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ON THE VALUE DISTRIBUTION OF ERROR SUMS FOR APPROXIMATIONS WITH RATIONAL NUMBERS

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Abstract

Let α be a real number with convergents p_m/q_m from the continued fraction expansion of α. In this paper we investigate the functions $\mathcal{E}(\alpha) := \sum_{m\geq 0} |\alpha q_m - p_m|$ and $\mathcal{E}^*(\alpha) := \sum_{m\geq 0}(\alpha q_m - p_m)$ depending only on α and prove that they take every value in $[0,(1+\sqrt{5})/2]$ and $[0,1]$, respectively. For any sequence $(\alpha_\mu)_{\mu\geq 1}$, which is uniformly distributed modulo 1, we show that both sequences $(\mathcal{E}(\alpha_\mu))_{\mu\geq 1}$ and $(\mathcal{E}^*(\alpha_\mu))_{\mu\geq 1}$ are not uniformly distributed. Among other things the proofs rely on an inequality for the function $\mathcal{E}(\alpha)$, which improves a former result of the first named author.

1. Introduction

For any real number α and its regular continued fraction expansion

$$\begin{aligned} \alpha &= \langle a_0; a_1, \ldots, a_n \rangle, && (\alpha \in \mathbb{Q} \setminus \mathbb{Z}), \\ \alpha &= \langle a_0; a_1, \ldots \rangle, && (\alpha \in \mathbb{R} \setminus \mathbb{Q}), \end{aligned}$$

where $a_0 \in \mathbb{Z}, a_\nu \in \mathbb{N}$ for $\nu \geq 1, a_n > 1$, we investigate the sums

$$\mathcal{E}(\alpha) := \mathcal{E}(a_1, a_2, \ldots) := \sum_{m\geq 0} |\alpha q_m - p_m| \tag{1}$$

and

$$\mathcal{E}^*(\alpha) := \mathcal{E}^*(a_1, a_2, \ldots) := \sum_{m\geq 0} (\alpha q_m - p_m). \tag{2}$$

Moreover, let $\mathcal{E}(\alpha) = \mathcal{E}^*(\alpha) = 0$ for $\alpha \in \mathbb{Z}$. Here, p_m/q_m denotes the m-th convergent of α. In case of $\alpha \in \mathbb{Q}$ these functions are finite sums, since α has a finite continued fraction expansion. Conversely, for a finite sequence $a_0, a_1, \ldots, a_{n-1}, 1$ ending with 1 we define $\mathcal{E}(a_1, \ldots, a_{n-1}, 1)$ and $\mathcal{E}^*(a_1, \ldots, a_{n-1}, 1)$ by

$$\mathcal{E}(a_1, \ldots, a_{n-1}, 1) := \mathcal{E}(a_1, \ldots, a_{n-1}+1) + |\alpha q_{n-1} - p_{n-1}|\,, \tag{3}$$

$$\mathcal{E}^*(a_1, \ldots, a_{n-1}, 1) := \mathcal{E}^*(a_1, \ldots, a_{n-1}+1) + (-1)^{n-1}|\alpha q_{n-1} - p_{n-1}| \tag{4}$$

with $p_{n-1}/q_{n-1} = \langle a_0; a_1, \ldots, a_{n-1}\rangle$ and $\alpha = \langle a_0; a_1, \ldots, a_{n-1}+1\rangle$. The additional term $|\alpha q_{n-1} - p_{n-1}|$ in (3) and (4) plays an essential role for the inequalities of error sums stated below in Lemma 8 and Lemma 12, respectively. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the error sums become infinite series converging absolutely. Set

$$\rho := \frac{1+\sqrt{5}}{2}\,, \qquad \tilde{\rho} := \frac{1-\sqrt{5}}{2}\,,$$

and let

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0)$$

denote the Fibonacci numbers.

The main focus in this paper relies on the function $\mathcal{E}(\alpha)$. Generally speaking, $\mathcal{E}(\alpha)$ is a measure of quality for the approximation of a real number α by convergents with small denominators. For more applications of $\mathcal{E}(\alpha)$ see [1], where the first named author has also proven that for any $\alpha \in \mathbb{R}$ the inequalities

$$0 \leq \mathcal{E}(\alpha) \leq \rho, \tag{5}$$

$$0 \leq \mathcal{E}^*(\alpha) \leq 1 \tag{6}$$

hold. We are now interested in a more detailed investigation of the value distribution of $\mathcal{E}(\alpha)$ and $\mathcal{E}^*(\alpha)$ in the intervals given by (5) and (6).

Proposition 1. *Let $n \in \mathbb{N}$ and let $a_1, a_2, \ldots$ be positive integers. Then we have*

$$\mathcal{E}(a_1, \ldots, a_n, \ldots) \leq \mathcal{E}(a_1, \ldots, a_n, 1, 1, \ldots).$$

Since $\mathcal{E}(\alpha) = \rho$ if and only if $\alpha \equiv \rho \pmod 1$ (see [1]), this proposition improves the inequality (5) effectively in case $a_1 \cdots a_n > 1$. The main results in this paper concerning the value distribution of the error sums $\mathcal{E}(\alpha)$ and $\mathcal{E}^*(\alpha)$ are given by the subsequent Theorems 2 to 5. As usual we write

$$\mathcal{E}(\mathbb{R}) := \{\mathcal{E}(\alpha)\,|\,\alpha \in \mathbb{R}\} \qquad \text{and} \qquad \mathcal{E}^*(\mathbb{R}) := \{\mathcal{E}^*(\alpha)\,|\,\alpha \in \mathbb{R}\}.$$

Theorem 2. *We have $\mathcal{E}(\mathbb{R}) = [0, \rho]$.*

Theorem 3. *We have $\mathcal{E}^*(\mathbb{R}) = [0, 1]$.*

The result of Theorem 3 is already known: By using the concept of mediants, J.N. Ridley and G. Petruska [4] proved that for every $0 < y < 1$ there exists an irrational number x such that $\mathcal{E}^*(x) = y$. Our proof of Theorem 3 is based on an algorithmic construction similar to the proof of Theorem 2. For this we use auxiliary lemmas which describe the local behaviour of the error sum functions.

In contrast to the above density results we also considered an error sum related to $\mathcal{E}(\alpha)$, defined by

$$\mathcal{E}_2(\alpha) := \sum_{m\geq 0} (\alpha q_m - p_m)^2 .$$

Both sums, $\mathcal{E}$ and $\mathcal{E}_2$, neglect the sign of the error terms in a simple way. But the value distribution of $\mathcal{E}_2$ differs essentially from that one of $\mathcal{E}$. In particular, there are subintervals of $[0, 1]$ where the values of $\mathcal{E}_2$ are not dense. We can show for $\alpha \in \mathbb{R}$ that

$$\mathcal{E}_2(\alpha) \notin \left(\frac{1}{4}, \frac{1}{2}\right) .$$

Let $\alpha_n := \langle 0; 1, 1, n\rangle$, $(n > 1)$. We have $\alpha_n \to 1/2$ for $n \to \infty$, and

$$\mathcal{E}_2(\alpha_n) > \frac{1}{2} \qquad \text{whereas} \qquad \mathcal{E}_2\left(\frac{1}{2}\right) = \frac{1}{4} .$$

In general, error sums are discontinuous functions.

Next, one may ask whether the values of $\mathcal{E}(\alpha)$ (of $\mathcal{E}^*(\alpha)$, respectively) are *uniformly distributed* in $[0, \rho]$ (in $[0, 1]$, respectively). The negative answer is given by the following theorems. For this purpose let $J \subseteq [0, \rho]$, $(\alpha_\mu)_{\mu\geq 1}$, be a sequence of real numbers, and

$$A(J, M) := \#\{1 \leq m \leq M \,:\, \mathcal{E}(\alpha_m) \in J\} \qquad (M \in \mathbb{N}),$$
$$A^*(J, M) := \#\{1 \leq m \leq M \,:\, \mathcal{E}^*(\alpha_m) \in J\} \qquad (M \in \mathbb{N}).$$

Theorem 4. *Let $(\alpha_\mu)_{\mu\geq 1}$ be a sequence of real numbers, which is uniformly distributed modulo one. For $N \in \mathbb{N}$ let $J_1 = (1, 1 + \rho^2/N)$ and $J_2 = (1 - \rho^2/N, 1)$. Then we have*

$$\liminf_{M\to\infty} \frac{A(J_1, M)}{M} \geq \frac{\log N}{30N} \qquad (N \in \mathbb{N}) ,$$

and

$$\limsup_{M\to\infty} \frac{A(J_2, M)}{M} \leq \frac{16\rho^4}{N^2} \qquad (N \in \mathbb{N},\, N \geq 32) .$$

This shows that there are more points $\mathcal{E}(\alpha)$ in J_1 than we would expect in the case of uniform distribution, and too little points in J_2. This is because of

$$\frac{|J_1|}{\rho} = \frac{\rho}{N} < \frac{\log N}{30N} \qquad \text{for} \quad N > \exp(30\rho) ,$$

and

$$\frac{|J_2|}{\rho} = \frac{\rho}{N} > \frac{16\rho^4}{N^2} \qquad \text{for } N \geq 68 .$$

Theorem 5. *Let* $(\alpha_\mu)_{\mu\geq 1}$ *be a sequence of real numbers, which is uniformly distributed modulo one. For* $N \geq 3$ *let* $J_3 = (1-1/N, 1]$. *Then we have*

$$\limsup_{M\to\infty} \frac{A^*(J_3, M)}{M} < \frac{5}{6N} + \frac{1}{N^2}.$$

In particular, this is less than $1/N$ *for* $N \geq 6$.

For the proof of Theorem 4 we need the inequality from Proposition 1. Therefore, we shall prove Proposition 1 in Section 4 separately. The proofs of Theorem 4 and Theorem 5 are given in the final Section 5. The appendix contains four plots illustrating the functions $\mathcal{E}$ and $\mathcal{E}^*$. Figure 1 and Figure 2 show the graphs of $\mathcal{E}$ and $\mathcal{E}^*$, respectively. To illustrate the value distribution of $\mathcal{E}$ and $\mathcal{E}^*$, we use 50 000 at random generated numbers $x_1, \ldots, x_{50000} \in [0,1]$ and plot the points $(i, \mathcal{E}(x_i))$ (Figure 3) and $(i, \mathcal{E}^*(x_i))$ (Figure 4) for $i = 1, \ldots, 50\,000$. The plots were computed using a standard computer algebra system. The value distribution of the error sums seems to be a little mystic due to some visible lines inside the plots. We could not prove a general result explaining the existence of these lines.

2. Proof of Theorem 2

2.1. Auxiliary Lemmas

In the following let n and $a_0, a_1, \ldots, a_n$ denote positive integers.

Lemma 6. *Let* $N \in \mathbb{N}$. *In the case of* $n = 1$, *further let* $a_1 > 1$. *Then, with* $\langle a_0; a_1, \ldots a_n\rangle = p_n/q_n$, *we have*

$$0 < \mathcal{E}(a_1, \ldots, a_n, N) - \mathcal{E}(a_1, \ldots, a_n) < \frac{1}{Nq_n}.$$

In particular we get the limits

$$\begin{aligned} \lim_{n\to\infty} \big(\mathcal{E}(a_1, \ldots, a_n, N) - \mathcal{E}(a_1, \ldots, a_n)\big) &= 0 \qquad (N \in \mathbb{N}), \\ \lim_{N\to\infty} \big(\mathcal{E}(a_1, \ldots, a_n, N) - \mathcal{E}(a_1, \ldots, a_n)\big) &= 0 \qquad (n \in \mathbb{N}). \end{aligned}$$

Proof. Let

$$\begin{aligned} \beta &:= \langle a_0; a_1, \ldots a_n\rangle, \\ \gamma &:= \langle a_0; a_1, \ldots a_n, N\rangle, \\ \frac{p_\nu}{q_\nu} &:= \langle a_0; a_1, \ldots a_\nu\rangle \qquad (0 \leq \nu \leq n). \end{aligned}$$

Then we have the identities

$$\beta = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}, \qquad \gamma = \frac{N p_n + p_{n-1}}{N q_n + q_{n-1}},$$

and

$$\begin{aligned}\mathcal{E} := \mathcal{E}(a_1, \ldots, a_n, N) - \mathcal{E}(a_1, \ldots, a_n) &= \sum_{\nu=0}^{n} (-1)^\nu (\gamma q_\nu - p_\nu) - \sum_{\nu=0}^{n-1} (-1)^\nu (\beta q_\nu - p_\nu) \\ = (-1)^n (\gamma q_n - p_n) + \sum_{\nu=0}^{n-1} (-1)^\nu (\gamma - \beta) q_\nu .\end{aligned}$$

With the above identities for β and γ we obtain

$$\gamma - \beta = \frac{N p_n + p_{n-1}}{N q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(N q_n + q_{n-1})},$$

$$\gamma q_n - p_n = q_n(\gamma - \beta) = \frac{(-1)^n}{N q_n + q_{n-1}},$$

and therefore

$$\mathcal{E} = \frac{1}{N q_n + q_{n-1}} + \frac{1}{q_n(N q_n + q_{n-1})} \sum_{\nu=0}^{n-1} (-1)^{n+\nu} q_\nu . \tag{7}$$

To estimate the sum

$$S := \sum_{\nu=0}^{n-1} (-1)^{n+\nu} q_\nu$$

we need to distinguish two cases according to the parity of n. For even n (with $n \geq 2$) we have

$$S = (q_0 - q_1) + (q_2 - q_3) + \cdots + (q_{n-2} - q_{n-1}) \leq 0,$$

and

$$S = q_0 + (q_2 - q_1) + (q_4 - q_3) + \cdots + (q_{n-2} - q_{n-3}) - q_{n-1} \geq -q_{n-1}.$$

For any odd n (with $n \geq 1$ by the assumption of the lemma) we get

$$S = -q_0 + (q_1 - q_2) + (q_3 - q_4) + \cdots + (q_{n-2} - q_{n-1}) \leq 0,$$

and

$$S = (q_1 - q_0) + (q_3 - q_2) + \cdots + (q_{n-2} - q_{n-3}) - q_{n-1} \geq -q_{n-1}.$$

The result of the distinction of cases is $-q_{n-1} \le S \le 0$. Hence we obtain from (7), regarding $n \ge 1$ and $q_{n-1} \ge q_0 = 1$,

$$\mathcal{E} \le \frac{1}{Nq_n + q_{n-1}} < \frac{1}{Nq_n}$$

and

$$\mathcal{E} \ge \frac{1}{Nq_n + q_{n-1}} - \frac{q_{n-1}}{q_n(Nq_n + q_{n-1})} = \frac{1}{Nq_n + q_{n-1}} \left(1 - \frac{q_{n-1}}{q_n}\right) > 0.$$

This proves the lemma. □

Lemma 7. *Let $b, c, n \in \mathbb{N}$, where $n \ge 2$ and $1 \le b < c$. Then, with $\langle a_0; a_1, \dots a_n\rangle = p_n/q_n$, we have*

$$0 \, < \, \mathcal{E}(a_1, \dots, a_n, b) - \mathcal{E}(a_1, \dots, a_n, c) \le \frac{c-b}{bcq_n}.$$

Proof. Let

$$\begin{aligned} \beta &:= \langle a_0; a_1, \dots a_n, b\rangle, \\ \gamma &:= \langle a_0; a_1, \dots a_n, c\rangle, \\ \frac{p_\nu}{q_\nu} &:= \langle a_0; a_1, \dots a_\nu\rangle \qquad (0 \le \nu \le n). \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{E} := \ & \mathcal{E}(a_1, \dots, a_n, b) - \mathcal{E}(a_1, \dots, a_n, c) = \sum_{\nu=0}^{n} (-1)^\nu (\beta q_\nu - p_\nu) - \sum_{\nu=0}^{n} (-1)^\nu (\gamma q_\nu - p_\nu) \\ &= (\beta - \gamma) \sum_{\nu=0}^{n} (-1)^\nu q_\nu. \end{aligned}$$

With

$$\beta = \frac{bp_n + p_{n-1}}{bq_n + q_{n-1}} \qquad \text{and} \qquad \gamma = \frac{cp_n + p_{n-1}}{cq_n + q_{n-1}}$$

we conclude that

$$\beta - \gamma = \frac{(b-c)(-1)^{n-1}}{(bq_n + q_{n-1})(cq_n + q_{n-1})}$$

and

$$\mathcal{E} = \frac{c-b}{(bq_n + q_{n-1})(cq_n + q_{n-1})} \sum_{\nu=0}^{n} (-1)^{n+\nu} q_\nu. \tag{8}$$

By similar arguments as used in the proof of Lemma 6, we obtain the bounds

$$1 \le \sum_{\nu=0}^{n} (-1)^{n+\nu} q_\nu \le q_n$$

for the alternating sum of the q_ν. This leads to

$$0 < \mathcal{E} \leq \frac{(c-b)q_n}{(bq_n + q_{n-1})(cq_n + q_{n-1})} \leq \frac{(c-b)q_n}{bcq_n^2} = \frac{c-b}{bcq_n}.$$

Hence, the lemma is proven. $\square$

Lemma 8. *Let $a_n \geq 3$. Then we have*

$$\mathcal{E}(a_1, \dots, a_{n-1}, a_n - 1) \leq \mathcal{E}(a_1, \dots, a_n, 1).$$

Proof. Replacing n by $n-1$ and setting $b = a_n - 1$, $c = a_n + 1$ in (8), we obtain

$$\begin{aligned} &\mathcal{E}(a_1, \dots, a_{n-1}, a_n - 1) - \mathcal{E}(a_1, \dots, a_{n-1}, a_n + 1) \\ &\leq \frac{2}{\big((a_n - 1)q_{n-1} + q_{n-2}\big)\big((a_n + 1)q_{n-1} + q_{n-2}\big)} \sum_{\nu=0}^{n-1} (-1)^{n+\nu-1} q_\nu, \end{aligned} \tag{9}$$

where $p_\nu / q_\nu = \langle a_0; a_1, \dots, a_\nu \rangle$ for $0 \leq \nu \leq n$. Now let

$$\gamma := \langle a_0; a_1, \dots, a_n, 1 \rangle = \langle a_0; a_1, \dots, a_n + 1 \rangle.$$

Substituting (3) into (9) with $n-1$ replaced by n, we get

$$\begin{aligned} &\mathcal{E}(a_1, \dots, a_{n-1}, a_n - 1) \leq \mathcal{E}(a_1, \dots, a_{n-1}, a_n, 1) + \\ &+ \frac{2}{\big((a_n - 1)q_{n-1} + q_{n-2}\big)\big((a_n + 1)q_{n-1} + q_{n-2}\big)} \sum_{\nu=0}^{n-1} (-1)^{n+\nu-1} q_\nu - |\gamma q_n - p_n|. \end{aligned}$$

By the expression

$$\gamma = \langle a_0; a_1, \dots, a_n + 1 \rangle = \frac{(a_n + 1)p_{n-1} + p_{n-2}}{(a_n + 1)q_{n-1} + q_{n-2}}$$

we compute

$$|\gamma q_n - p_n| = \frac{1}{(a_n + 1)q_{n-1} + q_{n-2}}$$

and obtain

$$\begin{aligned} &\mathcal{E}(a_1, \dots, a_{n-1}, a_n - 1) \\ &\qquad \leq \mathcal{E}(a_1, \dots, a_{n-1}, a_n, 1) - \frac{\big((a_n - 1)q_{n-1} + q_{n-2}\big) - 2\sum_{\nu=0}^{n-1} (-1)^{n+\nu-1} q_\nu}{\big((a_n - 1)q_{n-1} + q_{n-2}\big)\big((a_n + 1)q_{n-1} + q_{n-2}\big)}. \end{aligned}$$

By similar arguments as in the proofs of the two preceding lemmas and by using the conditions $a_n \geq 3$ and $n \geq 2$, we get

$$2 \sum_{\nu=0}^{n-1} (-1)^{n+\nu-1} q_\nu \leq (a_n - 1)q_{n-1} + q_{n-2},$$

which yields

$$\mathcal{E}(a_1,\dots,a_{n-1},a_n-1) \le \mathcal{E}(a_1,\dots,a_{n-1},a_n,1).$$

Therefore, Lemma 8 is proven. □

Lemma 9. *Let $n \ge 2$. Then there is a positive integer k such that the inequality*

$$\mathcal{E}(a_1,\dots,a_{n-1},2,\underbrace{1,1\dots,1}_{k}) > \mathcal{E}(a_1,\dots,a_{n-1},1)$$

holds.

Proof. Let

$$\begin{aligned}\beta &:= \langle a_0;a_1,\dots,a_{n-1},1\rangle = \langle a_0;a_1,\dots,a_{n-1}+1\rangle,\\ \delta &:= \langle a_0;a_1,\dots,a_{n-1},2,\underbrace{1,1\dots,1}_{k}\rangle,\end{aligned}$$

and let p_ν/q_ν for $0 \le \nu \le n+k$ be the convergents of δ. We express β,γ, and δ by

$$\begin{aligned}\beta &= \frac{(a_{n-1}+1)p_{n-2}+p_{n-3}}{(a_{n-1}+1)q_{n-2}+q_{n-3}} = \frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}},\\ \delta &= \frac{p_{n+k}}{q_{n+k}}.\end{aligned}$$

By induction one proves the formulas

$$p_{n+\nu} = F_{\nu+1}p_n + F_\nu p_{n-1} \qquad \text{and} \qquad q_{n+\nu} = F_{\nu+1}q_n + F_\nu q_{n-1}, \qquad (1 \le \nu \le k).$$

Hence, we get the following error sums:

$$\mathcal{E}(a_1,\dots,a_{n-1},1) = \sum_{m=0}^{n-1}(-1)^m(\beta q_m - p_m) = \sum_{m=0}^{n-1}(-1)^m\left(\frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}}q_m - p_m\right),$$

$$\begin{aligned}\mathcal{E}(a_1,\dots,a_{n-1},2,\underbrace{1,1\dots,1}_{k}) &= \sum_{m=0}^{n+k-1}(-1)^m(\delta q_m - p_m)\\ &= \sum_{m=0}^{n+k-1}(-1)^m\left(\frac{p_{n+k}}{q_{n+k}}q_m - p_m\right)\\ &= \mathcal{E}_1(k) + \mathcal{E}_2(k)\end{aligned}$$

with

$$\mathcal{E}_1(k) = \sum_{m=0}^{n}(-1)^m\left(\frac{F_{k+1}p_n + F_k p_{n-1}}{F_{k+1}q_n + F_k q_{n-1}}q_m - p_m\right)$$

and

$$\begin{aligned}\mathcal{E}_2(k) &= \sum_{m=n+1}^{n+k-1} (-1)^m \\ &\times \left(\frac{F_{k+1}p_n + F_k p_{n-1}}{F_{k+1}q_n + F_k q_{n-1}} (F_{m-n+1}q_n + F_{m-n}q_{n-1}) - (F_{m-n+1}p_n + F_{m-n}p_{n-1}) \right).\end{aligned}$$

Thus, we intend to prove the existence of a positive integer k satisfying

$$\mathcal{E}_1(k) + \mathcal{E}_2(k) - \sum_{m=0}^{n-1} (-1)^m \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} q_m - p_m \right) \geq 0. \tag{10}$$

Using the identities

$$\sum_{m=n+1}^{n+k-1} (-1)^m F_{m-n+1} = (-1)^{n+1} \sum_{m=2}^{k} (-1)^m F_m = (-1)^{n+k-1} F_{k-1}$$

and

$$\sum_{m=n+1}^{n+k-1} (-1)^m F_{m-n} = (-1)^{n+1} \sum_{m=2}^{k} (-1)^m F_{m-1} = (-1)^{n+k-1} F_{k-2} + (-1)^{n+1},$$

we get the following expression for $\mathcal{E}_2(k)$:

$$\begin{aligned}\mathcal{E}_2(k) &= (-1)^{n+k-1} \Bigg[\frac{F_{k+1}p_n + F_k p_{n-1}}{F_{k+1}q_n + F_k q_{n-1}} \left(F_{k-1}q_n + F_{k-2}q_{n-1} + (-1)^k q_{n-1} \right) \\ &\quad - \left(F_{k-1}p_n + F_{k-2}p_{n-1} + (-1)^k p_{n-1} \right) \Bigg] \\ &= (-1)^{n+k-1} \frac{\left(F_{k-2}F_{k+1} + (-1)^k F_{k+1} - F_{k-1}F_k \right) (p_n q_{n-1} - p_{n-1}q_n)}{F_{k+1}q_n + F_k q_{n-1}} \\ &= (-1)^k \frac{F_{k-2}F_{k+1} - F_{k-1}F_k + (-1)^k F_{k+1}}{F_{k+1}q_n + F_k q_{n-1}} \\ &= (-1)^k \frac{(-1)^{k-1} + (-1)^k F_{k+1}}{F_{k+1}q_n + F_k q_{n-1}} \\ &= \frac{F_{k+1} - 1}{F_{k+1}q_n + F_k q_{n-1}} \ .\end{aligned}$$

With $p_n = 2p_{n-1} + p_{n-2}$ and $q_n = 2q_{n-1} + q_{n-2}$ we obtain

$$\begin{aligned}
&\mathcal{E}_1(k) + \mathcal{E}_2(k) \\
&= \sum_{m=0}^{n} (-1)^m \left(\frac{F_{k+1}(2p_{n-1} + p_{n-2}) + F_k p_{n-1}}{F_{k+1}(2q_{n-1} + q_{n-2}) + F_k q_{n-1}} q_m - p_m \right) \\
&\quad + \frac{F_{k+1} - 1}{F_{k+1}(2q_{n-1} + q_{n-2}) + F_k q_{n-1}} \\
&= \sum_{m=0}^{n} (-1)^m \left(\frac{F_{k+3} p_{n-1} + F_{k+1} p_{n-2}}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} q_m - p_m \right) + \frac{F_{k+1} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \\
&= (-1)^n \left(\frac{(F_{k+3} p_{n-1} + F_{k+1} p_{n-2})(2q_{n-1} + q_{n-2})}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} - (2p_{n-1} + p_{n-2}) \right) \\
&\quad + \sum_{m=0}^{n-1} (-1)^m \left(\frac{F_{k+3} p_{n-1} + F_{k+1} p_{n-2}}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} q_m - p_m \right) + \frac{F_{k+1} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \\
&= (-1)^n \frac{(F_{k+3} - 2F_{k+1})(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \\
&\quad + \sum_{m=0}^{n-1} (-1)^m \left(\frac{F_{k+3} p_{n-1} + F_{k+1} p_{n-2}}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} q_m - p_m \right) + \frac{F_{k+1} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \\
&= \sum_{m=0}^{n-1} (-1)^m \left(\frac{F_{k+3} p_{n-1} + F_{k+1} p_{n-2}}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} q_m - p_m \right) + \frac{F_{k+2} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \, .
\end{aligned}$$

This can be used to express the left-hand side of (10):

$$\begin{aligned}
&\mathcal{E}_1(k) + \mathcal{E}_2(k) - \sum_{m=0}^{n-1} (-1)^m \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} q_m - p_m \right) \\
&= \sum_{m=0}^{n-1} (-1)^m \left(\frac{F_{k+3} p_{n-1} + F_{k+1} p_{n-2}}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} - \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \right) q_m + \frac{F_{k+2} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \\
&= \sum_{m=0}^{n-1} (-1)^m \frac{(F_{k+3} - F_{k+1})(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})}{(F_{k+3} q_{n-1} + F_{k+1} q_{n-2})(q_{n-1} + q_{n-2})} q_m + \frac{F_{k+2} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \\
&= \frac{F_{k+2}}{(F_{k+3} q_{n-1} + F_{k+1} q_{n-2})(q_{n-1} + q_{n-2})} \sum_{m=0}^{n-1} (-1)^{n+m} q_m + \frac{F_{k+2} - 1}{F_{k+3} q_{n-1} + F_{k+1} q_{n-2}} \, .
\end{aligned}$$

To prove (10) for some $k \geq 1$ it is sufficient to show that

$$F_{k+2} \left(1 + \frac{1}{q_{n-1} + q_{n-2}} \sum_{m=0}^{n-1} (-1)^{n+m} q_m \right) \geq 1. \tag{11}$$

From the proof of Lemma 6 we know that

$$-q_{n-1} \leq \sum_{m=0}^{n-1} (-1)^{n+m} q_m \leq 0.$$

By the condition $n \geq 2$ we have $q_{n-2} \geq q_0 = 1$, which gives

$$0 < 1 + \frac{1}{q_{n-1} + q_{n-2}} \sum_{m=0}^{n-1} (-1)^{n+m} q_m \leq 1.$$

Thus, for any large positive integer k, the inequality (11) holds. Moreover, the smallest k satisfying (11) can be computed effectively. This completes the proof of Lemma 9. ☐

Lemma 10. *Let M be a positive integer with $M \geq 3$. Then there is a positive integer k such that the inequality*

$$\mathcal{E}(M, \underbrace{1, 1 \ldots, 1}_{k}) \geq \frac{2}{M}$$

holds. For $M = 2$ we have

$$\mathcal{E}(2, \underbrace{1, 1 \ldots, 1}_{k}) = 1 - \frac{1}{F_{k+3}}.$$

Proof. Let $\beta = \langle 0; M, \underbrace{1, 1 \ldots, 1}_{k} \rangle$. By p_ν / q_ν we denote the convergents of β given by

$$\begin{array}{lllll} p_{-1} = 1, & p_0 = a_0, & p_1 = 1, & p_\nu = F_\nu & (2 \leq \nu \leq k+1), \\ q_{-1} = 0, & q_0 = 1, & q_1 = M, & q_\nu = MF_\nu + F_{\nu-1} & (2 \leq \nu \leq k+1). \end{array}$$

One gets

$$\begin{aligned} \mathcal{E}(M, \underbrace{1, 1 \ldots, 1}_{k}) &= \sum_{\nu=0}^{k+1} (-1)^\nu \left(q_\nu \frac{p_{k+1}}{q_{k+1}} - p_\nu \right) \\ = & \quad (q_0 - q_1) \frac{p_{k+1}}{q_{k+1}} - (p_0 - p_1) + \frac{p_{k+1}}{q_{k+1}} \sum_{\nu=2}^{k+1} (-1)^\nu q_\nu - \sum_{\nu=2}^{k+1} (-1)^\nu p_\nu \\ = & \quad (1 - M) \frac{F_{k+1}}{MF_{k+1} + F_k} + 1 + \frac{F_{k+1}}{MF_{k+1} + F_k} \left(M \sum_{\nu=2}^{k+1} (-1)^\nu F_\nu + \sum_{\nu=2}^{k+1} (-1)^\nu F_{\nu-1} \right) \\ & \quad - \sum_{\nu=2}^{k+1} (-1)^\nu F_\nu \, . \end{aligned}$$

Taking into account some identities for alternating sums of Fibonacci numbers, we find that

$$\mathcal{E}(M,\underbrace{1,1\ldots,1}_{k}) = \frac{F_{k+3}-1}{MF_{k+1}+F_k} \ .$$

Then, the inequality from the lemma is equivalent to

$$M > \frac{2F_k}{F_{k+3}-2F_{k+1}-1} = 2 + \frac{2}{F_k-1} \ ,$$

which is fulfilled for $M \geq 3$ and a sufficient large integer k. (More precisely: Choose $k \geq 4$ for $M \geq 4$ and $k \geq 5$ for $M = 3$). For $M = 2$ we have

$$\mathcal{E}(2,\underbrace{1,1\ldots,1}_{k}) = \frac{F_{k+3}-1}{2F_{k+1}+F_k} = 1 - \frac{1}{F_{k+3}} \ .$$

This proves the lemma. □

2.2. Algorithmic Proof of Theorem 2

In the following we describe an algorithm, which produces a number η with an error sum $\mathcal{E}(\eta) = \alpha$ for any given $\alpha \in [0,\rho]$. Moreover, we can choose an arbitrary $a_0 \in \mathbb{Z}$, since $\mathcal{E}(\eta)$ does not depend on $a_0 = [\eta]$. Since $\mathcal{E}(\beta) = 0$ for $\beta = 0$, $\mathcal{E}(\rho) = \rho$ and $\mathcal{E}(1,1) = 1$, we may assume that $0 < \alpha < \rho$ and $\alpha \neq 1$.

Step 1: We consider two cases:

`Case 1.1`: $1 < \alpha < \rho$. We know from Lemma 6 that there is a unique integer $k \geq 2$ satisfying

$$\mathcal{E}(1,\underbrace{1,1\ldots,1}_{k-1}) \leq \alpha < \mathcal{E}(1,\underbrace{1,1\ldots,1}_{k}).$$

`Case 1.2`: $0 < \alpha < 1$. There is a unique integer $M \geq 2$ with

$$\mathcal{E}(M,1) = \frac{2}{M+1} \leq \alpha < \frac{2}{M} = \mathcal{E}(M-1,1).$$

By Lemma 10 there is a unique $k \geq 2$ with

$$\mathcal{E}(M,\underbrace{1,1\ldots,1}_{k-1}) \leq \alpha < \mathcal{E}(M,\underbrace{1,1\ldots,1}_{k}).$$

In any case, step 1 of the algorithm provides a sequence $a_1, a_2, \ldots, a_{n_1}$ of positive integers with $n_1 \geq 2$ and

$$\mathcal{E}(a_1,\ldots,a_{n_1}) \leq \alpha < \mathcal{E}(a_1,\ldots,a_{n_1},1).$$

$\mathcal{E}(a_1,\dots,a_{n_1}) = \alpha$ holds. If this is true, the algorithm terminates with $\eta = \langle a_1,\dots,a_{n_1}\rangle$. If not, we go to step 2.

Step 2: We have

$$\mathcal{E}(a_1,\dots,a_{n_1}) < \alpha < \mathcal{E}(a_1,\dots,a_{n_1},1)$$

with $n_1 \geq 2$. By Lemma 7 there is a unique integer $L \geq 2$ satisfying

$$\mathcal{E}(a_1,\dots,a_{n_1}) < \mathcal{E}(a_1,\dots,a_{n_1},L) \leq \alpha < \mathcal{E}(a_1,\dots,a_{n_1},L-1).$$

In case of $\alpha = \mathcal{E}(a_1,\dots,a_{n_1},L)$ the algorithm terminates with the number $\eta = \langle a_0; a_1,\dots,a_{n_1},L\rangle$. Otherwise, the inequalities

$$\mathcal{E}(a_1,\dots,a_{n_1}) < \mathcal{E}(a_1,\dots,a_{n_1},L) < \alpha < \mathcal{E}(a_1,\dots,a_{n_1},L-1) \tag{12}$$

hold. Then we have to distinguish two cases.

`Case 2.1`: $L \geq 3$. Since $n_1 \geq 2$, we get from (12) and Lemma 8 with $n = 1 + n_1$ and $a_n = L \geq 3$:

$$\mathcal{E}(a_1,\dots,a_{n_1},L) < \alpha < \mathcal{E}(a_1,\dots,a_{n_1},L,1).$$

Step 2 ends with $n_2 = 1 + n_1$, $a_{n_2} = L$, and

$$\mathcal{E}(a_1,\dots,a_{n_2}) < \alpha < \mathcal{E}(a_1,\dots,a_{n_1},a_{n_2},1). \tag{13}$$

`Case 2.2`: $L = 2$. If $\mathcal{E}(a_1,\dots,a_{n_1},1) \leq \mathcal{E}(a_1,\dots,a_{n_1},2,1)$, we finish step 2 with error terms satisfying (13), where $a_{n_2} = L = 2$. Otherwise, i.e., for

$$\mathcal{E}(a_1,\dots,a_{n_1},2,1) < \mathcal{E}(a_1,\dots,a_{n_1},1),$$

we have to distinguish the following two cases:

`Case 2.2.1`: $\alpha < \mathcal{E}(a_1,\dots,a_{n_1},2,1)$;

`Case 2.2.2`: $\mathcal{E}(a_1,\dots,a_{n_1},2,1) \leq \alpha < \mathcal{E}(a_1,\dots,a_{n_1},1)$.

In Case 2.2.1 we finish step 2 with error terms satisfying (13) with $a_{n_2} = L = 2$. In Case 2.2.2 the algorithm either terminates with $\eta = \langle a_0, a_1,\dots,a_{n_1},2,1\rangle$, or we apply Lemma 9 with $n = 1 + n_1$. For a unique $k \geq 2$ we get

$$\mathcal{E}(a_1,\dots,a_{n_1},2,\underbrace{1,1\dots,1}_{k-1}) < \alpha \leq \mathcal{E}(a_1,\dots,a_{n_1},2,\underbrace{1,1\dots,1}_{k}).$$

If $\mathcal{E}(a_1,\dots,a_{n_1},2,\underbrace{1,1\dots,1}_{k}) = \alpha$, the algorithm terminates with

$$\eta = \langle a_0, a_1,\dots,a_{n_1},2,\underbrace{1,1\dots,1}_{k}\rangle.$$

Otherwise, we finish step 2 with $n_2 = k + n_1$, $a_{n_1+1} = 2, a_{n_1+2} = \cdots = a_{n_2} = 1$ and

$$\mathcal{E}(a_1, \ldots, a_{n_2}) < \alpha < \mathcal{E}(a_1, \ldots, a_{n_2}, 1).$$

Again, this is equivalent to (13) with $2 \leq n_1 < n_2$, since $k \geq 2$.
Step 3: We repeat step 2 starting with n_2, which satisfies (13). If the algorithm does not terminate in this step, we construct positive integers $n_3 > n_2$ and $a_1, \ldots, a_{n_3}$ with

$$\mathcal{E}(a_1, \ldots, a_{n_3}) < \alpha < \mathcal{E}(a_1, \ldots, a_{n_3}, 1).$$

The above method can be iterated. Either the algorithm will terminate, or Lemma 6 guarantees that

$$\lim_{n \to \infty} \left(\mathcal{E}(a_1, \ldots, a_n, 1) - \mathcal{E}(a_1, \ldots, a_n) \right) = 0\,,$$

such that by $\mathcal{E}(a_1, \ldots a_n) < \alpha < \mathcal{E}(a_1, \ldots, a_n, 1)$ the number $\eta = \langle a_0, a_1, a_2, \ldots \rangle$ satisfies $\mathcal{E}(\eta) = \alpha$. □

Example. Let $\alpha = 202/157$. Then the above algorithm produces the number

$$\eta = \langle 1; 1, 1, 2, 1, 89 \rangle = \frac{987}{628}\,.$$

3. Proof of Theorem 3

3.1. Auxiliary Lemmas

As in Section 2.1, let $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n$ denote positive integers.

Lemma 11. *Put* $p_n/q_n = \langle a_0; a_1, \ldots, a_n \rangle$.
(i) Let n *be even. Then, the sequence of rationals* $(\mathcal{E}^*(a_1, \ldots, a_n, N))_{N \geq 1}$ *is strictly decreasing and*

$$0 < \mathcal{E}^*(a_1, \ldots, a_n, N) - \mathcal{E}^*(a_1, \ldots, a_n) < \frac{1+n}{Nq_n + q_{n+1}}$$

holds for $N \geq 1$.
(ii) Let n *be odd. Then, the sequence of rationals* $(\mathcal{E}^*(a_1, \ldots, a_n, N))_{N \geq 1}$ *is strictly increasing and*

$$0 < \mathcal{E}^*(a_1, \ldots, a_n) - \mathcal{E}^*(a_1, \ldots, a_n, N) < \frac{1+n}{Nq_n + q_{n+1}}$$

holds for $N \geq 1$. *In particular we have*

$$\lim_{N \to \infty} \mathcal{E}^*(a_1, \ldots, a_n, N) = \mathcal{E}^*(a_1, \ldots, a_n) \qquad (n \in \mathbb{N}).$$

Proof. Let

$$\begin{aligned}\beta &:= \langle a_0; a_1, \dots a_n\rangle,\\ \gamma &:= \langle a_0; a_1, \dots a_n, N\rangle,\\ \frac{p_\nu}{q_\nu} &:= \langle a_0; a_1, \dots a_\nu\rangle \qquad (0 \le \nu \le n).\end{aligned}$$

Then we have the identities

$$\beta = \frac{p_n}{q_n}, \qquad \gamma = \frac{Np_n + p_{n-1}}{Nq_n + q_{n-1}},$$

and

$$\begin{aligned}\mathcal{E} &:= \mathcal{E}(a_1, \dots, a_n, N) - \mathcal{E}(a_1, \dots, a_n) = \sum_{\nu=0}^{n} (\gamma q_\nu - p_\nu) - \sum_{\nu=0}^{n-1} (\beta q_\nu - p_\nu)\\ &= \gamma q_n - p_n + \sum_{\nu=0}^{n-1} (\gamma - \beta) q_\nu.\end{aligned}$$

With

$$\begin{aligned}\gamma - \beta &= \frac{Np_n + p_{n-1}}{Nq_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(Nq_n + q_{n-1})},\\ \gamma q_n - p_n &= q_n(\gamma - \beta) = \frac{(-1)^n}{Nq_n + q_{n-1}}\end{aligned}$$

we get

$$\mathcal{E} = \frac{(-1)^n}{Nq_n + q_{n-1}} + \frac{(-1)^n}{q_n(Nq_n + q_{n-1})} \sum_{\nu=0}^{n-1} q_\nu = \frac{(-1)^n}{Nq_n + q_{n-1}} \left(1 + \frac{1}{q_n} \sum_{\nu=0}^{n-1} q_\nu\right). \quad (14)$$

Setting $\gamma' := \langle a_0; a_1, \dots a_n, N + 1\rangle$, we obtain

$$\begin{aligned}\mathcal{E}' &:= \mathcal{E}(a_1, \dots, a_n, N + 1) - \mathcal{E}(a_1, \dots, a_n, N) = \sum_{\nu=0}^{n} (\gamma' q_\nu - p_\nu) - \sum_{\nu=0}^{n} (\gamma q_\nu - p_\nu)\\ &= \sum_{\nu=0}^{n} (\gamma' - \gamma) q_\nu.\end{aligned}$$

Using

$$\gamma' - \gamma = \frac{(N + 1)p_n + p_{n-1}}{(N + 1)q_n + q_{n-1}} - \frac{Np_n + p_{n-1}}{Nq_n + q_{n-1}} = \frac{(-1)^{n+1}}{((N + 1)q_n + q_{n-1})(Nq_n + q_{n-1})},$$

we express $\mathcal{E}'$ by

$$\mathcal{E}' = \frac{(-1)^{n+1}}{((N + 1)q_n + q_{n-1})(Nq_n + q_{n-1})} \sum_{\nu=0}^{n} q_\nu. \quad (15)$$

It is clear that

$$\sum_{\nu=0}^{n} q_\nu > 0 \qquad \text{and} \qquad \frac{1}{q_n}\sum_{\nu=0}^{n-1} q_\nu < \frac{1}{q_n}\cdot nq_n = n. \tag{16}$$

For even n we get from (14), (15), and (16) that

$$\mathcal{E}' < 0 \qquad \text{and} \qquad 0 < \mathcal{E} < \frac{1+n}{Nq_n + q_{n-1}}.$$

For odd n we get from (14), (15), and (16) that

$$\mathcal{E}' > 0 \qquad \text{and} \qquad 0 < -\mathcal{E} < \frac{1+n}{Nq_n + q_{n-1}}.$$

This completes the proof of the lemma. □

Lemma 12. *(i) Let n be even. Then we have*

$$\mathcal{E}^*(a_1,\dots,a_n,1) > \mathcal{E}^*(a_1,\dots,a_n+1).$$

(ii) Let n be odd. Then we have

$$\mathcal{E}^*(a_1,\dots,a_n,1) < \mathcal{E}^*(a_1,\dots,a_n+1).$$

Proof. The lemma is an obvious consequence of the identity stated in (4). □

3.2. Algorithmic Proof of Theorem 3

As in Section 2 we will prove Theorem 3 by the algorithmic construction of a number $\eta = \langle a_0; a_1,\dots,a_n\rangle$ with $\mathcal{E}^*(\eta) = \alpha \in [0,1]$ for some arbitrary $\alpha \in [0,1]$. By $\mathcal{E}^*(\beta) = 0$ for $\beta = 0$ and $\mathcal{E}^*(1,1) = 1$ it suffices to assume that $\alpha \in (0,1)$. Let again a_0 be an arbitrary integer. We shall compute a_k in step k of the following algorithm. Depending on the parity of k the constructions differ.

Step 1: There is a unique positive integer M with

$$\mathcal{E}^*(M+1) = \frac{1}{M+1} < \alpha \le \frac{1}{M} = \mathcal{E}^*(M).$$

Set $a_1 = M$. We consider two cases:

`Case 1`: $\mathcal{E}^*(a_1) = \alpha \rightarrow$ the algorithm terminates.

`Case 2`: $\mathcal{E}^*(a_1) > \alpha \rightarrow$ go to step 2.

Step 2: From Lemma 12 we know that

$$\mathcal{E}^*(a_1, 1) < \mathcal{E}^*(a_1 + 1) < \alpha < \mathcal{E}^*(a_1)\,.$$

Therefore, Lemma 11 guarantees the existence of a unique positive integer M with

$$\mathcal{E}^*(a_1, M) < \alpha \leq \mathcal{E}^*(a_1, M + 1).$$

Again we consider two cases:

`Case 1`: $\mathcal{E}^*(a_1, M + 1) = \alpha \rightarrow$ the algorithm terminates with $a_2 = M + 1$.

`Case 2`: $\mathcal{E}^*(a_1, M + 1) > \alpha \rightarrow$ set $a_2 = M$ and go to step 3.

Step 3: From Lemma 12 we know that

$$\mathcal{E}^*(a_1, a_2, 1) > \mathcal{E}^*(a_1, a_2 + 1) > \alpha > \mathcal{E}^*(a_1, a_2)\,.$$

Therefore, Lemma 11 guarantees the existence of a unique positive integer M with

$$\mathcal{E}^*(a_1, a_2, M + 1) < \alpha \leq \mathcal{E}^*(a_1, a_2, M).$$

Set $a_3 = M$. We consider two cases:

`Case 1`: $\mathcal{E}^*(a_1, a_2, a_3) = \alpha \rightarrow$ the algorithm terminates.

`Case 2`: $\mathcal{E}^*(a_1, a_2, a_3) > \alpha \rightarrow$ go to step 4.

$$\vdots$$

Step $2k$: As a result of the above $2k - 1$ cycles we have the numbers $a_1, \ldots, a_{2k-1}$. If the algorithm is still at work, α satisfies

$$\mathcal{E}^*(a_1, \ldots, a_{2k-1} + 1) < \alpha < \mathcal{E}^*(a_1, \ldots, a_{2k-1}).$$

From Lemma 12 we know that

$$\mathcal{E}^*(a_1, \ldots, a_{2k-1}, 1) < \mathcal{E}^*(a_1, \ldots, a_{2k-1} + 1) < \alpha < \mathcal{E}^*(a_1, \ldots, a_{2k-1})\,.$$

Therefore, Lemma 11 guarantees the existence of a unique positive integer M with

$$\mathcal{E}^*(a_1, \ldots, a_{2k-1}, M) < \alpha \leq \mathcal{E}^*(a_1, \ldots, a_{2k-1}, M + 1).$$

We consider two cases:

`Case 1`: $\mathcal{E}^*(a_1, \ldots, a_{2k-1}, M+1) = \alpha \rightarrow$ the algorithm terminates with $a_{2k} = M+1$.

`Case 2`: $\mathcal{E}^*(a_1, \ldots, a_{2k-1}, M + 1) > \alpha \rightarrow$ set $a_{2k} = M$ and go to step $2k + 1$.

Step $2k + 1$: Here we have

$$\mathcal{E}^*(a_1, \ldots, a_{2k}) < \alpha < \mathcal{E}^*(a_1, \ldots, a_{2k} + 1)\,.$$

From Lemma 12 we know that

$$\mathcal{E}^*(a_1, \ldots, a_{2k}, 1) \geq \mathcal{E}^*(a_1, \ldots, a_{2k} + 1) > \alpha > \mathcal{E}^*(a_1, \ldots, a_{2k})\,.$$

Therefore, Lemma 11 guarantees the existence of a unique positive integer M with

$$\mathcal{E}^*(a_1, \ldots, a_{2k}, M + 1) < \alpha \leq \mathcal{E}^*(a_1, \ldots, a_{2k}, M).$$

Set $a_{2k+1} = M$. We consider two cases:

`Case 1`: $\mathcal{E}^*(a_1, \ldots, a_{2k+1}) = \alpha \rightarrow$ the algorithm terminates.

`Case 2`: $\mathcal{E}^*(a_1, \ldots, a_{2k+1}) > \alpha \rightarrow$ go to step $2k + 2$.

Either the algorithm will terminate, or for every $N \in \mathbb{N}$ Lemma 11 gives the limit

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} |\mathcal{E}^*(a_1, \ldots, a_n, N) - \mathcal{E}^*(a_1, \ldots, a_n, N + 1)| \\ &\leq \lim_{n\to\infty} \left(\frac{1 + n}{N q_n + q_{n-1}} + \frac{1 + n}{(N + 1) q_n + q_{n-1}} \right) = 0\,, \end{aligned}$$

such that by $\mathcal{E}^*(a_1, \ldots, a_{2n-1} + 1) < \alpha < \mathcal{E}^*(a_1, \ldots, a_{2n-1})$ and $\mathcal{E}^*(a_1, \ldots, a_{2n}) < \alpha < \mathcal{E}^*(a_1, \ldots, a_{2n} + 1)$, the irrational number $\eta = \langle 0, a_1, a_2, \ldots \rangle$ satisfies $\mathcal{E}^*(\eta) = \alpha$. □

Example. Let

$$\alpha = \frac{3846888972029}{31159800925831}\,.$$

Then the above algorithm computes

$$\eta = \langle 1; 8, 90, 82, 17120, 30781 \rangle \quad \text{with} \quad \alpha = \mathcal{E}(\eta)\,.$$

4. Proof of Proposition 1

We need two auxiliary lemmas.

Lemma 13. *Let* $\beta := \langle 0; a_1, \ldots, a_n, 1, 1, \ldots \rangle$. *Moreover, let* p_ν / q_ν $(\nu \geq 0)$ *be the convergents of* β. *Then we have*

$$\beta = \frac{\rho p_n + p_{n-1}}{\rho q_n + q_{n-1}}, \tag{17}$$

and

$$\mathcal{E}(\beta) = \sum_{\nu=0}^{n} (-1)^\nu \left(\frac{\rho p_n + p_{n-1}}{\rho q_n + q_{n-1}} q_\nu - p_\nu \right) + \frac{\rho}{\rho q_n + q_{n-1}}\,. \tag{18}$$

Proof. From the definition of β we obtain the identities

$$p_{n+\nu} = F_{\nu+1}p_n + F_\nu p_{n-1}, \qquad q_{n+\nu} = F_{\nu+1}q_n + F_\nu q_{n-1} \qquad (\nu \geq 1). \tag{19}$$

Hence we have, for ν tending to infinity,

$$\frac{p_{n+\nu}}{q_{n+\nu}} = \frac{F_{\nu+1}/F_\nu p_n + p_{n-1}}{F_{\nu+1}/F_\nu q_n + q_{n-1}} \longrightarrow \frac{\rho p_n + p_{n-1}}{\rho q_n + q_{n-1}},$$

which proves (17). It remains to show the formula

$$\sum_{\nu=n+1}^{\infty} (-1)^\nu \left(\beta q_\nu - p_\nu\right) = \frac{\rho}{\rho q_n + q_{n-1}}\,. \tag{20}$$

Using (17) and (19), we express the left-hand side of (20) by

$$\begin{aligned}
&\sum_{\nu=1}^{\infty}(-1)^{\nu+n}\left(\frac{\rho p_n + p_{n-1}}{\rho q_n + q_{n-1}}(F_{\nu+1}q_n + F_\nu q_{n-1}) - (F_{\nu+1}p_n + F_\nu p_{n-1})\right)\\
&= \frac{(-1)^n}{\rho q_n + q_{n-1}}\sum_{\nu=1}^{\infty}(-1)^\nu\big((\rho p_n + p_{n-1})(F_{\nu+1}q_n + F_\nu q_{n-1})\\
&\qquad\qquad - (\rho q_n + q_{n-1})(F_{\nu+1}p_n + F_\nu p_{n-1})\big)\\
&= \frac{1}{\rho q_n + q_{n-1}}\sum_{\nu=1}^{\infty}(-1)^\nu(F_{\nu+1} - \rho F_\nu)\\
&= \frac{1}{\sqrt{5}(\rho q_n + q_{n-1})}\sum_{\nu=1}^{\infty}(-1)^\nu\big((\rho^{\nu+1} - \tilde{\rho}^{\nu+1}) - \rho(\rho^\nu - \tilde{\rho}^\nu)\big)\\
&= \frac{1}{\sqrt{5}(\rho q_n + q_{n-1})}\sum_{\nu=1}^{\infty}(-1)^\nu(\rho\tilde{\rho}^\nu - \tilde{\rho}^{\nu+1}) = \frac{\rho - \tilde{\rho}}{\sqrt{5}(\rho q_n + q_{n-1})}\sum_{\nu=1}^{\infty}(-\tilde{\rho})^\nu\\
&= \frac{\rho}{\rho q_n + q_{n-1}},
\end{aligned}$$

which equals the right-hand side of (20). Therefore, (18) is proven. ☐

Lemma 14. *Let* $\alpha := \langle 0; a_1, \ldots, a_n, a_{n+1}, 1, 1, \ldots\rangle$ *and* $\beta := \langle 0; a_1, \ldots, a_n, 1, 1, \ldots\rangle$. *Then we have*

$$\mathcal{E}(\beta) - \mathcal{E}(\alpha) \geq 0\,.$$

Proof. Let p_ν/q_ν be the convergents of α. Then, applying (18), we split $\mathcal{E}(\beta) - \mathcal{E}(\alpha)$ into three parts:

$$\mathcal{E}(\beta) - \mathcal{E}(\alpha) = S_1 + S_2 + S_3, \tag{21}$$

where

$$\begin{aligned}
S_1 &:= \left(\frac{\rho p_n + p_{n-1}}{\rho q_n + q_{n-1}} - \frac{(\rho a_{n+1}+1)p_n + \rho p_{n-1}}{(\rho a_{n+1}+1)q_n + \rho q_{n-1}} \right) \sum_{\nu=0}^{n} (-1)^\nu q_\nu \,, \\
S_2 &:= \frac{\rho}{\rho q_n + q_{n-1}} - \frac{\rho}{(\rho a_{n+1}+1)q_n + \rho q_{n-1}} \,, \\
S_3 &:= (-1)^n \left(\frac{(\rho a_{n+1}+1)p_n + \rho p_{n-1}}{(\rho a_{n+1}+1)q_n + \rho q_{n-1}} (a_{n+1}q_n + q_{n-1}) - (a_{n+1}p_n + p_{n-1}) \right) .
\end{aligned}$$

We observe for S_1 on the one hand the identity

$$\begin{aligned}
&\frac{\rho p_n + p_{n-1}}{\rho q_n + q_{n-1}} - \frac{(\rho a_{n+1}+1)p_n + \rho p_{n-1}}{(\rho a_{n+1}+1)q_n + \rho q_{n-1}} \\
&= (-1)^n \frac{\rho(a_{n+1}-1)}{(\rho q_n + q_{n-1})((\rho a_{n+1}+1)q_n + \rho q_{n-1})} \,,
\end{aligned}$$

and the other hand the inequality

$$(-1)^n \sum_{\nu=0}^{n} (-1)^\nu q_\nu \geq 0,$$

such that we obtain $S_1 \geq 0$. Moreover, we find the expressions

$$S_2 = \frac{\rho\big((\rho a_{n+1} - \rho + 1)q_n + (\rho - 1)q_{n-1}\big)}{(\rho q_n + q_{n-1})\big((\rho a_{n+1}+1)q_n + \rho q_{n-1}\big)}$$

and

$$S_3 = \frac{-1}{(\rho a_{n+1}+1)q_n + \rho q_{n-1}}.$$

This yields

$$S_2 + S_3 = \frac{\rho^2(a_{n+1}-1)}{(\rho q_n + q_{n-1})\big((\rho a_{n+1}+1)q_n + \rho q_{n-1}\big)} \geq 0.$$

We have shown that $S_1 + S_2 + S_3 \geq 0$. Hence, the lemma follows by (21). □

Proof of Proposition 1. Let $\alpha := \langle 0; a_1, \ldots, a_n, \ldots \rangle$, $\beta := \langle 0; a_1, \ldots, a_n, 1, 1, \ldots \rangle$,

$$\mathcal{E}_\nu := \mathcal{E}(a_1, \ldots, a_\nu, 1, 1, \ldots) \qquad (\nu \geq 1), \qquad \text{and} \qquad \mathcal{E}_\infty := \mathcal{E}(\alpha) = \mathcal{E}(a_1, a_2, \ldots).$$

From Lemma 14 we know that $\mathcal{E}_\nu - \mathcal{E}_{\nu+1} \geq 0$ for $\nu \geq 1$. Summing up these inequalities, we obtain

$$\mathcal{E}_n - \mathcal{E}_N = \sum_{\nu=n}^{N-1} (\mathcal{E}_\nu - \mathcal{E}_{\nu+1}) \geq 0 \qquad (N > n) \,.$$

For N tending to infinity it turns out that

$$\mathcal{E}(\alpha) = \mathcal{E}_\infty = \lim_{N\to\infty} \mathcal{E}_N \leq \mathcal{E}_n = \mathcal{E}(\beta),$$

which proves the statement in Proposition 1. □

5. Proofs of Theorem 4 and Theorem 5

Proof of Theorem 4. Let $\alpha := \langle 0; 1, a_2, a_3, \ldots\rangle$ and $\beta := \langle 0; 1, a_2, a_3, 1, 1 \ldots\rangle$. Then, by Proposition 1 and Lemma 13, we have

$$1 < \mathcal{E}(\alpha) \leq \mathcal{E}(\beta) = 1 + \frac{1+2\rho}{\rho a_2 a_3 + \rho a_3 + a_2 + \rho + 1} < 1 + \frac{\rho^2}{a_2 a_3}.$$

For fixed $a_2, a_3 \in \mathbb{N}$ the real number α lies in the interval $\mathcal{M}(a_2, a_3)$ given by

$$\mathcal{M}(a_2, a_3) := [\langle 0; 1, a_2, a_3 + 1\rangle, \langle 0; 1, a_2, a_3\rangle] \tag{22}$$

(see [2]). Then, the numbers α with $a_2 a_3 \geq N$ for some $N \in \mathbb{N}$ form the set

$$\mathcal{I} := \bigcup_{\substack{i,j\geq 1\\ ij \geq N}} \mathcal{M}(i,j).$$

Now, $\mathcal{E}(\alpha)$ satisfies the inequalities

$$1 < \mathcal{E}(\alpha) < 1 + \frac{\rho^2}{N}.$$

It is well-known that $\mathcal{M}(a_2, a_3)$ and $M(a_2', a_3')$ do not intersect for any $(a_2, a_3) \neq (a_2', a_3')$. Using (22), we compute the length of $\mathcal{I}$:

$$|\mathcal{I}| = \sum_{\substack{i,j\geq 1\\ ij \geq N}} |\mathcal{M}(i,j)| = \sum_{\substack{i,j\geq 1\\ ij \geq N}} \frac{1}{(ij+j+1)(ij+i+j+2)}.$$

Since

$$ij + j + 1 \leq 3ij \qquad \text{and} \qquad ij + i + j + 2 \leq 5ij$$

hold for $i, j \geq 1$, we find a lower bound for $|\mathcal{I}|$ by

$$|\mathcal{I}| \geq \frac{1}{15} \sum_{i=1}^{\infty} \sum_{j=\max(1,N/i)} \frac{1}{i^2 j^2} = \frac{1}{15} \left(\sum_{i=1}^{N} \frac{1}{i^2} \sum_{j \geq N/i} \frac{1}{j^2} + \sum_{i=N+1}^{\infty} \frac{1}{i^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \right).$$

Moreover, we have

$$\sum_{j \geq N/i} \frac{1}{j^2} \geq \int_{1+[N/i]}^{\infty} \frac{dt}{t^2} = \frac{1}{1+[N/i]} \geq \frac{i}{2N},$$

which leads to

$$\sum_{i=1}^{N} \frac{1}{i^2} \sum_{j \geq N/i} \frac{1}{j^2} \geq \frac{1}{2N} \sum_{i=1}^{N} \frac{1}{i} \geq \frac{\log(N+1)}{2N} > \frac{\log N}{2N}.$$

Finally, this yields

$$|\mathcal{I}| > \frac{1}{15}\sum_{i=1}^{N}\frac{1}{i^2}\sum_{j\geq N/i}\frac{1}{j^2} > \frac{\log N}{30N}\,.$$

Now, let $(\alpha_\mu)_{\mu\geq 1}$ be a sequence of uniformly distibuted real numbers modulo 1. Let us assume that the sequence $(\mathcal{E}(\alpha_\mu))_{\mu\geq 1}$ is also uniformly distributed in $[0,\rho]$. With the notation for $A(J_1,M)$ introduced in Section 1, we then have

$$\lim_{M\to\infty}\frac{A(J_1,M)}{M} = \frac{|J_1|}{\rho} = \frac{\rho^2}{N\rho} = \frac{\rho}{N}\,.$$

But this does not hold for large N, since the above inequality for $|\mathcal{I}|$ shows that

$$\liminf_{M\to\infty}\frac{A(J_1,M)}{M} \geq \frac{\log N}{30N}\,.$$

To prove the second statement in Theorem 4, we first note that $\mathcal{E}(\alpha)\geq 1$ holds for $1/2<\alpha<1$, so that $\mathcal{E}(1,a_2,a_3,\dots)\geq 1$. Next, let $a_1\geq 3$ and $N\geq 32$. By Proposition 1, Lemma 6, and Lemma 13 we have

$$\mathcal{E}(a_1,a_2,a_3,\dots) \leq \mathcal{E}(a_1,1,1,\dots) = \frac{1+\rho}{a_1-1+\rho} \leq \frac{1+\rho}{2+\rho} = 1-\frac{1}{2+\rho} < 1-\frac{\rho^2}{N}\,.$$

(Lemma 6 is needed if a rational α corresponds to a finite sequence $a_1,a_2,a_3\dots$.) It follows, with $N\geq 32$, that

$$\mathcal{E}(\alpha)\in J_2 \wedge \alpha=\langle 0;a_1,a_2,\dots\rangle \quad\Longrightarrow\quad a_1=2\,.$$

Therefore, we may write $\alpha=\langle 0;2,\underbrace{1,1,\dots,1}_{k},a_{k+2},a_{k+3},\dots\rangle$ $(k\geq 0)$ for a number α satisfying $\mathcal{E}(\alpha)\in J_2$. If α is a rational number, $0,2,\underbrace{1,1,\dots,1}_{k},a_{k+2},a_{k+3},\dots$ becomes a finite sequence. By Lemma 13 it follows that $\mathcal{E}(2,1,1,\dots)=1\notin J_2$. We assume that

$$F_{k+3} < \frac{N}{4\rho^2}\,. \tag{23}$$

By Lemma 10 and (23),

$$\mathcal{E}(2,\underbrace{1,1,\dots,1}_{k}) = 1-\frac{1}{F_{k+3}} < 1-\frac{4\rho^2}{N} < 1-\frac{\rho^2}{N}\,,$$

and hence $\mathcal{E}(2,\underbrace{1,1,\dots,1}_{k})\notin J_2$, a contradiction. Thus it remains to consider the case $\mathcal{E}(\alpha)\in J_2$ with

$$\alpha=\langle 0;2,\underbrace{1,1,\dots,1}_{k},a_{k+2},a_{k+3},\dots\rangle$$

and $a_{k+2} \geq 2$, where $\alpha \in \mathbb{Q}$ is possible. Again Lemma 6 and Proposition 1 give

$$\mathcal{E}(\alpha) \leq \mathcal{E}(2, \underbrace{1, 1, \ldots, 1}_{k}, a_{k+2}, 1, 1 \ldots). \tag{24}$$

In order to compute $\mathcal{E}(\beta)$ for $\beta := \langle 0; 2, \underbrace{1, 1, \ldots, 1}_{k}, a_{k+2}, 1, 1 \ldots \rangle$, we apply Lemma 13 with $n = k + 2$. Let p_ν/q_ν $(\nu \geq 0)$ be the convergents of β. We find that

$$\begin{array}{rclcrcl} p_\nu & = & F_\nu & (0 \leq \nu \leq k+1), & p_{k+2} & = & a_{k+2}F_{k+1} + F_k, \\ q_\nu & = & F_{\nu+2} & (0 \leq \nu \leq k+1), & q_{k+2} & = & a_{k+2}F_{k+3} + F_{k+2}. \end{array}$$

By straightforward computations including the application of the identities $F_{-1} = 1$,

$$\begin{array}{rcll} F_kF_{\nu+2} - F_{k+2}F_\nu & = & (-1)^\nu F_{k-\nu} & (0 \leq \nu \leq k+1), \\ F_0 + F_1 + F_2 + \cdots + F_m & = & F_{m+2} - 1 & (m \geq 0), \end{array}$$

we obtain

$$\begin{aligned} \mathcal{E}(\beta) &= \sum_{\nu=0}^{k+1} (-1)^\nu \Big(\frac{\rho(a_{k+2}F_{k+1} + F_k) + F_{k+1}}{\rho(a_{k+2}F_{k+3} + F_{k+2}) + F_{k+3}} F_{\nu+2} - F_\nu \Big) \\ &\quad + (-1)^{k+2} \Big(\frac{\rho(a_{k+2}F_{k+1} + F_k) + F_{k+1}}{\rho(a_{k+2}F_{k+3} + F_{k+2}) + F_{k+3}} \big(a_{k+2}F_{k+3} + F_{k+2}\big) \\ &\quad - \big(a_{k+2}F_{k+1} + F_k\big) \Big) + \frac{\rho}{\rho(a_{k+2}F_{k+3} + F_{k+2}) + F_{k+3}} \\ &= 1 - \frac{(a_{k+2} - 1)\rho}{\rho(a_{k+2}F_{k+3} + F_{k+2}) + F_{k+3}} \\ &\leq 1 - \frac{(a_{k+2} - 1)\rho}{\rho a_{k+2} + \rho + 1} \cdot \frac{1}{F_{k+3}}. \end{aligned}$$

The function $(\rho x - \rho)/(\rho x + \rho + 1)$ increases strictly for $x \geq 2$. Thus we obtain

$$\mathcal{E}(\beta) \leq 1 - \frac{\rho}{3\rho + 1} \cdot \frac{1}{F_{k+3}} < 1 - \frac{1}{4F_{k+3}},$$

and consequently, by (23) and (24),

$$\mathcal{E}(\alpha) < 1 - \frac{1}{4F_{k+3}} < 1 - \frac{\rho^2}{N}.$$

This contradicts our hypothesis $\mathcal{E}(\alpha) \in J_2$. We have disproved (23), so that we may assume

$$F_{k+3} \geq \frac{N}{4\rho^2}$$

for $\alpha = \langle 0; 2, \underbrace{1, 1, \dots, 1}_{k}, a_{k+2}, a_{k+3}, \dots \rangle$ with $\mathcal{E}(\alpha) \in J_2$. We have already shown for $N \geq 32$ that

$$\mathcal{I} := \left\{\gamma \in [0,1] : \mathcal{E}(\gamma) \in J_2\right\} \subseteq \left\{\langle 0; 2, \underbrace{1, 1, \dots, 1}_{k}, a_{k+2}, a_{k+3}, \dots \rangle : F_{k+3} \geq \frac{N}{4\rho^2}\right\}.$$

Let k_0 denote the smallest positive integer satisfying $F_{k_0+3} \geq N/(4\rho^2)$. Note that $k_0 \geq 1$ by $N \geq 32$. Then we have

$$\begin{aligned} |\mathcal{I}| &\leq \left|\left\{\langle 0; 2, \underbrace{1, 1, \dots, 1}_{k}, a_{k+2}, a_{k+3}, \dots \rangle : F_{k+3} \geq \frac{N}{4\rho^2}\right\}\right| \\ &= \left|\langle 0; 2, \underbrace{1, 1, \dots, 1}_{k_0} \rangle - \langle 0; 2, \underbrace{1, 1, \dots, 1}_{k_0-1}, 2 \rangle\right| \\ &= \left|\frac{F_{k_0+1}}{F_{k_0+3}} - \frac{F_{k_0+2}}{F_{k_0+4}}\right| = \frac{|F_{k_0+1}F_{k_0+4} - F_{k_0+2}F_{k_0+3}|}{F_{k_0+3}F_{k_0+4}} \\ &= \frac{1}{F_{k_0+3}F_{k_0+4}} \leq \frac{1}{F_{k_0+3}^2} < \frac{16\rho^4}{N^2}\,. \end{aligned} \tag{25}$$

Let $(\alpha_\mu)_{\mu \geq 1}$ be a sequence of uniformly distributed real numbers modulo 1. Then, in case of uniform distribution of $(\mathcal{E}(\alpha_\mu))_{\mu \geq 1}$ in $[0, \rho]$, we have

$$\lim_{M \to \infty} \frac{A(J_2, M)}{M} = \frac{|J_2|}{\rho} = \frac{\rho}{N}\,.$$

But (25) shows that

$$\limsup_{M \to \infty} \frac{A(J_2, M)}{M} \leq \frac{16\rho^4}{N^2} < \frac{\rho}{N}\,,$$

where the right-hand inequality holds for $N \geq 68$. □

Proof of Theorem 5. Let $\alpha := \langle 0; a_1, a_2, \dots \rangle$. Then, for $a_1 \geq 2$, we find that

$$\mathcal{E}^*(\alpha) \leq \mathcal{E}^*(a_1) = \frac{1}{a_1} \leq \frac{1}{2} < 1 - \frac{1}{N} \qquad (N \geq 3).$$

Therefore, $a_1 = 1$ is a necessary condition for $\mathcal{E}^*(\alpha) \in J_3$. Next, by Lemma 11 we have the following upper bound for $\mathcal{E}^*(\alpha) \in J_3$:

$$1 - \frac{1}{N} < \mathcal{E}^*(\alpha) \leq \mathcal{E}^*(1, a_2, a_3) = \frac{a_2a_3 - a_3 + 2}{a_2a_3 + a_3 + 1}.$$

From this inequality we conclude that $a_2 > (2N-1) - (N+1)/a_3$. Therefore, for any positive integers N and a_3,

$$\mathcal{E}^*(\alpha) \in J_3 \qquad \Longrightarrow \qquad a_2 \geq A := \left[(2N-1) - \frac{N+1}{a_3}\right] + 1 \in \mathbb{N}.$$

Combining this with (22), it turns out that

$$\mathcal{E}^*(\alpha) \in J_3 \qquad \Longrightarrow \qquad \alpha \in \mathcal{I} := \bigcup_{a_3=1}^{\infty} \bigcup_{a_2=A}^{\infty} \mathcal{M}(a_2, a_3).$$

Since

$$|\mathcal{M}(a_2, a_3)| = \frac{1}{(a_2a_3 + a_3 + 1)(a_2a_3 + a_2 + a_3 + 2)}\,,$$

we get

$$|\mathcal{I}| = \sum_{\nu=N+1}^{\infty} \frac{1}{\nu(2\nu-1)} + \sum_{a_3=2}^{\infty} \sum_{\nu=A+2}^{\infty} \frac{1}{((\nu-1)a_3+1)((\nu-1)a_3+\nu)}.$$

Since $c > 0$ and $a_3 \geq 2$, we find a lower bound for $A+2$:

$$A + 2 > (2N-1) - \frac{N+1}{a_3} + 2 \geq (2N-1) - \frac{N+1}{2} + 2 > \frac{3N}{2}.$$

This yields

$$\begin{aligned}
|\mathcal{I}| &\leq \sum_{\nu=N+1}^{\infty} \frac{1}{\nu(2\nu-1)} + \sum_{a_3=2}^{\infty} \sum_{\nu\geq 3N/2} \frac{1}{((\nu-1)a_3+1)((\nu-1)a_3+\nu)} \\
&= \sum_{\nu=N+1}^{\infty} \frac{1}{\nu(2\nu-1)} + \sum_{\nu\geq 3N/2} \sum_{a_3=2}^{\infty} \frac{1}{((\nu-1)a_3+1)((\nu-1)a_3+\nu)} \\
&= \sum_{\nu=N+1}^{\infty} \frac{1}{\nu(2\nu-1)} + \sum_{\nu\geq 3N/2} \frac{1}{(\nu-1)(2\nu-1)} \\
&< \frac{1}{2} \sum_{\nu=N+1}^{\infty} \frac{1}{\nu(\nu-1)} + \frac{1}{2} \sum_{\nu\geq 3N/2} \frac{1}{(\nu-1)^2}\,.
\end{aligned}$$

Using the identity

$$\sum_{\nu=N+1}^{\infty} \frac{1}{\nu(\nu-1)} = \frac{1}{N}$$

and the estimate

$$\sum_{\nu\geq 3N/2} \frac{1}{(\nu-1)^2} \leq \int_{(3N-2)/2}^{\infty} \frac{dx}{(x-1)^2} = \frac{2}{3N-4} \qquad (N \geq 2)\,,$$

we finish the proof of Theorem 5 for $N \geq 3$ by

$$|\mathcal{I}| < \frac{1}{2N} + \frac{1}{3N-4} < \frac{1}{2N} + \frac{1}{3N} + \frac{1}{N^2} = \frac{5}{6N} + \frac{1}{N^2}\,.$$

□

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Appendix: Plots

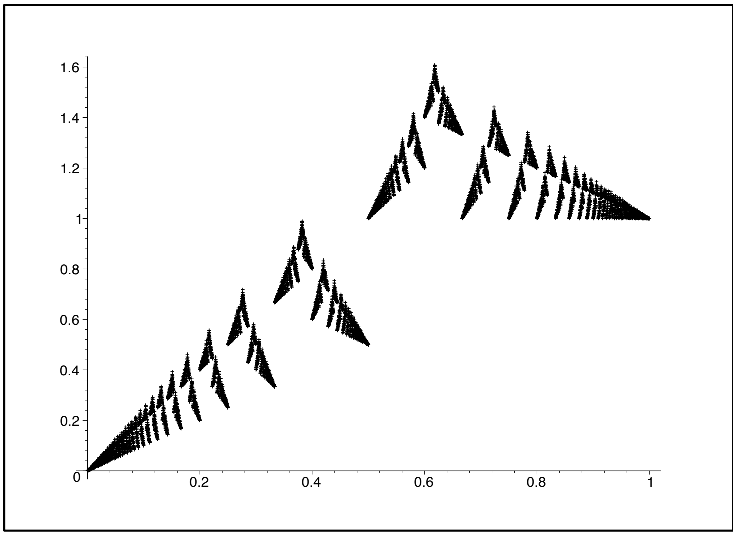

Figure 1: The graph of $\mathcal{E}$

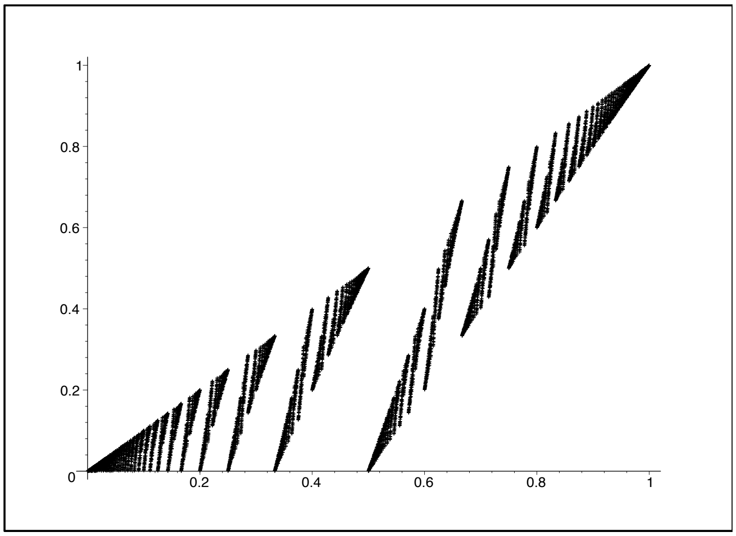

Figure 2: The graph of $\mathcal{E}^*$

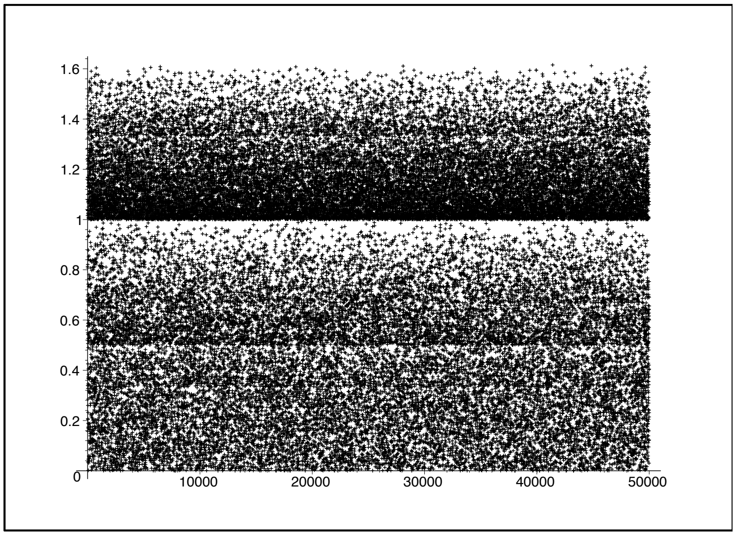

Figure 3: The values of $\mathcal{E}(\alpha)$ for 50 000 at random generated points

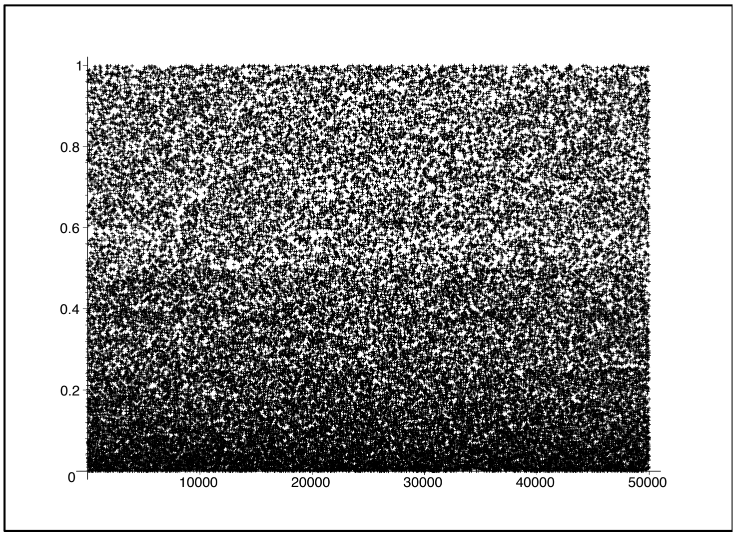

Figure 4: The values of $\mathcal{E}^*(\alpha)$ for 50 000 at random generated points

Figure 1 implies that the inequalities $1 \le \mathcal{E}(\alpha)/\alpha \le \rho^2$ hold for every $\alpha \in (0,1)$. Indeed we have $\mathcal{E}(\alpha) = \alpha$ for $\alpha = 1/k$ and $\mathcal{E}(\alpha) = \rho^2\alpha$ for $\alpha = \langle 0; k, 1, 1, 1, \ldots \rangle = 1/(k-1+\rho)$ with $k \in \mathbb{N}$, where the latter equation follows from Lemma 13. Moreover, $\mathcal{E}(\alpha) \ge 1$ for $\alpha > 1/2$, and $\mathcal{E}(1,1) = \mathcal{E}(1,k-1) = \mathcal{E}((k-1)/k) = 1$ for every integer $k \ge 3$.

Concerning Figure 2 one may guess that $0 \le \mathcal{E}(\alpha) \le \alpha$ holds for every $\alpha \in (0,1)$. More precisely we have $\mathcal{E}^*(\alpha) = \alpha$ for $\alpha = 1/k$ with $k \in \mathbb{N}$, $\mathcal{E}^*(1,k-1,1) = k/(k+1) = \langle 0; 1, k-1, 1 \rangle$ for every integer $k \ge 2$, and $\mathcal{E}^*(k,1) = 0$ for $k \in \mathbb{N}$. Moreover, $\mathcal{E}^*(\alpha) \ge 2\alpha - 1$ for $\alpha \ge 1/2$, and we have $\mathcal{E}^*((k-1)/k) = (k-2)/k = 2(k-1)/k - 1$ for every integer $k \ge 3$.

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