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SOME COMBINATORIAL SERIES AND RECIPROCAL RELATIONS INVOLVING MULTIFOLD CONVOLUTIONS

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Abstract

The present paper considers a kind of combinatorial series and its allied reciprocal relations which are determined by discrete multi-fold convolutions. Furthermore, their various formal and analytic expressions in explicit forms are obtained. Constructive applications to some well-known sequences such as the Bell numbers, the Fibonacci numbers, the Stirling numbers and some others given by integer partition functions are also presented.

1. Introduction

In this paper, we obtain various explicit constructive results for combinatorial sums or series. These results are closely related to what we will call the discrete multi-fold convolutions. Our paper consists of two main parts. The first part, composed of Sections 2, 3, and 4, deals with one kind of discrete multi-fold convolution which is determined by an arbitrary function on the set of positive integers. Some basic identities and delta operator summation formulae are investigated and illustrated with examples. The second part, consisting of Sections 5 and 6, discusses another general class of discrete multi-fold convolutions which are formed by finitely many different functions with discrete variables. Constructive applications to some well-known number sequences are expounded in detail.

In this paper, we will denote the sets of positive integers, non-negative integers, real numbers and complex numbers by $\mathbb{N}_+, \mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ respectively. Also, we will use the following notation:

- $\sigma(n)$: the set of partitions of $n \in \mathbb{N}_+$, usually written as $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1 + 2k_2 + \cdots + nk_n = n$, $k_i \in \mathbb{N}$.

- $\sigma(n,k)$: the subset of $\sigma(n)$ consisting of the partitions of n with k parts, i.e., partitions $1^{k_1}2^{k_2}\cdots n^{k_n}$ subject to $k_1+k_2+\cdots+k_n=k$.
- $(t)_k=t(t-1)(t-2)\cdots(t-k+1)$: the kth falling factorial of t with $(t)_0=1$.
- Δ : the difference operator defined by $\Delta f(t)=f(t+1)-f(t)$, and $\Delta\Delta^k=\Delta^{k+1}(k\in\mathbb{N}_+)$ with $\Delta^0=1$ denoting the identity operator.
- D : the differentiation operator with $DD^k=D^{k+1}$ and $D^0=1$.
- E : the shift operator defined by $E=1+\Delta$ and $E^x f(t)=f(t+x), x\in\mathbb{R}$.

It should be pointed out that both Δ and D, nowadays known as delta operators, could be generally applied to formal power series to deduce certain formal results. Throughout the present paper, we will make frequent use of formal power series, and provide suitable convergence conditions so that the formal results actually are exact and analytic under these conditions.

2. A Kind of Multi-Fold Convolution

As usual, we will say that $(x_1,x_2,\cdots,x_k)$ is a *k-composition* of n with non-negative parts, if $x_1+x_2+\cdots+x_k=n$, where $x_i\in\mathbb{N}, 1\le i\le k$. The set of all such compositions of n may be denoted by $[n,k,0]$, i.e.,

$$[n,k,0]:=\big\{(x_1,x_2,\cdots,x_k)\big|\sum_{i=1}^{k}x_i=n, x_i\ge 0\big\}.$$

Also, we will use the standard representation for the set $\sigma(n,k)$, i.e.,

$$\sigma(n,k):=\big\{(x_1,x_2,\cdots,x_n)\big|\sum_{i=1}^{n}ix_i=n,\sum_{i=1}^{n}x_i=k, x_i\ge 0\big\}.$$

Definition 2.1. Let $f(x)$ be a real-valued or complex-valued function defined on $\mathbb{N}$ with $f(0)=1$. Then the k-fold convolution and the n/k-partition sum associated with $f(x)$ are respectively defined by the following summations:

$$S_n^k(f) = \sum_{[n,k,0]} f(x_1)f(x_2)\cdots f(x_k) \tag{2.1}$$

$$T_n^k(f) = \sum_{\sigma(n,k)} \frac{f^{k_1}(1)f^{k_2}(2)\cdots f^{k_n}(n)}{k_1!k_2!\cdots k_n!} \tag{2.2}$$

where the sums on the right-hand sides (in short, RHS) of (2.1) and (2.2) range over the sets $[n,k,0]$ and $\sigma(n,k)$ respectively.

Note that $[n,k,0]$ and $\sigma(n,k)$ have no meaning for $k=0$. For convenience, we define $S_n^0(f)=T_n^0(f)=0$, and thereby have sequences $\{S_n^k(f)\}_{n,k\geq 0}$ and $\{T_n^k(f)\}_{n,k\geq 0}$. Also it is obvious that $S_n^1(f)=f(n), S_1^k(f)=kf(1)$ and

$$T_n^1(f)=f(n), T_1^1(f)=f(1);\ T_1^k(f)=0 \text{ for } k>1.$$

Moreover, it is easy to see that the RHS of (2.2) with replacement $f(n)\to t_n (n\in \mathbb{N}_+)$ is in agreement with the incomplete Bell polynomial in $t_i (1\leq i\leq n)$, up to a constant factor $n!$ (cf.[2]).

In what follows we assume that

$$G(t)=\sum_{k=0}^{\infty} g(k)t^k \tag{2.3}$$

is a formal power series with complex coefficients. As usual, $G^{(k)}(t)=D^kG(t)$ denotes the kth formal derivative of $G(t)$.

Theorem 2.2. *For $m,n\in\mathbb{N}_+$, the following identities hold:*

$$S_n^m(f) = \sum_{k=1}^{m}(m)_k T_n^k(f) \tag{2.4}$$

$$T_n^k(f) = \frac{1}{k!}\Delta^k S_n^t(f)\big|_{t=0} \tag{2.5}$$

$$\sum_{k=1}^{\infty} g(k)S_n^k(f)t^k = \sum_{k=1}^{n} G^{(k)}(t)T_n^k(f)t^k, \tag{2.6}$$

where the left-hand side (in short, LHS) of (2.6) is a formal power series.

Proof. To justify (2.4), according to Definition 2.1, we only need to compute the LHS of (2.4) in this way: for $1\leq k\leq m$, consider first the finite sum

$$\sum_{[n,m,0]_k} f(x_1)f(x_2)\cdots f(x_m), \tag{2.7}$$

where $[n,m,0]_k$ denotes the subset of $[n,m,0]$, being composed of all compositions $(x_1,x_2,\cdots,x_m)$ of n with just k components $x_i\geq 1$. In other words, there are $m-k$ components $x_i=0$ in $(x_1,x_2,\cdots,x_m)$. Recall that $f(0)=1$ and such factors will take $m-k$ ordered places in $\binom{m}{m-k}=\binom{m}{k}$ different ways. Meanwhile, the number of all possible permutations of the factors in the product

$$f^{k_1}(1)f^{k_2}(2)\cdots f^{k_n}(n)$$

over the set $\sigma(n,k)$ is enumerated by $k!/(k_1!k_2!\cdots k_n!)$. Thus the sum (2.7) boils down to

$$\binom{m}{k}\sum_{\sigma(n,k)}\frac{k!}{k_1!k_2!\cdots k_n!}f^{k_1}(1)f^{k_2}(2)\cdots f^{k_n}(n)=(m)_kT_n^k(f).$$

Summing on $k, 1 \le k \le m$, we therefore obtain (2.4).

With (2.4) in hand we are able to reformulate $S_n^t(f)$ in the form

$$S_n^t(f) = \sum_{j\ge 1} j!\binom{t}{j} T_n^j(f),\ S_n^0(f) = 0.$$

Then it follows that

$$\begin{aligned}\Delta^k S_n^t(f)\big|_{t=0} &= \sum_{j\ge 1} j!\Delta^k \binom{t}{j}\big|_{t=0} T_n^j(f) \\ &= \sum_{j\ge 1} j!\binom{0}{j-k} T_n^j(f) = k!\, T_n^k(f).\end{aligned}$$

Thus (2.5) is proved.

Once again, by use of (2.4) we may compute the LHS of (2.6) formally as follows:

$$\begin{aligned}\sum_{m=1}^{\infty} g(m) S_n^m(f) t^m &= \sum_{m=1}^{\infty} g(m) t^m \sum_{k=1}^{m} (m)_k T_n^k(f) \\ &= \sum_{k=1}^{\infty} \Bigg(\sum_{m=k}^{\infty} (m)_k g(m) t^m \Bigg) T_n^k(f) \\ &= \sum_{k=1}^{\infty} D^k \Bigg(\sum_{m=k}^{\infty} g(m) t^m \Bigg) T_n^k(f) t^k \\ &= \sum_{k=1}^{n} G^{(k)}(t) T_n^k(f) t^k.\end{aligned}$$

The last equality follows from the fact that $T_n^k(f) = 0$ for $k > n$. This completes the proof of (2.6). □

Put $G(t) = 1/(1-t)$ for $|t| < 1$ and e^t for $t \in \mathbb{R}$ in (2.6) in succession. Then we may get the following

Corollary 2.3. *For $n \in \mathbb{N}$, the sequence $\{S_n^k(f)\}_{k\ge 0}$ has the ordinary and exponential generating functions, respectively, as follows:*

$$\sum_{k=0}^{\infty} S_n^k(f) t^k = \sum_{k=1}^{n} \frac{k!}{1-t}\left(\frac{t}{1-t}\right)^k T_n^k(f) \tag{2.8}$$

$$\sum_{k=0}^{\infty} S_n^k(f) \frac{t^k}{k!} = e^t \sum_{k=1}^{n} T_n^k(f) t^k. \tag{2.9}$$

It should be mentioned that as an extension of a basic result contained in [15] by Savits and Constantine, formula (2.6) was first given by Hsu in his short note

[10]. Afterwards, it was recovered by Constantine [3] via a different approach. In fact, formula (2.6) is actually a consequence implied by the general transformation formula of series (cf.[8])

$$\sum_{k=0}^{\infty} f(k)\phi^{(k)}(0)\frac{t^k}{k!} = \sum_{k=0}^{\infty} \Delta^k f(0)\phi^{(k)}(t)\frac{t^k}{k!}.$$

To see this, it suffices to substitute $f(t)$ with $S_n^t(f)$ and $\phi(t)$ with $G(t)$, and then simplify the resulting identity by (2.5).

3. A Convergence Theorem

In this section, we will provide a simple and verifiable convergence condition for (2.6) in order to make it an available exact formula.

Theorem 3.1. *If $G(t) = \sum_{k\geq 0} g(k)t^k$ is absolutely convergent for $|t| < r$, then so is the infinite series on the LHS of (2.6), thereby (2.6) is an exact formula for $|t| < r$.*

Proof. Comparing the LHS of (2.6) with $G(t)$ and using Cauchy's root test for the convergence of infinite series, we only need to show that for every $n \in \mathbb{N}_+$,

$$\overline{\lim_{k\to\infty}} \left|S_n^k(f)\right|^{1/k} \leq 1.$$

To this end, assume $\max\limits_{0\leq x\leq n} |f(x)| = \rho$. By the definition of $S_n^k(f)$, it is easily found that every product $f(x_1)f(x_2)\cdots f(x_k)$ restricted by

$$x_1 + x_2 + \cdots + x_k = n \tag{3.1}$$

contains at most n factors $f(x_i)$ with $x_i \geq 1$. Owing to the facts that $f(0) = 1$ and $|f(x)| < \rho$ for $x \neq 0$, it follows directly that $|f(x_1)f(x_2)\cdots f(x_k)| \leq \rho^n$. Meanwhile, the total number of such terms is $\binom{n+k-1}{n}$, a fact coming from the number of solutions in nonnegative integers for the diophantine equation (3.1). Thus we have

$$\left|S_n^k(f)\right| \leq \binom{n+k-1}{n} \rho^n,$$

that leads to

$$\overline{\lim_{k\to\infty}} \left|S_n^k(f)\right|^{1/k} \leq \overline{\lim_{k\to\infty}} \left(\binom{n+k-1}{n} \rho^n\right)^{1/k} = 1.$$

Hence the theorem is proved. $\square$

Corollary 3.2. *Identity (2.8) is analytic for $|t| < 1$ and so is (2.9) for $|t| < +\infty$.*

Example 3.3. Evidently, the following power series

$$G(t) = \frac{1}{\sqrt{1-4t}} = \sum_{k=0}^{\infty} \binom{2k}{k} t^k$$

converges absolutely for $|t| < 1/4$. In this case, we have

$$G^{(k)}(t) = \frac{(2k)!}{k!}(1-4t)^{-k-1/2}, \quad g(k) = \binom{2k}{k}.$$

As a consequence of (2.6), we get the following exact formula

$$\sum_{k=0}^{\infty} \binom{2k}{k} S_n^k(f) t^k = \sum_{k=1}^{n} \frac{(2k)!}{k!} \frac{t^k}{(1-4t)^{k+1/2}} T_n^k(f) \tag{3.2}$$

for $|t| < 1/4$.

Example 3.4. It is well known that the Fibonacci number sequence $\{F_n\}_{n\geq 0}$ is generated by the generating function

$$G(t) = \frac{1}{1-t-t^2} = \sum_{k=0}^{\infty} F_k t^k,$$

which is convergent absolutely for $|t| < (\sqrt{5}-1)/2$. To ease notations, we let $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. We therefore have

$$G(t) = \frac{1}{(1-at)(1-bt)} = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (a^{k+1} - b^{k+1}) t^k$$

and $g(k) = F_k = (a^{k+1} - b^{k+1})/\sqrt{5}$. Simple computation gives

$$G^{(k)}(t) = \frac{1}{\sqrt{5}} D^k \left(\frac{a}{1-at} - \frac{b}{1-bt} \right) = \frac{k!}{\sqrt{5}} \left[\left(\frac{a}{1-at} \right)^{k+1} - \left(\frac{b}{1-bt} \right)^{k+1} \right],$$

that reduces (2.6) to

$$\sum_{k=0}^{\infty} F_k S_n^k(f) t^k = \sum_{k=1}^{n} \frac{k!}{\sqrt{5}} \left[\left(\frac{a}{1-at} \right)^{k+1} - \left(\frac{b}{1-bt} \right)^{k+1} \right] T_n^k(f) t^k, \tag{3.3}$$

which is an exact formula for $|t| < (\sqrt{5}-1)/2$.

Example 3.5. Setting $G(t) = (1+t)^\alpha$ for $\alpha \in \mathbb{R}$ and $|t| < 1$, we have $g(k) = \binom{\alpha}{k}$ and $G^{(k)}(t) = (\alpha)_k (1+t)^{\alpha-k}$. A direct substitution of these facts into (2.6) yields

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} S_n^k(f) t^k = \sum_{k=1}^{n} (\alpha)_k \frac{t^k}{(1+t)^{k-\alpha}} T_n^k(f). \tag{3.4}$$

The further substitution $\alpha \to -\alpha - 1$ leads us to

$$\sum_{k=0}^{\infty} \binom{\alpha+k}{k} S_n^k(f)(-t)^k = \sum_{k=1}^{n} (\alpha+k)_k \frac{(-t)^k}{(1+t)^{k+\alpha+1}} T_n^k(f). \tag{3.5}$$

Obviously, both (3.4) and (3.5) are analytic for $|t| < 1$. It is also clear that (2.8) can be deduced from (3.4) via the substitutions $\alpha \to -1$ and $\alpha \to -t$.

As indicated above, formula (2.6) can be used to construct various special exact formulae or identities via the choices of $G(t)$ and $f(t)$, provided that they are subject to the convergence conditions given by Theorem 3.1.

4. Some Operator Summation Formulae

It is known that both D and Δ are delta operators, so that by use of Mullin-Rota's substitution rule (cf.[9]), we may deduce some special operator summation formulae from (2.9), (3.2), (3.4) and (3.5) respectively.

For any function $\phi(t)$ over $\mathbb{R}$, it is easy to check that

$$(1+\Delta)^\alpha \phi(t) = E^\alpha \phi(t) = \phi(t+\alpha).$$

Thus, under the substitution $t \to \Delta$, we see that (3.4) and (3.5) together yields a pair of Δ-type operator summation formulae. The results are as follows:

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{\alpha}{k} S_n^k(f) \Delta^k \phi(0) &= \sum_{k=1}^{n} (\alpha)_k T_n^k(f) \Delta^k \phi(\alpha - k) & (4.1) \\ \sum_{k=0}^{\infty} \binom{\alpha+k}{k} S_n^k(f)(-\Delta)^k \phi(0) &= \sum_{k=1}^{n} (\alpha+k)_k T_n^k(f)(-\Delta)^k \phi(-\alpha-k-1). & (4.2) \end{aligned}$$

Alternatively, the substitution $t \to -\frac{1}{4}\Delta$ reduces (3.2) to a Δ-type summation formula of the form

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}} \binom{2k}{k} S_n^k(f) \Delta^k \phi(0) = \sum_{k=1}^{n} \frac{(-1)^k (2k)!}{2^{2k} k!} T_n^k(f) \Delta^k \phi(-k - 1/2). \tag{4.3}$$

As above, substituting t by D in (2.9) and simplifying the result by the relations that $e^D = E = 1+\Delta$, we come up with a D-type summation formula for $\phi(t) \in C^\infty$ (the set of infinitely differentiable real functions over $\mathbb{R}$) evaluated at $t = 0$:

$$\sum_{k=0}^{\infty} \frac{1}{k!} S_n^k(f) D^k \phi(0) = \sum_{k=1}^{n} T_n^k(f) D^k \phi(1). \tag{4.4}$$

In accordance with Example 3.5 in Section 3, it is clear that (4.1) and (4.2) are exact formulae under the condition

$$\overline{\lim_{k\to\infty}}\left|\Delta^k\phi(0)\right|^{1/k} < 1. \tag{4.5}$$

Actually condition (4.5) also ensures the validity of formula (4.3) inasmuch as it is just equivalent to the condition (see Example 3.3)

$$\overline{\lim_{k\to\infty}}\left|\frac{1}{4^k}\Delta^k\phi(0)\right|^{1/k} < \frac{1}{4}.$$

Moreover, Corollary 3.2 states that (4.4) is an exact formula under the condition

$$\overline{\lim_{k\to\infty}}\left|D^k\phi(0)\right|^{1/k} < +\infty. \tag{4.6}$$

It may happen that $\lim_{k\to\infty}\left|\Delta^k\phi(0)\right|^{1/k} = 1$. In this case, the convergence conditions for (4.1), (4.2), and (4.3) should be investigated separately.

As one might expect, all formulae from (4.1) to (4.4) can be employed to produce various special formulae and identities, because $f(x)$ and $\phi(t)$ are free to choose. In what follows we will detail how to find concrete identities by considering some interesting examples.

Example 4.1. Substitute $\phi(t)$ with $\phi_1(t) = \binom{t+\beta}{m}$ and $\phi_2(t) = 1/(t+\beta)$ in turn, $m \in \mathbb{N}_+$ and $\beta > 0$. Then we have

$$\Delta^k\phi_1(t) = \binom{t+\beta}{m-k}, \quad \Delta^k\phi_2(t) = \frac{(-1)^k}{t+\beta}\binom{t+\beta+k}{k}^{-1}.$$

From now on, we write briefly $\binom{t+\beta+k}{k}^{-1}$ for $1/\binom{t+\beta+k}{k}$. Under these two choices, it is easy to deduce the following six formulae respectively from (4.1), (4.2), and (4.3):

$$\sum_{k=0}^{m}\binom{\alpha}{k}\binom{\beta}{m-k}S_n^k(f) = \sum_{k=1}^{n}(\alpha)_k\binom{\alpha+\beta-k}{m-k}T_n^k(f) \tag{4.7}$$

$$\sum_{k=0}^{m}(-1)^k\binom{\alpha+k}{k}\binom{\beta}{m-k}S_n^k(f) = \sum_{k=1}^{n}(-1)^k(\alpha+k)_k\binom{\beta-\alpha-k-1}{m-k}T_n^k(f) \tag{4.8}$$

$$\sum_{k=0}^{m}\frac{(-1)^k}{4^k}\binom{2k}{k}\binom{\beta}{m-k}S_n^k(f) = \sum_{k=1}^{n}\frac{(-1)^k(2k)!}{4^k k!}\binom{\beta-k-1/2}{m-k}T_n^k(f) \tag{4.9}$$

$$\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\binom{\beta+k}{k}^{-1}S_n^k(f)=\beta\sum_{k=1}^{n}\frac{(-1)^k(\alpha)_k}{\alpha+\beta-k}\binom{\alpha+\beta}{k}^{-1}T_n^k(f) \quad (4.10)$$

$$\sum_{k=0}^{\infty}\binom{\alpha+k}{k}\binom{\beta+k}{k}^{-1}S_n^k(f)=\beta\sum_{k=1}^{n}\frac{(\alpha+k)_k}{\beta-\alpha-k-1}\binom{\beta-\alpha-1}{k}^{-1}T_n^k(f) \quad (4.11)$$

$$\sum_{k=0}^{\infty}\frac{1}{4^k}\binom{2k}{k}\binom{\beta+k}{k}^{-1}S_n^k(f)=\beta\sum_{k=1}^{n}\frac{(2k)!}{4^k k!(\beta-k-1/2)}\binom{\beta-1/2}{k}T_n^k(f). \quad (4.12)$$

We remark that all infinite series involved in (4.10), (4.11), and (4.12) are assumed to be convergent under suitable conditions for α,β and $S_n^k(f)$.

Example 4.2. The most simple case of $S_n^k(f)$ is when $n=1$ with $f(1)=1$. In such a case, it is easy to check that

$$S_1^k(f)=k, T_1^1(f)=1, T_1^k(f)=0 \text{ for } k>1.$$

Consequently, each identity from (4.7) to (4.12) yields correspondingly a special identity for $n=1$. The results are stated as follows:

$$\sum_{k=0}^{m}k\binom{\alpha}{k}\binom{\beta}{m-k}=\alpha\binom{\alpha+\beta-1}{m-1} \quad (Vandermonde) \quad (4.13)$$

$$\sum_{k=0}^{m}(-1)^k k\binom{\alpha}{k}\binom{\beta}{m-k}=-(\alpha+1)\binom{\beta-\alpha-2}{m-1} \quad (4.14)$$

$$\sum_{k=0}^{m}\frac{(-1)^k k}{4^k}\binom{2k}{k}\binom{\beta}{m-k}=-\frac{1}{2}\binom{\beta-3/2}{m-1} \quad (4.15)$$

$$\sum_{k=0}^{\infty}(-1)^k k\binom{\alpha}{k}\binom{\beta+k}{k}^{-1}=-\frac{\alpha\beta}{(\beta+\alpha)(\alpha+\beta-1)} \quad (4.16)$$

$$\sum_{k=0}^{\infty}k\binom{\alpha+k}{k}\binom{\beta+k}{k}^{-1}=\frac{(1+\alpha)\beta}{(\beta-\alpha-1)(\beta-\alpha-2)} \quad (4.17)$$

$$\sum_{k=0}^{\infty}\frac{k}{4^k}\binom{2k}{k}\binom{\beta+k}{k}^{-1}=\frac{\beta}{2(\beta-1/2)(\beta-3/2)}. \quad (4.18)$$

Note that for $\alpha>0$ and $\beta>0$ we have the estimates

$$|\binom{\alpha}{k}|\le\frac{\lfloor\alpha\rfloor+1}{k}=O(1/k),\ \binom{2k}{k}/4^k\sim\frac{1}{\sqrt{k\pi}}=O(1/\sqrt{k})$$

and

$$\binom{\beta+k}{k}\ge\binom{k+\lfloor\beta\rfloor}{[\beta]}=O(k^{\lfloor\beta\rfloor}) \quad (k\to\infty),$$

where $|\binom{\alpha}{k}|$ is the absolute value of $\binom{\alpha}{k}$, $\lfloor x\rfloor$ denotes the largest integer not greater than x, the big O notation takes the usual meaning of asymptotic. Thus all infinite series appearing in (4.16), (4.17), and (4.18) are absolutely convergent under their respective conditions, i.e.,

$$(4.16) : \alpha > 0, \beta \geq 2; \quad (4.17) : \alpha > 0, \beta \geq \alpha + 3; \quad (4.18) : \beta \geq 2.$$

Both (4.16) and (4.17) are comparable with the series

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \binom{\beta + k}{k}^{-1} = \frac{\beta}{\alpha + \beta} \tag{4.19}$$

for $\alpha > 0, \beta \geq 1$. It is of interest to note that the special case of (4.19) when α is an integer is recorded as a "theorem" in Wilf [17, p.134,Theorem], being employed as an example of the well-known WZ method. Thus we believe that (4.16), (4.17), and (4.18) may also be verified by means of the WZ method.

Example 4.3. By the multivariate Vandermonde convolution formula it is easily found that for $\alpha \in \mathbb{R}$ and $r \in \mathbb{N}_+$,

$$\begin{aligned} \sum_{[n,k,0]} \binom{\alpha}{x_1}\binom{\alpha}{x_2}\cdots\binom{\alpha}{x_k} &= \binom{k\alpha}{n} \\ \sum_{[n,k,0]} \binom{x_1}{r}\binom{x_2}{r}\cdots\binom{x_k}{r} &= \binom{n+k-1}{kr+k-1}. \end{aligned}$$

Now, replace $f(x)$ with $\binom{\alpha}{x}$ and $\binom{r+x}{r}$ in turn. For both cases, we therefore obtain

$$S_n^k\left(\binom{\alpha}{x}\right) = \binom{k\alpha}{n}, \; T_n^k\left(\binom{\alpha}{x}\right) = \sum_{\sigma(n,k)} \frac{\binom{\alpha}{1}^{k_1}\binom{\alpha}{2}^{k_2}\cdots\binom{\alpha}{n}^{k_n}}{k_1!k_2!\cdots k_n!}; \tag{4.20}$$

$$S_n^k\left(\binom{r+x}{r}\right) = \binom{k(r+1)+n-1}{n}$$

$$T_n^k\left(\binom{r+x}{r}\right) = \sum_{\sigma(n,k)} \frac{\binom{r+1}{r}^{k_1}\binom{r+2}{r}^{k_2}\cdots\binom{r+n}{r}^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{4.21}$$

Accordingly, (2.6) leads us to a pair of combinatorial series as follows:

$$\sum_{k=0}^{\infty} g(k)\binom{k\alpha}{n}t^k = \sum_{k=1}^{n} G^{(k)}(t)t^k T_n^k\left(\binom{\alpha}{x}\right) \tag{4.22}$$

$$\sum_{k=0}^{\infty} g(k)\binom{k(r+1)+n-1}{n}t^k = \sum_{k=1}^{n} G^{(k)}(t)t^k T_n^k\left(\binom{r+x}{r}\right). \tag{4.23}$$

As is expected, a variety of special identities can be deduced from (4.7) and (4.12) with $S_n^k(f)$ and $T_n^k(f)$ being replaced by those of (4.20) and (4.21) . For instance, by virtue of (4.11) and (4.20), we may find a combinatorial series

$$\sum_{k=0}^{\infty}\binom{\alpha+k}{k}\binom{\beta+k}{k}^{-1}\binom{k\delta}{n}=\sum_{k=1}^{n}\frac{(\alpha+k)_k\beta}{\beta-\alpha-k-1}\binom{\beta-\alpha-1}{k}^{-1}T_n^k\left(\binom{\delta}{x}\right). \tag{4.24}$$

A bit of analysis shows that this series is absolutely convergent under the conditions $\alpha, \delta > 0, \beta \geq \alpha + n + 2$. Other identities are left to the interested reader to work out.

5. Reciprocal Relations

Let us begin with a basic result originally due to Hsu and his coauthors of [1].

Lemma 5.1. ([1, Corollary 1]) *Let $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ be two sequences such that*

$$\exp\left(\sum_{n=1}^{\infty}\alpha_n\, t^n\right)=1+\sum_{n=1}^{\infty}\beta_n\, t^n \tag{5.1}$$

or, equivalently,

$$\log\left(1+\sum_{n=1}^{\infty}\beta_n\, t^n\right) = \sum_{n=1}^{\infty}\alpha_n\, t^n. \tag{5.2}$$

Then the system of relations

$$\beta_n=\sum_{\sigma(n)}\frac{\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n}}{k_1!k_2!\cdots k_n!} \tag{5.3}$$

holds if and only if so does the system

$$\alpha_n=\sum_{\sigma(n)}(-1)_{k-1}\frac{\beta_1^{k_1}\beta_2^{k_2}\cdots\beta_n^{k_n}}{k_1!k_2!\cdots k_n!}, \tag{5.4}$$

where the sum on the right ranges over $\sigma(n) = \{1^{k_1}2^{k_2}\cdots n^{k_n}\}, k = k_1+k_2+\cdots+k_n$.

It is worth mentioning that in the context of combinatorial analysis, such a pair of equivalent relations is said to be a reciprocal relation. We refer the reader to [4] and [1, Sections 1 and 2] for its precise definition and applications. Meanwhile, the factor $(-1)_{k-1}$ is just the Möbius function of the partition lattice [16, (30)]. For $k > 1$, it can be written explicitly as

$$(-1)_{k-1}=(-1)^{k_1+k_2+\cdots+k_n-1}(k_1+k_2+\cdots+k_n-1)!.$$

Keeping Lemma 5.1 in mind, we now consider an m-fold convolution of the form

$$S_n^m(f) = \sum_{[n,m,0]} f_1(x_1)f_2(x_2)\cdots f_m(x_m), \tag{5.5}$$

where $f_j(x)$ are m known functions with $f_j(0) = 1$, the sum ranges over all the m-compositions $(x_1, x_2, \cdots, x_m)$ of n with $x_1 + x_2 + \cdots + x_m = n$, $x_j \geq 0, 1 \leq j \leq m$.

We set

$$\log\left(1 + \sum_{n=1}^{\infty} f_j(n)\, t^n\right) = \sum_{n=1}^{\infty} \phi_j(n)\, t^n \tag{5.6}$$

or, equivalently

$$\exp\left(\sum_{n=1}^{\infty} \phi_j(n)\, t^n\right) = 1 + \sum_{n=1}^{\infty} f_j(n)\, t^n, \tag{5.7}$$

where each $\phi_j(n)$ is defined on $\mathbb{N}$. All that we are interested in is a reciprocal relation between $S_n^m(f)$ and $\sum_{j=1}^m \phi_j(n)$. The relevant conclusion can be given as follows.

Theorem 5.2. *Let $S_n^m(f)$ and $\phi_m(n)$ be given as above. The following reciprocal relation holds:*

$$S_n^m(f) = \sum_{\sigma(n)} \prod_{r=1}^{n} \frac{\left(\sum_{j=1}^m \phi_j(r)\right)^{k_r}}{k_r!} \tag{5.8}$$

$$\sum_{j=1}^{m} \phi_j(n) = \sum_{\sigma(n)} (-1)_{k-1} \prod_{r=1}^{n} \frac{\left(S_r^m(f)\right)^{k_r}}{k_r!}. \tag{5.9}$$

In particular, set $f_j(x) = f(x)$ with $f(0) = 1, 1 \leq j \leq m$. Then $\phi_j(x) = \phi(x)$. Therefore, (5.8) and (5.9) together yield a reciprocal relation as below.

Corollary 5.3. *We have*

$$S_n^m(f) = \sum_{\sigma(n)} m^k \frac{\phi(1)^{k_1}\phi(2)^{k_2}\cdots\phi(n)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{5.10}$$

$$\phi(n) = \frac{1}{m}\sum_{\sigma(n)} (-1)_{k-1} \frac{\left(S_1^m(f)\right)^{k_1}\left(S_2^m(f)\right)^{k_2}\cdots\left(S_n^m(f)\right)^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{5.11}$$

Proof. Observe first that (5.8) and (5.9) are special consequences of (5.3) and (5.4) under the choices that

$$\alpha_n = \sum_{j=1}^{m} \phi_j(n), \quad \beta_n = S_n^m(f).$$

Thus, by Lemma 5.1 it suffices to verify (5.1) for such choices. To this end, one only needs to compute

$$\begin{aligned}
\exp\left(\sum_{n=1}^{\infty} t^n \sum_{j=1}^{m} \phi_j(n)\right) &= \prod_{j=1}^{m} \exp\left(\sum_{n=1}^{\infty} \phi_j(n)\, t^n\right) = \prod_{j=1}^{m} \left(\sum_{n=0}^{\infty} f_j(n)\, t^n\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{[n,m,0]} f_1(x_1) f_2(x_2) \cdots f_m(x_m)\right) t^n \\
&= \sum_{n=0}^{\infty} S_n^m(f)\, t^n.
\end{aligned}$$

Note that $S_0^m(f) = f_1(0)f_2(0)\cdots f_m(0) = 1$. Hence identity (5.1) is confirmed. By Lemma 5.1 again, (5.7) and (5.8) follow directly from (5.3) and (5.4), respectively. Moreover, when $\phi_j(x) = \phi(x)$, the product in (5.8) becomes

$$\prod_{r=1}^{n} (m\phi(r))^{k_r} = m^{k_1+k_2+\cdots+k_n} \prod_{r=1}^{n} \phi(r)^{k_r} = m^k \prod_{r=1}^{n} \phi(r)^{k_r}.$$

Hence (5.10) and (5.11) are deduced from (5.7) and (5.8) correspondingly. Thus the theorem and the corollary are proved. □

As already shown in [11], many classical special functions such as the Genenbauer-Humbert polynomials and the Sheffer polynomials can be represented by the cycle indicator of the form

$$C_n(t_1, t_2, \cdots, t_n) = \sum_{\sigma(n)} \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{t_1}{1}\right)^{k_1} \left(\frac{t_2}{2}\right)^{k_2} \cdots \left(\frac{t_n}{n}\right)^{k_n}. \tag{5.12}$$

Such a representation has an advantage that the sequence $\{C_n(t_1, t_2, \cdots, t_n)\}_{n\geq 0}$ satisfies a kind of recurrence relation. We refer the reader to [11] for further details. Comparing (5.10) with (5.12) we may rewrite

$$S_n^m(f) = \frac{1}{n!} C_n\big(1m\phi(1), 2m\phi(2), \cdots, nm\phi(n)\big) \tag{5.13}$$

with each $\phi(r)$ being given by the coefficient of t^r in the power series expansion of $\log(1 + \sum_{n=1}^{\infty} f(n)t^n)$, i.e.,

$$\phi(r) = [t^r] \log\left(1 + \sum_{n=1}^{\infty} f(n)\, t^n\right). \tag{5.14}$$

As a benefit of doing so, $\{S_n^m(f)\}_{m,n\geq 1}$ may be computed via the recurrence relations of $C_n(t_1, t_2, \cdots, t_n)$.

It may be of interest to compare the expression (5.10) with the basic identity (2.4). Nevertheless, there does not seem to exist any available recurrence relation for the sum on the RHS of (2.4).

6. Further Examples Involving Some Classical Number Sequences

In this section we will only focus on some applications of Corollary 5.3. As for the reciprocal relation (5.8) and (5.9) in Theorem 5.2, we have not found any good example so far.

Example 6.1. Recall that the Bell numbers ϖ are defined by the exponential generating function

$$\exp(e^t-1)=\exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\varpi(n)\frac{t^n}{n!}. \tag{6.1}$$

Define the m-fold convolution

$$S_n^m(\varpi)=\sum_{[n,m,0]}\frac{\varpi(x_1)\varpi(x_2)\cdots\varpi(x_m)}{x_1!x_2!\cdots x_m!}. \tag{6.2}$$

Then by taking $f(n)=\varpi(n)/n!$ and using (5.14) we find that

$$\phi(n)=[\,t^n](e^t-1)=1/n!.$$

In such a case, a combination of the above result with (5.10) and (5.11) finally leads to a reciprocal relation:

$$S_n^m(\varpi) \quad = \quad \sum_{\sigma(n)} m^k \frac{\left(1/1!\right)^{k_1}\left(1/2!\right)^{k_2}\cdots\left(1/n!\right)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{6.3}$$

$$\frac{m}{n!} \quad = \quad \sum_{\sigma(n)}(-1)_{k-1}\frac{\left(S_1^m(\varpi)\right)^{k_1}\left(S_2^m(\varpi)\right)^{k_2}\cdots\left(S_n^m(\varpi)\right)^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{6.4}$$

It is of interest to note that the case $m=1$ of (6.3) yields a well-known identity

$$\frac{\varpi(n)}{n!}=\sum_{\sigma(n)}\frac{\left(1/1!\right)^{k_1}\left(1/2!\right)^{k_2}\cdots\left(1/n!\right)^{k_n}}{k_1!k_2!\cdots k_n!}.$$

Example 6.2. Let $p(n)$ and $M_2(n)$ be the numbers of partitions and plane partitions of n, respectively. Recall further that in the book [7, p. 310] by Hardy and Wright, the divisor function $\sigma_k(n)$ is defined by

$$\sigma_k(n)=\sum_{d|n}d^k$$

for $n,k\geq 0$. We may write $\sigma(n)$ for $\sigma_1(n)$. As is well-known to us, the Euler formula [7] and the MacMahon function [14] state that the generating functions for $\{p(n)\}_{n\geq 0}$ and $\{M_2(n)\}_{n\geq 0}$ are respectively given by

$$\prod_{k=1}^{\infty}(1-t^k)^{-1} \qquad \text{and} \qquad \prod_{k=1}^{\infty}(1-t^k)^{-k},$$

yielding

$$\begin{aligned}\log\left(1+\sum_{n=1}^{\infty}p(n)\,t^n\right) &= \sum_{n=1}^{\infty}\sigma(n)\frac{t^n}{n}\\ \log\left(\sum_{n=1}^{\infty}M_2(n)\,t^n\right) &= \sum_{n=0}^{\infty}\sigma_2(n)\frac{t^n}{n}.\end{aligned}$$

To proceed further, let us consider the m-fold convolutions

$$S_n^m(p)=\sum_{[n,m,0]}p(x_1)p(x_2)\cdots p(x_m)$$

and

$$S_n^m(M_2)=\sum_{[n,m,0]}M_2(x_1)M_2(x_2)\cdots M_2(x_m).$$

Taking these into account and making use of both (5.10) and (5.11), we obtain two pairs of reciprocal relations as follows:

$$S_n^m(p) = \sum_{\sigma(n)}m^k\frac{\big(\sigma(1)/1\big)^{k_1}\big(\sigma(2)/2\big)^{k_2}\cdots\big(\sigma(n)/n\big)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{6.5}$$

$$\frac{m}{n}\sigma(n) = \sum_{\sigma(n)}(-1)_{k-1}\frac{\big(S_1^m(p)\big)^{k_1}\big(S_2^m(p)\big)^{k_2}\cdots\big(S_n^m(p)\big)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{6.6}$$

and

$$S_n^m(M_2) = \sum_{\sigma(n)}m^k\frac{\big(\sigma_2(1)/1\big)^{k_1}\big(\sigma_2(2)/2\big)^{k_2}\cdots\big(\sigma_2(n)/n\big)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{6.7}$$

$$\frac{m}{n}\sigma_2(n) = \sum_{\sigma(n)}(-1)_{k-1}\frac{\big(S_1^m(M_2)\big)^{k_1}\big(S_2^m(M_2)\big)^{k_2}\cdots\big(S_n^m(M_2)\big)^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{6.8}$$

In particular, when $m=1$, we recover two known identities (cf.[7, (4.6)/(4.7)]):

$$p(n) = \sum_{\sigma(n)}\frac{\big(\sigma(1)/1\big)^{k_1}\big(\sigma(2)/2\big)^{k_2}\cdots\big(\sigma(n)/n\big)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{6.9}$$

$$M_2(n) = \sum_{\sigma(n)}\frac{\big(\sigma_2(1)/1\big)^{k_1}\big(\sigma_2(2)/2\big)^{k_2}\cdots\big(\sigma_2(n)/n\big)^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{6.10}$$

Example 6.3. It is clear from [16, Vol.II, p.76, Exer. 5.13(b)] that for $j\geq 1$

$$\exp\left(\sum_{n\geq 1}\chi_n(F_j)\frac{x^n}{n}\right)=\sum_{n\geq 0}n!^{j-1}x^n,$$

where F_j denotes the free group on j generators[1] and $\chi_n(F_j)$ denotes the number of subgroups of F_j of index n. On account of this, we thereby obtain the following reciprocal relation:

$$S_n^m(F) = \sum_{\sigma(n)} \prod_{r=1}^{n} \frac{\left(\sum_{j=1}^{m} \chi_r(F_j)\right)^{k_r}}{k_r!\, r^{k_r}} \tag{6.11}$$

$$\frac{1}{n}\sum_{j=1}^{m} \chi_n(F_j) = \sum_{\sigma(n)} (-1)_{k-1} \prod_{r=1}^{n} \frac{\left(S_r^m(F)\right)^{k_r}}{k_r!} \tag{6.12}$$

with the m-fold convolution

$$S_n^m(F) = \sum_{[n,m,0]} (x_1!)^0 (x_2!)^1 \cdots (x_m!)^{m-1}.$$

Both (6.11) and (6.12) contain the well-known Cauchy identity and its dual form as the special case when $m = 1$ (F_1 a cyclic group):

$$1 = \sum_{\sigma(n)} \frac{1}{k_1!k_2!\cdots k_n!} \prod_{r=1}^{n} \frac{1}{r^{k_r}} \tag{6.13}$$

$$\frac{1}{n} = \sum_{\sigma(n)} \frac{(-1)_{k-1}}{k_1!k_2!\cdots k_n!}. \tag{6.14}$$

We can put the preceding results in a general context. To do this, replace F_j with a free Abelian group G_j which is finitely generated by j generators and let $\mu_n(G_j)$ be the number of conjugacy classes, i.e., $g^{-1}G_j g$ for all $g \in F_j$, of subgroups of G_j of index n. According to Group theory or [16, Vol. II, Exer.5.13(a)], we have

$$\exp\left(\sum_{n\geq 1} \chi_n(G_j)\frac{x^n}{n}\right) = \prod_{n\geq 1} \frac{1}{(1-x^n)^{\mu_n(G_j)}}.$$

So it is easily found that

$$\chi_n(G_j) = \sum_{d|n} \mu_d(G_j)\, d,$$

since

$$\sum_{n\geq 1} \chi_n(G_j)\frac{x^n}{n} = \log \prod_{n\geq 1} \frac{1}{(1-x^n)^{\mu_n(G_j)}} = \sum_{n\geq 1}\left(\sum_{d|n} \mu_d(G_j)\, d\right)\frac{x^n}{n}.$$

[1]which means that every element of the group F_j can be expressed as a combination (under the group operation) of elements of certain j-subset of F_j and their inverses.

Thus we obtain

$$S_n^m(G) = \sum_{\sigma(n)} \prod_{r=1}^{n} \frac{\big(\sum_{j=1}^{m} \chi_r(G_j)\big)^{k_r}}{k_r!\, r^{k_r}} \tag{6.15}$$

$$\frac{1}{n}\sum_{j=1}^{m} \chi_n(G_j) = \sum_{\sigma(n)} (-1)_{k-1} \prod_{r=1}^{n} \frac{\big(S_r^m(G)\big)^{k_r}}{k_r!}, \tag{6.16}$$

where the m-fold convolution

$$S_n^m(G) = \sum_{[n,m,0]} \big(\sum_{d|x_1} \mu_d(G_1)\, d\big)^1 \big(\sum_{d|x_2} \mu_d(G_2)\, d\big)^2 \cdots \big(\sum_{d|x_m} \mu_d(G_m)\, d\big)^m.$$

Example 6.4. Recall that in Example 3.4, we write $a = (1+\sqrt{5})/2, b = (1-\sqrt{5})/2$. Thus, the Fibonacci numbers F_n may be written as $F_n = (a^{n+1} - b^{n+1})/\sqrt{5}$. Starting with the generating function for $\{F_n\}_{n\geq 0}$, namely

$$\sum_{n=0}^{\infty} F_n\, t^n = \frac{1}{(1-at)(1-bt)},$$

it is not hard to find that

$$\log\left(\sum_{n=0}^{\infty} F_n\, t^n\right) = \sum_{n=1}^{\infty} \frac{a^n + b^n}{n}\, t^n.$$

On considering the m-fold convolution

$$S_n^m(F) = \sum_{[n,m,0]} F_{x_1} F_{x_2} \cdots F_{x_m},$$

we immediately obtain the following reciprocal relation via (5.10) and (5.11):

$$S_n^m(F) = \sum_{\sigma(n)} m^k \frac{\big((a+b)/1\big)^{k_1} \big((a^2+b^2)/2\big)^{k_2} \cdots \big((a^n+b^n)/n\big)^{k_n}}{k_1!k_2!\cdots k_n!} \tag{6.17}$$

$$m\,\frac{a^n+b^n}{n} = \sum_{\sigma(n)} (-1)_{k-1} \frac{\big(S_1^m(F)\big)^{k_1} \big(S_2^m(F)\big)^{k_2} \cdots \big(S_n^m(F)\big)^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{6.18}$$

It is worth mentioning that the case $m = 1$ of (6.18) is correspondent to the well-known identity (cf.[7, Example 6])

$$\sum_{\sigma(n)} (-1)^{k-1}(k-1)! \frac{F_1^{k_1} F_2^{k_2} \cdots F_n^{k_n}}{k_1!k_2!\cdots k_n!} = \frac{1}{n}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

Example 6.5. For $p \in \mathbb{N}_+$, as is known to us, there holds a rational generating function for the Stirling numbers of the second, namely

$$\sum_{n=0}^{\infty} \left\{ \begin{matrix} n+p \\ p \end{matrix} \right\} t^n = \frac{1}{(1-t)(1-2t)\cdots(1-pt)},$$

where we have adopted Knuth's notation for the Stirling numbers (cf.[12] or [17]). By this we at once obtain

$$\begin{aligned} \log\left(\sum_{n=0}^{\infty} \left\{ \begin{matrix} n+p \\ p \end{matrix} \right\} t^n \right) &= \sum_{j=1}^{p} \log(1-jt)^{-1} \\ &= \sum_{n=1}^{\infty} (1^n + 2^n + \cdots + p^n) \frac{t^n}{n} = \sum_{n=1}^{\infty} s_p^n \frac{t^n}{n}, \end{aligned}$$

where $s_p^n = \sum_{j=1}^{p} j^n$ is the sum of arithmetic progression of degree n. Given m and $p \in \mathbb{N}_+$, let us consider the m-fold convolution

$$S_n^m(S) = \sum_{[n,m,0]} \left\{ \begin{matrix} x_1+p \\ p \end{matrix} \right\} \left\{ \begin{matrix} x_2+p \\ p \end{matrix} \right\} \cdots \left\{ \begin{matrix} x_m+p \\ p \end{matrix} \right\}.$$

In view of (5.10) and (5.11), it is easy to find the following reciprocal relation:

$$S_n^m(S) = \sum_{\sigma(n)} m^k \frac{\left(s_p^1/1\right)^{k_1} \left(s_p^2/2\right)^{k_2} \cdots \left(s_p^n/n\right)^{k_n}}{k_1! k_2! \cdots k_n!} \tag{6.19}$$

$$\frac{m}{n} s_p^n = \sum_{\sigma(n)} (-1)_{k-1} \frac{\left(S_1^m(S)\right)^{k_1} \left(S_2^m(S)\right)^{k_2} \cdots \left(S_n^m(S)\right)^{k_n}}{k_1! k_2! \cdots k_n!}. \tag{6.20}$$

Certainly (6.19) may also be written as

$$S_n^m(S) = \frac{1}{n!} C_n\left(ms_p^1, ms_p^2, \cdots, ms_p^n\right). \tag{6.21}$$

Evidently, the case $m = 1$ is just the cycle indicator of the Stirling numbers

$$\left\{ \begin{matrix} n+p \\ p \end{matrix} \right\} = \frac{1}{n!} C_n\left(s_p^1, s_p^2, \cdots, s_p^n\right).$$

7. Concluding Remarks

In our preceding discussion, we have illustrated the applications of Theorem 2.2, Theorem 5.2 and Corollary 5.3 to Analytic Combinatorics by establishing some summation and transformation formulae. All these examples demonstrate that the k-fold convolution and $n/k-$partition given by Definition 2.1 are closely related with the generating functions of *admissible combinatorial constructions* (cf.[5, Def.1.5; I.2]), such as the integer partition, the rooted or labeled tree, etc. Perhaps most noteworthy is that two summations often occur in the ring of symmetric functions [13]. For example, the elementary symmetric functions and the kth power sum in n independent variables x_i are respectively defined by

$$\begin{aligned} e_k &= \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} x_{i_1}x_{i_2}\cdots x_{i_k} \\ s_k &= x_1^k + x_2^k + \cdots + x_n^k. \end{aligned}$$

In these two expressions, the integer n is purposely suppressed for clarity. Recall that Girard's famous formula states

$$s_m = m \sum_{\sigma(m,n)} (-1)^{m+k_1+k_2+\cdots+k_n} \frac{(k_1+k_2+\cdots+k_n-1)!}{k_1!k_2!\cdots k_n!} e_1^{k_1} e_2^{k_2} \cdots e_n^{k_n}. \tag{7.1}$$

In [14, Section I, Chapter I, Sections 5 and 6], MacMahon generalized this formula by establishing

$$\sum_{m=1}^{\infty} s_m t^m = \frac{e_1 t - 2e_2 t^2 + 3e_3 t^3 + \cdots}{1 - e_1 t + e_2 t^2 - e_3 t^3 + \cdots}. \tag{7.2}$$

This clearly states that as a reciprocal of (7.1), each e_k can be expressed in terms of s_m

$$e_m = \sum_{\sigma(m,n)} \frac{(-1)^{m+k_1+k_2+\cdots+k_n}}{k_1!k_2!\cdots k_n!} \left(\frac{s_1}{k_1}\right)^{k_1} \left(\frac{s_2}{k_2}\right)^{k_2} \cdots \left(\frac{s_n}{k_n}\right)^{k_n}. \tag{7.3}$$

Another pair of reciprocal relations for e_m and s_m via the use of determinants can be found in [13, p.28, Example 8]. Thus, it is interesting to study such reciprocal relations as Theorem 5.2 and Corollary 5.3 in full generality in the ring of symmetric functions. This will be discussed in our forthcoming paper.

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