IMPROVING THE CHEN AND CHEN RESULT FOR ODD PERFECT NUMBERS

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Abstract

If q^α is the Euler factor of an odd perfect number N, then we prove that its so-called index $\sigma(N/q^\alpha)/q^\alpha \geq 3^2 \times 5 \times 7 = 315$. It follows that for any odd perfect number, the ratio of the non-Euler part to the Euler part is greater than $3^2 \times 5 \times 7/2$.

1. Introduction

The main motivation for studying the structure of an odd perfect number is ultimately to establish that such a number cannot exist. It is known that any odd perfect number N must have at least 9 distinct prime factors [10], be larger than 10^{1500} [12], have a squarefree core which is less than $2N^{\frac{17}{26}}$ [9], and every prime divisor is less than $(3N)^{\frac{1}{3}}$ [1]. These results represent recent progress on what must be one of the oldest current problems in mathematics.

Following Dris [5], in this paper we define the index m of a prime power dividing N. Using a lower bound for the index one can derive an upper bound, in terms of N, for the Euler factor of N. Dris found the bound $m \geq 3$; then Dris and Luca [6] improved this to $m \geq 6$. In [4] a list of forms in terms of products of prime powers, which includes the results of Dris and Dris-Luca, is derived. We improve the method of [4], obtaining an expanded list of prime power products which cannot occur as the value of an index. This enables us to conclude, in the case of the Euler factor, that $m \geq 315$; for any other prime, if the Euler factor divides N to a power at least 2 then $m \geq 630$, and if the Euler factor divides N to the power 1 then $m \geq 210$.

Notations: $\Omega(n)$ is the total number of prime divisors of n counted with multiplicity, $\omega(n)$ the number of distinct prime divisors of n, $\omega_0(n)$ is the number of distinct odd prime divisors of n, $\sigma(n)$ the sum of the divisors of n, $d(n)$ the number of divisors of n, $\log_2 n$ the logarithm to base 2, $(a,\ b)$ the greatest common divisor, $p^e \| n$ means p^e divides n but p^{e+1} does not, $\nu_p(n)$ the highest power of p which divides n, and $\operatorname{ord}_p a$ is the smallest power of a which is congruent to 1 modulo p. The symbol $\Box$, when not being used to denote the end of a proof, represents the square of an integer.

Let N denote an odd perfect number, and q a prime divisor with $q^\alpha \| N$ say. We write the standard factorization of N as

$$N \ = \ q^\alpha \times \prod_{i=1}^{k} p_i^{\lambda_i} \times \prod_{j=k+1}^{s} p_j^{\lambda_j}$$

where for $1 \leq i \leq k$ we have

$$\sigma\left(p_i^{\lambda_i}\right) = m_i q^{\beta_i},\ \beta_i \geq 0,\ (m_i, q) = 1,\ m_i > 1. \tag{1}$$

These prime numbers p_i are called primes of **type 1**. For $k+1 \leq j \leq s$

$$\sigma\left(p_j^{\lambda_j}\right) = q^{\beta_j},\ \ \beta_j > 0 \tag{2}$$

and the p_j are called primes of **type 2**.

One defines the **index** or **perfect number index at prime** q to be the integer

$$m := \frac{\sigma\left(N/q^\alpha\right)}{q^\alpha}\ ; \tag{3}$$

in particular $m = m_1 \cdots m_k$.

In fact $4 \nmid m$, $q \nmid m$, and if an odd prime p satisfies $p^e \mid m$ then $p^e \mid N$. Furthermore if q is the Euler prime, then m is odd and each m corresponding to any other prime is even. Lastly we have the fundamental equation

$$m \times \sigma(q^\alpha) \ = \ 2 \times \prod_{i=1}^{k} p_i^{\lambda_i} \times \prod_{j=k+1}^{s} p_j^{\lambda_j} \ = \ \frac{2N}{q^\alpha}. \tag{4}$$

2. Preliminary Results

First we state the theorem of Chen and Chen [4].

Theorem 1 *If N is an odd perfect number with a prime power $q^{\alpha}\|N$, then the index $m := \sigma(N/q^{\alpha})/q^{a}$ is not equal to any of the six forms*

$$\{p_1,\ p_1^2,\ p_1^3,\ p_1^4,\ p_1p_2,\ p_1^2p_2\}$$

where p_1 and p_2 are any distinct primes.

The following lemma comes from [6]. Here we give an alternative proof.

Lemma 2 *If for some j with $k+1 \le j \le s$ (so p_j is a prime of type 2) and for some γ with $2 \le \gamma \le \lambda_j$ we have $p_j^{\gamma} \mid (q^{\alpha+1}-1)/(q-1)$, then $p_j^{\gamma-1} \mid \alpha+1$.*

Proof. Because $p_j\big(1+p_j+\cdots+p_j^{\lambda_j-1}\big) = q^{\beta_j}-1$ one deduces $p_j^1\|q^{\beta_j}-1$, in which case $p_j^1\|q^{\operatorname{ord}_{p_j}(q)}-1$. However

$$2 \ \le\ \gamma\ \le\ \nu_{p_j}\left(\frac{q^{\alpha+1}-1}{q-1}\right) = \nu_{p_j}\left(\frac{q^{\operatorname{ord}_{p_j}(q)}-1}{q-1}\right) + \nu_{p_j}\left(\frac{\alpha+1}{\operatorname{ord}_{p_j}(q)}\right). \qquad (5)$$

If $\operatorname{ord}_{p_j}(q) = 1$ then $\gamma \le \nu_{p_j}(\alpha+1)$ and $p_j^{\gamma} \mid \alpha+1$, whereas if $\operatorname{ord}_{p_j}(q) > 1$ one has $\gamma \le 1+\nu_{p_j}(\alpha+1)$ and therefore $p_j^{\gamma-1} \mid \alpha+1$. □

Lemma 3 (Ljunggren, see [7]) *The only integer solutions $(x,\ n,\ y)$ with $|x| > 1,\ n > 2,\ y > 0$ to the equation $(x^n-1)/(x-1) = y^2$ are $(7,\ 4,\ 20)$ and $(3,\ 5,\ 11)$, i.e. $(7^4-1)/(7-1) = 20^2$ and $(3^5-1)/(3-1) = 11^2$.*

Lemma 4 *[7] The only solutions in non-zero integers with $n > 1$ to the equation $y^n = x^2+x+1$ are $n = 3$, $y = 7$ and $x = 18$ or $x = -19$.*

The following well known result [2, 3, 13] guarantees the existence of primitive prime divisors for expressions of the form a^n-1 with fixed $a > 1$.

Lemma 5 *Let a and n be integers greater than 1. Then there exists a prime $p \mid a^n-1$ which does not divide any of a^m-1 for each $m \in \{2,\dots,n-1\}$, except possibly in the two cases $n = 2$ and $a = 2^{\beta}-1$ for some $\beta \ge 2$, or $n = 6$ and $a = 2$. Such a prime is called a* **primitive prime factor**.

We complete this set of preliminary results by filling in the missing case from the proof of the fundamental lemma [4, Lemma 2.4].

Lemma 6 *Let N be an odd perfect number. Then $d(\alpha+1) \le \omega(N)$ whenever a prime power $q^{\alpha}\|N$.*

Proof. Let $n_1,\ n_2,\dots,\ n_w$ denote all the distinct positive divisors of $\alpha+1$ which are greater than 1.

If $2 \mid \alpha+1$ then α is odd, and thus $q \equiv \alpha \equiv 1 \bmod 4$. Therefore q cannot be of the form $2^\beta - 1$ and must be odd. By Lemma 5 there exists a primitive prime factor $q_i \mid q^{n_i} - 1$; since $2 \mid q^1 - 1$ the q_i are all odd, and as they are primitive, one finds $q_i \nmid q^1 - 1$ also. Hence

$$q_i \;\mid\; \frac{q^{n_i}-1}{q-1} \;\mid\; \frac{q^{\alpha+1}-1}{q-1}$$

so that $q_{n_1} \cdots q_{n_w} \mid (q^{\alpha+1}-1)/(q-1)$. But $m \times \sigma(q^\alpha) = 2N/q^\alpha$ thus, including the divisor 1 and recalling $2 \mid \sigma(q^\alpha)$, one obtains the inequalities

$$d(\alpha+1) = w+1 \;\le\; \omega\big(\sigma(q^\alpha)\big) \;\le\; \omega\big(m\sigma(q^\alpha)\big) = \omega\left(\frac{2N}{q^\alpha}\right) = \omega(N).$$

Alternatively if $2 \nmid \alpha+1$ then α is even so, again by Lemma 5, we obtain distinct odd primes q_{n_i} with

$$q_{n_1} \cdots q_{n_w} \mid \frac{q^{\alpha+1}-1}{q-1}.$$

Because in this case $2 \mid m$ and $2 \nmid \sigma(q^\alpha)$, we deduce that

$$d(\alpha+1) = 1+w \;\le\; 1+\omega\big(\sigma(q^\alpha)\big) \;\le\; \omega\big(m\sigma(q^\alpha)\big) = \omega\left(\frac{2N}{q^a}\right) = \omega(N)$$

which completes the proof of the lemma. □

3. The Proof

We now amend the proof of Theorem 1.1 of [4].

Lemma 7 *Let N be an odd perfect number, and m the index at some prime divisor of N. Then*

$$\Omega(m) + \omega_0(m) \;\ge\; \omega(N) - \log_2 \sqrt{\omega(N)} - \eta$$

where $\eta = 1$ if m is odd, $\eta = \frac{1}{2}$ if m is even and the Euler prime divides N to a power greater than 1, and $\eta = \frac{3}{2}$ if m is even and the Euler prime divides N exactly to the power 1.

Proof. Whenever $(m,\ p_{k+1} \cdots p_s) =\ p_{k+1} \cdots p_s$, one has an inequality

$$s - k \;\le\; \omega_0(m) = t$$

and it follows that

$$k + t \;\ge\; s = \omega(N) - 1.$$

Because $k \le \Omega(m)$, $t = \omega_0(m)$ and $\omega(N) \ge 9$, we quickly deduce

$$\Omega(m) + \omega_0(m) \;\ge\; k + t \;\ge\; \omega(N) - 2 \;\ge\; \omega(N) - \log_2 \sqrt{\omega(N)} - 0.42.$$

The non-trivial case occurs when $(m,\ p_{k+1}\cdots p_s) \neq\ p_{k+1}\cdots p_s$. By suitably reordering the p_i, we can always write for some l with $k \le l < s$:

$$\frac{p_{k+1}\cdots p_s}{(m, p_{k+1}\cdots p_s)} \;=\; p_{l+1}\cdots p_s. \tag{6}$$

Applying [4] Equation (2.2) and (6), we see that

$$p_{l+1}^{\lambda_{l+1}}\cdots p_s^{\lambda_s} \mid \sigma(q^\alpha).$$

Moreover using [4] Equation (2.1) and [4] Lemma 2.3,

$$p_i^{\lambda_i - 1} \mid \alpha + 1, \quad l+1 \le i \le s$$

hence

$$p_{l+1}^{\lambda_{l+1}-1}\cdots p_s^{\lambda_s - 1} \mid \alpha + 1.$$

Now for $k+1 \le i \le s$ one knows $\sigma(p_i^{\lambda_i}) = q^{\beta_i}$, and q is odd so we must have λ_i even. It follows for $l+1 \le i \le s$ each $\lambda_i \ge 2$, thus $p_{l+1}\cdots p_s \mid \alpha+1$. Note also that $l < s$ in which case $s - l \ge 1$.

If $s - l = 1$ then because $\omega(N) \ge 9$,

$$\Omega(m) + \omega_0(m) \ge k + t \ge l = s - 1 \ge \omega(N) - 2 \ge \omega(N) - \log_2 \sqrt{\omega(N)} - 0.42$$

as in the previous case.

If $s - l \ge 2$ then we **claim** at most one of the $\lambda_i = 2$ and the remainder have $\lambda_i \ge 4$. To see this, consider the equations

$$p_i^2 + p_i + 1 = q^{\beta_i}.$$

If $\beta_i > 1$ then, by Lemma 4, the only solution is $\beta_i = 3$, $q = 7$ and $p_i = 18$ which is not prime, so the solution cannot occur in this context. Hence $\beta_i = 1$ and the form of the equation is $q = x^2 + x + 1$. But this, for given q, has at most one positive integer solution, therefore at most one prime solution p_i.

By renumbering the p_i if necessary, when $s - l \ge 2$ we can write

$$p_{l+1}^3 p_{l+2}^3 \cdots p_{s-1}^3 p_s \mid \alpha + 1.$$

Case 1. Suppose that the index m is odd. Then q is the Euler prime, and consequently $2 \mid \alpha + 1$. Hence

$$2p_{l+1}^3 p_{l+2}^3 \cdots p_{s-1}^3 p_s \mid \alpha + 1,$$

and thus, by Lemma 6, we have

$$2^{2s-2l} \leq d(\alpha+1) \leq \omega(N),$$

or in other words $s-l \leq \log_2 \sqrt{\omega(N)}$, which implies

$$l \geq \omega(N) - \log_2 \sqrt{\omega(N)} - 1.$$

As $\omega_0(m) = t$ then by Equation (6) we have $l-k \leq t$, so $l \leq \Omega(m) + \omega_0(m)$. Lastly because $\omega(N) \geq 9$,

$$6.41 \leq \omega(N) - \log_2 \sqrt{\omega(N)} - 1 \leq l \leq \Omega(m) + \omega_0(m).$$

`Case 2.` Here we assume the Euler prime divides N to a power at least 2. Let m be even. Now $m = m_1 \cdots m_k$ and $2 \| m$ so, for a unique i, one knows that $2 \mid m_i$. We **claim** that $2 \neq m_i$. If not, then

$$\sigma(p_i^{\lambda_i}) = 2q^{\beta_i}$$

whence p_i is the Euler prime and $\lambda_i + 1$ is even; we can write

$$\frac{p_i^{\lambda_i+1} - 1}{2(p_i-1)} = \left(\frac{p_i^{\frac{\lambda_i+1}{2}} - 1}{p_i - 1} \right) \times \left(\frac{p_i^{\frac{\lambda_i+1}{2}} + 1}{2} \right) = q^{\beta_i}$$

but this cannot hold since the two factors in the middle term are coprime and greater than 1, thus $2 \neq m_i$.

It follows that $k \leq \Omega(m) - 1$. In this scenario with $s - l \geq 2$, we also know

$$p_{l+1}^3 p_{l+2}^3 \cdots p_{s-1}^3 p_s \mid \alpha + 1$$

thus

$$2^{2s-2l-1} \leq d(\alpha+1) \leq \omega(N),$$

which in turn implies

$$l \geq s - \frac{1}{2} - \log_2 \sqrt{\omega(N)} = \omega(N) - \log_2 \sqrt{\omega(N)} - \frac{3}{2}.$$

It follows from the discussion that

$$\omega(N) - \log_2 \sqrt{\omega(N)} - \frac{3}{2} \leq l \leq k + t \leq \Omega(m) - 1 + \omega_0(m)$$

and therefore

$$6.91 \leq \omega(N) - \log_2 \sqrt{\omega(N)} - \frac{1}{2} \leq \Omega(m) + \omega_0(m).$$

`Case 3.` We shall now assume the Euler prime divides N exactly to the power 1 and that m is even. Here we have only the weaker inequality $k \leq \Omega(m)$, and using an identical argument to Case 2:

$$5.91 \leq \omega(N) - \log_2 \sqrt{\omega(N)} - \frac{3}{2} \leq \Omega(m) + \omega_0(m).$$

□

Lemma 8 *If the index m is a square, then $\alpha = 1$.*

Proof. If $m = \square$ then necessarily q is the Euler prime. We must have $\sigma(q^\alpha) = 2\square$ and α is odd. Assuming $\alpha > 1$ then

$$\frac{1}{2}\left(\frac{q^{\alpha+1}-1}{q-1}\right) = \left(\frac{q^{(\alpha+1)/2}-1}{q-1}\right) \times \left(\frac{q^{(\alpha+1)/2}+1}{2}\right) = \square$$

and the two factors in the penultimate term are coprime, in which case

$$\frac{q^{(\alpha+1)/2}-1}{q-1} = \square.$$

By Lemma 3 one has $(\alpha+1)/2 \leq 2$, and as $\alpha \equiv 1 \bmod 4$ we deduce $\alpha = 1$, thereby yielding a contradiction. $\square$

Lemma 9 *If the index m is odd, then it cannot be the sixth power of a prime.*

Proof. Firstly the index being odd means it corresponds to the Euler prime. Assume $m = p^6 = \square$. By Lemma 8, we have $\alpha = 1$. If $p = p_I$ is of type 1 then $\sigma(p_I^{\lambda_I}) = p_I^\theta q^{\beta_I}$ for some $\theta > 0$, which is false. Hence p_I will be type 2. If any other prime p_j were also of type 2, then due to the equality $\sigma(q^\alpha) = \frac{2N}{q^\alpha p_I^6}$ we would have $p_j^2 \mid \sigma(q^\alpha)$ and also $q^{\beta_j} = \frac{p_j^{\lambda_j+1}-1}{p_j-1}$; however from Lemma 2 there is a divisibility $p_j \,\big|\, \alpha + 1 = 2$, which is clearly false as $p_j \geq 3$.

Consequently there exists exactly one type 2 prime, p_I. Note that $\lambda_I \geq 6$. If $\lambda_I \neq 6$ we would have λ_I even and greater than 6, implying $p_I^2 \mid \sigma(q^\alpha)$ and by Lemma 2, $p_I \mid \alpha + 1$ which is false. Hence $\lambda_I = 6$ and we can write $\sigma(q^\alpha) = 2p_1^{\lambda_1} \cdots p_k^{\lambda_k}$. But $m = p^6 = m_1 \cdots m_k$ has at most 6 factors, in which case $k \leq 6$; therefore $9 \leq \omega(N) = k + 2 \leq 8$ a clear contradiction, completing the proof that $m \neq p^6$. $\square$

Applying Lemmas 7 and 9, we have shown

Theorem 10 *If N is an odd perfect number and the odd prime $q^\alpha \| N$ then the index $\sigma(N/q^\alpha)/q^\alpha$ is either odd when q is the Euler prime, or even but not divisible by 4 when q is not the Euler prime.*
(i) If q is the Euler prime, it cannot take any of the 11 forms $\{p, p^2, p^3, p^4, p^5, p^6, p_1p_2,\ p_1^2p_2,\ p_1^3p_2,\ p_1^2p_2^2,\ p_1p_2p_3\}$ where p is any odd prime and p_1, p_2, p_3 are any distinct odd primes.

(ii) If q is not the Euler prime and the Euler prime divides N to a power greater than 1, it cannot take any of the 7 forms $\{2,\ 2p,\ 2p^2,\ 2p^3,\ 2p^4,\ 2p_1p_2,\ 2p_1^2p_2\}$.

(iii) If q is not the Euler prime and the Euler prime divides N to the power 1, it cannot take any of the 5 forms $\{2,\ 2p,\ 2p^2,\ 2p^3,\ 2p_1p_2\}$.

Therefore the smallest possible value of the index m is, respectively:

$$3^2 \times 5 \times 7 = 315 \quad \textit{in case (i)},$$

$$2 \times 3^2 \times 5 \times 7 = 630 \quad \textit{in case (ii)},$$

$$\textit{and} \quad 2 \times 3 \times 5 \times 7 = 210 \quad \textit{in case (iii)}.$$

Corollary 11 *It follows directly that for any odd perfect number, the ratio of the non-Euler part to the Euler part is greater than $315/2$.*

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