2}$ and therefore $A(n) = B(n)$ for $n \ge n_0$, a contradiction.

In this paper, we obtain some slightly weaker results in this direction.

Theorem 1. *If $\mathbb{N} = A \cup B$ and $A \cap B = \{4k : k \in \mathbb{N}\}$, then $R_2(A,n) = R_2(B,n)$ cannot hold for all sufficiently large integers n.*

Theorem 2. *If $\mathbb{N} = A \cup B$ and $A \cap B = \{4k : k \in \mathbb{N}\}$, then $R_3(A,n) = R_3(B,n)$ cannot hold for all sufficiently large integers n.*

Remark 3. For $l = 1,2,3$, we can prove that if $\mathbb{N} = A \cup B$ and $A \cap B = \{4k + l : k \in \mathbb{N}\}$, then $R_i(A,n) = R_i(B,n)$ cannot hold for all sufficiently large integers n, where $i = 2,3$.

Currently, we cannot complete the following conjecture.

Conjecture 4. Let $m \in \mathbb{N}$ and $R \subset \{0, 1, \cdots, m-1\}$. If $\mathbb{N} = A \cup B$ and $A \cap B = \{r + km : k \in \mathbb{N}, r \in R\}$, then $R_i(A,n) = R_i(B,n)$ cannot hold for all sufficiently large integers n, where $i = 2,3$.

2. Proofs

The proofs are very similar, we only present here the proof of Theorem 1.

Proof. Suppose that there exist an integer n_0 and $A, B \subset \mathbb{N}$ with $A \cup B = \mathbb{N}$ and $A \cap B = \{4k : k \in \mathbb{N}\}$ such that $R_2(A, n) = R_2(B, n)$ for all $n \geq n_0$. Without loss of generality, we may assume that $n_0 = 8N - 1$, N is a positive integer; then there exists a polynomial $p(x)$ of degree at most $8N - 2$ such that

$$\sum_{n=0}^{\infty}(R_2(A, n) - R_2(B, n))x^n = p(x). \tag{1}$$

Let

$$f(x) = \sum_{a \in A} x^a, \quad g(x) = \sum_{b \in B} x^b$$

and

$$\varepsilon_i = \begin{cases} 1, & i \in A \\ 0, & i \notin A. \end{cases}$$

Then we have

$$f(x) = \sum_{a \in A} x^a = \sum_{i=0}^{\infty} \varepsilon_i x^i,$$

and

$$\sum_{n=0}^{\infty} R_2(A, n)x^n = \frac{1}{2}(f^2(x) - f(x^2)).$$

Moreover,

$$\begin{aligned} g(x) &= \sum_{b \in B} x^b = \sum_{b \in (\mathbb{N} \setminus A) \cup (A \cap B)} x^b \\ &= \sum_{n=0}^{\infty} x^n - f(x) + \frac{1}{1 - x^4} \\ &= \frac{2 + x + x^2 + x^3}{1 - x^4} - f(x), \end{aligned}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} R_2(B, n)x^n = \frac{1}{2}(g^2(x) - g(x^2)) \\ = \;& \frac{1}{2}\left(\left(\frac{2 + x + x^2 + x^3}{1 - x^4} - f(x)\right)^2 - \left(\frac{2 + x^2 + x^4 + x^6}{1 - x^8} - f(x^2)\right)\right) \\ = \;& -\frac{1}{1 - x^2} - \frac{x + 1}{1 - x^4} + \frac{x^4}{(1 - x^4)^2(1 + x^4)} + \frac{3}{(1 - x)(1 - x^4)} \\ & -f(x)\frac{2 + x + x^2 + x^3}{1 - x^4} + \frac{1}{2}f^2(x) + \frac{1}{2}f(x^2). \end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(R_2(A,n)-R_2(B,n))x^n \\
&=\frac{1}{1-x^2}+\frac{x+1}{1-x^4}-\frac{x^4}{(1-x^4)^2(1+x^4)} \\
&-\frac{3}{(1-x)(1-x^4)}+f(x)\frac{2+x+x^2+x^3}{1-x^4}-f(x^2).
\end{aligned} \tag{2}$$

Write

$$p(x)(1-x^4)=\sum_{i=0}^{8N+2}\alpha_i x^i.$$

By (1) and (2), we have

$$\begin{aligned}
&(2+x+x^2+x^3)f(x) \\
&=f(x^2)-x^4f(x^2)+\frac{x^4}{1-x^8}+\frac{3}{1-x} \\
&-x^2-x-2+p(x)(1-x^4).
\end{aligned} \tag{3}$$

Write both sides of (3) in power series, we have

$$\begin{aligned}
&\sum_{i=0}^{\infty}2\varepsilon_i x^i+\sum_{i=0}^{\infty}\varepsilon_i x^{i+1}+\sum_{i=0}^{\infty}\varepsilon_i x^{i+2}+\sum_{i=0}^{\infty}\varepsilon_i x^{i+3} \\
&=\sum_{i=0}^{\infty}\varepsilon_i x^{2i}-\sum_{i=0}^{\infty}\varepsilon_i x^{2i+4}+\sum_{i=0}^{\infty}x^{8i+4}+\sum_{i=0}^{\infty}3x^i \\
&-x^2-x-2+\sum_{i=0}^{8N+2}\alpha_i x^i.
\end{aligned} \tag{4}$$

Comparing the coefficients of x^{8N+3}, x^{8N+4} on the both sides of (4), we have

$$\begin{cases} 2\varepsilon_{8N+3}+\varepsilon_{8N+2}+\varepsilon_{8N+1}+\varepsilon_{8N}=3, \\ 2\varepsilon_{8N+4}+\varepsilon_{8N+3}+\varepsilon_{8N+2}+\varepsilon_{8N+1}=\varepsilon_{4N+2}-\varepsilon_{4N}+4. \end{cases}$$

Since $A\cap B=\{4k : k\in\mathbb{N}\}$, we have $\varepsilon_{4N}=\varepsilon_{8N}=\varepsilon_{8N+4}=1$, thus

$$\begin{cases} 2\varepsilon_{8N+3}+\varepsilon_{8N+2}+\varepsilon_{8N+1}=2, \\ \varepsilon_{8N+3}+\varepsilon_{8N+2}+\varepsilon_{8N+1}=\varepsilon_{4N+2}+1, \end{cases}$$

hence

$$\varepsilon_{8N+1}=0, \varepsilon_{8N+2}=0, \varepsilon_{8N+3}=1, \varepsilon_{4N+2}=0,$$

or

$$\varepsilon_{8N+1}=1, \varepsilon_{8N+2}=1, \varepsilon_{8N+3}=0, \varepsilon_{4N+2}=1.$$

Comparing the coefficients of x^{8N+5} on the both sides of (4), we have

$$2\varepsilon_{8N+5} + \varepsilon_{8N+3} + \varepsilon_{8N+2} = 2,$$

then $\varepsilon_{8N+2} = \varepsilon_{8N+3} = 0$ or $\varepsilon_{8N+2} = \varepsilon_{8N+3} = 1$, a contradiction.

This completes the proof of Theorem 1. $\square$

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