$(k+1)$-SUMS VERSUS k-SUMS

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Abstract

A k-sum of a set $A \subseteq \mathbb{Z}$ is an integer that may be expressed as a sum of k distinct elements of A. How large can the ratio of the number of $(k+1)$-sums to the number of k-sums be? Writing $k \wedge A$ for the set of k-sums of A we prove that

$$\frac{|(k+1) \wedge A|}{|k \wedge A|} \leq \frac{|A| - k}{k+1}$$

whenever $|A| \geq (k^2+7k)/2$. The inequality is tight – the above ratio being attained when A is a geometric progression. This answers a question of Ruzsa.

1. Introduction

Given a set $A = \{a_1, ..., a_n\}$ of n integers we denote by $k \wedge A$ the set of integers which may be represented as a sum of k distinct elements of A. In this paper we consider the problem of how large the ratio $|(k+1) \wedge A|/|k \wedge A|$ can be. The upper bound

$$\frac{|(k+1) \wedge A|}{|k \wedge A|} \leq \frac{n}{k+1} \tag{1}$$

is easily obtained using a straightforward double-counting argument.

Ruzsa [1] asked whether this inequality may be strengthened to

$$\frac{|(k+1) \wedge A|}{|k \wedge A|} \leq \frac{n-k}{k+1}$$

whenever n is large relative to k. We confirm that this is indeed the case.

Theorem 1.1. *Let A be a set of n integers and suppose that $n \geq (k^2+7k)/2$. Then*

$$\frac{|(k+1) \wedge A|}{|k \wedge A|} \leq \frac{n-k}{k+1} \ . \tag{2}$$

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Furthermore, in the case that $n > (k^2+7k)/2$, *equality holds if and only if* $|k \wedge A| = \binom{n}{k}$ *and* $|(k+1) \wedge A| = \binom{n}{k+1}$.

Since the ratio $(n-k)/(k+1)$ is obtained for all k in the case that A is a geometric progression this result is best possible for each pair k, n covered by the theorem. However, we do not believe that $n \geq (k^2+7k)/2$ is a necessary condition for inequality (2). Indeed, we pose the following question.

Question 1.2. Does (2) hold whenever $n > 2k$?

The inequality $n > 2k$ is necessary. Indeed, for any pair k, n with $n/2 \leq k \leq n-1$ the inequality (2) fails for the set $A = \{1, \dots, n\}$ (or indeed any arithmetic progression of length n). To see this note that $|k \wedge A| = k(n-k)+1$ for each $k = 1, \dots, n$, and that the inequality

$$\frac{(k+1)(n-k)+1}{k(n-k)+1} \leq \frac{n-k}{k+1}$$

holds if and only if $k \leq (n-1)/2$. Thus, we have also verified for the case that A is an arithmetic progression that (2) holds whenever $n > 2k$. We also note for any set $A \subseteq \mathbb{Z}$ that (2) holds trivially (and with equality) in the case that $k = (n-1)/2$. Indeed this follows immediately from the symmetry $|k \wedge A| = |(n-k) \wedge A|, k = 1, \dots, n-1$.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is closely related to the double-counting argument one uses to prove (1). We recall that argument now.

Fix $k \in \{0, \dots, n-1\}$ and a set $A = \{a_1, \dots, a_n\}$ of n integers. We say that an element $s \in k \wedge A$ *extends* to $t \in (k+1) \wedge A$ if there exist distinct elements $a_1, \dots, a_{k+1}$ of A such that $s = a_1 + \cdots + a_k$ and $t = a_1 + \cdots + a_{k+1}$. Define the bipartite graph G with vertex sets $U = \{u_s : s \in k \wedge A\}$ and $V = \{v_t : t \in (k+1) \wedge A\}$ and edge set $E(G) = \{u_s v_t : s \,\text{extends to}\, t\}$. We prove (1) by counting $e(G)$ in two different ways:

(i) $e(G) \leq n|U|$, since each vertex $u_s \in U$ has at most n neighbours in V.

(ii) $e(G) \geq (k+1)|V|$ since each vertex $v_t \in V$ is adjacent to each vertex $u_{t-a_i} : i = 1, \dots, k+1$, where $a_1 + \cdots + a_{k+1}$ is a $(k+1)$-sum to t.

Since $|U| = |k \wedge A|$ and $|V| = |(k+1) \wedge A|$ we obtain that

$$(k+1)|(k+1) \wedge A| \leq e(G) \leq n|k \wedge A|,$$

completing the proof of (1).

The alert reader will note that the extremal cases of each of (i) and (ii) occur in rather different situations. The inequality $e(G) \leq n|U|$ may be tight only if each element $s \in k \wedge A$ extends to $s+a$ for all $a \in A$. Equivalently, for each $s \in k \wedge A$ and $a \in A$, s may be represented as a k-sum that does not use a, i.e., $s = a_1 + \cdots + a_k$ for distinct $a_1, \ldots, a_k \in A \setminus \{a\}$. In particular, the inequality in (i) may be tight only if each k-sum has at least two representations. On the contrary, the second inequality $e(G) \geq (k+1)|V|$ may be tight only if each $t \in (k+1) \wedge A$ may be represented as a $(k+1)$-sum in a unique way. This simple observation is the key to our proof.

We put the above observations into action by defining $Q_k \subseteq k \wedge A$ to be the set of $s \in k \wedge A$ that have a unique representation as a k-sum and $S = (k \wedge A) \setminus Q_k$ to be the set of s with at least two representations. We immediately obtain a new upper bound on $e(G)$, namely:

$$e(G) \;\leq\; (n-k)|U_k| + n|S| \;=\; (n-k)|k \wedge A| + k|S|\,. \tag{3}$$

Correspondingly, one may define Q_{k+1} to be the set of $t \in (k+1) \wedge A$ that are uniquely represented as a $(k+1)$-sum and $T = ((k+1) \wedge A) \setminus Q_{k+1}$ to be the set of t with at least two representations. It then follows (using Lemma 2.2 below) that

$$e(G) \;\geq\; (k+1)|U_{k+1}| + (k+3)|T| \;= (k+1)|(k+1) \wedge A| + 2|T|\,. \tag{4}$$

Unfortunately (3) and (4) do not directly imply Theroem 1.1 since it is non-trivial to relate $|S|$ and $|T|$. For this reason we define a subgraph H of G as follows. Recall that a pair $u_s v_t$ is an edge of G if there exists a representation $s = a_1 + \cdots + a_k$ of s as a k-sum of elements of A and $a \in A \setminus \{a_1, \ldots, a_k\}$ such that $t = s + a$. Include an edge $u_s v_t$ of G in H if and only if there exist two representations $s = a_1 + \cdots + a_k = b_1 + \cdots + b_k$ of s as a k-sum of elements of A and $a \in A \setminus (\{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\})$ such that $s + a = t$. (Note: if an edge $u_s v_t$ of G is included in H then in particular $s \in S$ and $t \in T$.)

We begin with two lemmas.

Lemma 2.1. *Let the set $S \subseteq k \wedge A$ and the graph $H \subseteq G$ be as defined above. Then $e(H) \geq (n-2k)|S|$.*

Proof. For each $s \in S$ the vertex u_s has degree at least $n - 2k$ in H. Indeed, writing $s = a_1 + \cdots + a_k = b_1 + \cdots + b_k$ we have that $u_s v_t \in E(H)$ for each $t \in \{s + a : a \in A \setminus (\{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\})\}$. □

Lemma 2.2. *Let the set $T \subseteq (k+1) \wedge A$ be as defined above. Then $d_G(v_t) \geq k+3$ for all $t \in T$.*

Proof. An element $t \in T$ has at least two representations $t = a_1 + \cdots + a_{k+1} = b_1 + \cdots + b_{k+1}$ as a $(k+1)$-sum of elements of A. Furthermore the sets $\{a_1, \ldots, a_{k+1}\}$ and

$\{b_1, \ldots, b_{k+1}\}$ cannot have precisely k common elements (as in that case they would have different sums). It follows that the set $B = \{a_1, \ldots, a_{k+1}\} \cup \{b_1, \ldots, b_{k+1}\}$ has cardinality at least $k+3$. The proof is now complete since $u_s v_t$ is an edge of G for each $s \in \{t - b : b \in B\}$. $\square$

Combining Lemma 2.2 with the trivial bound $d_G(v_t) \geq d_H(v_t)$ for each $t \in T$, we deduce that

$$d_G(v_t) \;\geq\; \frac{k+1}{k+3}(k+3) + \frac{2}{k+3} d_H(v_t) \;=\; k+1+\frac{2d_H(v_t)}{k+3}\,. \tag{5}$$

Consequently,

$$e(G) \;\geq\; (k+1)|Q_{k+1}| + \sum_{t\in T}\left((k+1)+\frac{2d_H(v_t)}{k+3}\right) \;=\; (k+1)|(k+1)\wedge A| + \frac{2e(H)}{k+3}\,.$$

The proof of the theorem is now nearly complete. Indeed, applying Lemma 2.1 we obtain the bound

$$e(G) \;\geq\; (k+1)|(k+1)\wedge A| + \frac{2(n-2k)|S|}{k+3}\,,$$

which combined with (3) yields

$$(k+1)|(k+1)\wedge A| + \frac{2(n-2k)|S|}{k+3} \leq (n-k)|k\wedge A| + k|S|\,. \tag{6}$$

Now, since $n \geq (k^2+7k)/2$ the second term on the left hand side is at least the second term on the right hand side. Thus, $(k+1)|(k+1)\wedge A| \leq (n-k)|k \wedge A|$, completing the proof of the inequality stated in Theorem 1.1.

In the case that $n > (k^2+7k)/2$, it follows from (6) that the equality $(k+1)|(k+1)\wedge A| = (n-k)|k\wedge A|$ may only occur if $|S| = 0$. In this case the k-sums of A are distinct, and so $|k\wedge A| = \binom{n}{k}$ and $|(k+1)\wedge A| = (n-k)|k\wedge A|/(k+1) = \binom{n}{k+1}$. This completes the proof of Theorem 1.1.

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References

[1] I. Ruzsa, Open problem included as Problem 30 (page 202) of the following reference.

[2] A. Geroldinger and I. Ruzsa, Combinatorial Number Theory and Additive Group Theory, *Advanced Courses in Mathematics - CRM Barcelona*, Birkhäuser Basel, 2008.