ON WEIGHTED ZERO SUM SUBSEQUENCES OF SHORT LENGTH

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Abstract

Let G be an additive finite abelian group with exponent m and A a non-empty subset of $\{1, 2, \ldots, m-1\}$. The constant $\eta_A(G)$ (respectively, $s_A(G)$) is defined to be the smallest positive integer t such that any sequence of length t of elements of G contains a non-empty A-weighted zero-sum subsequence of length at most m (respectively, of length equal to m). In this note we shall calculate the value of $\eta_{\pm}(G)$ and $s_{\pm}(G)$ for certain finite abelian groups G of rank 2 and rank 3. In 2007, Gao et al. conjectured that $s(G) = \eta(G) + \exp(G) - 1$ for any finite abelian group G. For weight $A = \{\pm 1\}$, we shall observe that the weighted analog of this conjecture does not hold for $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ for an odd integer $n > 7$. However, we show that the conjecture holds for any abelian group G of order 8 and 16.

1. Introduction

Let G be an additive finite abelian group with exponent m and A a non-empty subset of $\{1, 2, \ldots, m-1\}$. The notations that we use in this note were introduced by A. Geroldinger and F. Halter-Koch in [12]. Readers may also refer to the survey article by W. Gao and A. Geroldinger [11].

Let $\mathcal{F}(G)$ denote an abelian multiplicative monoid with a basis G. An element S of $\mathcal{F}(G)$ is called a *sequence* over G. A sequence of not necessarily distinct elements of G may be written in various ways: $S = (x_1, x_2, \cdots, x_t) = x_1 x_2 \cdots x_t = \prod_{i=1}^{t} x_i = \prod_{g \in G} g^{\mathsf{v}_g(S)}$, where $\mathsf{v}_g(S) \geq 0$ is called the *multiplicity* of g in S. We call $|S| = t$ the *length* of S and $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) | g \in G\}$ the *maximum multiplicity* of S. We say that S *contains* some $g \in G$ if $\mathsf{v}_g(S) \geq 1$. A sequence T is said to be a *subsequence* of a sequence S if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for every $g \in G$. If T is a subsequence of S, we write $T|S$ and ST^{-1} denotes the sequence obtained by deleting the terms of sequence T from the sequence S.

Let $S = x_1x_2\cdots x_t$ be a finite sequence over G. By $\sigma(S)$ we denote the sum of all the elements of S. That is, $\sigma(S) = \sum_{i=1}^{t} x_i$. For $\bar{a} = (a_1, a_2, \cdots, a_t)$ with $a_i \in A$, we define $\sigma_{\bar{a}}(S) = \sum_{i=1}^{t} a_i x_i$.

A sequence $S = x_1x_2\cdots x_t$ over G is said to be a *zero-sum sequence* with respect to the weight set A or an *A-weighted zero-sum sequence,* if there exists $\bar{a} = (a_1, a_2, \cdots, a_t) \in A^t$ such that $\sum_{i=1}^{t} a_i x_i = 0$. The *Davenport constant* of an abelian group G, denoted by $D(G)$, is the least positive integer k such that every sequence of length k contains a non-empty zero-sum subsequence. The constant $E(G)$ is the least positive integer k such that every sequence of length k contains a zero-sum subsequence of length $|G|$. The Davenport constant of an abelian group G with respect to weight A, denoted by $D_A(G)$, is the least positive integer k such that for every sequence of length k, there exists a non-empty subsequence T and $\bar{a} \in A^{|T|}$ such that $\sigma_{\bar{a}}(T) = 0$. Similarly, the constant $E_A(G)$ is the least positive integer k such that for every sequence of length k, there exists a subsequence T of length $|G|$ and $\bar{a} \in A^{|T|}$ such that $\sigma_{\bar{a}}(T) = 0$. These two weighted constants were introduced by Adhikari et al. in [2, 3, 4]. In 1961, Erdős, Ginzburg and Ziv [8] proved that every sequence of length $2n-1$ over the cyclic group C_n contains a zero sum subsequence of length n. This result is one of the starting points of research on zero-sum problems.

For a given positive integer k, $\eta_A^{km}(G)$ is defined to be the least positive integer t such that every sequence S over G of length t has a non-empty zero-sum subsequence of length at most km with respect to weight A. In the spirit of some combinatorial constants considered by Harborth [7] and others, for a positive integer k, we define $s_A^{km}(G)$ to be the least positive integer t such that every sequence S over G of length t has a zero-sum subsequence of length km with respect to weight A. When $k = 1$, we write $\eta_A(G)$ instead of $\eta_A^{km}(G)$ and $s_A(G)$ instead of $s_A^{km}(G)$. In this note, C_n denotes the cyclic group of order n. If $A = \{1\}$, $\eta_A(G)$ and $s_A(G)$ are respectively the well-known constants $\eta(G)$ and $s(G)$. For a finite cyclic group G, one has $\eta(G) = D(G)$ and $s(G) = E(G)$.

It was proved by Gao [9] that for any finite abelian group G, $E(G) = |G| + D(G) - 1$ and generalizing this in 2009, Grynkiewicz et al. [13] proved that $E_A(G) = |G| + D_A(G) - 1$ for any non-empty subset A of $\{1, 2, \ldots, m-1\}$. It was conjectured by W. Gao et al. [10] that $s(G) = \eta(G) + \exp(G) - 1$. That this conjecture is true for finite abelian groups of rank at most 2 was proved by Geroldinger and Halter-Koch [12]. It is very natural to ask whether the weighted analog $s_A(G) = \eta_A(G) + \exp(G) - 1$ holds. Here we consider this analog in the case of the weight $A = \{1, -1\}$ and observe that $s_{\pm}(C_n \oplus C_n) > \eta_{\pm}(C_n \oplus C_n) + \exp(C_n \oplus C_n) - 1$, for an odd integer $n > 7$. However, we prove that $s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$, for any abelian group G of order 8 and 16.

2. Theorems and Observations

First, we state the following theorem which we shall be using several times.

Theorem 1 (Adhikari et al. [6]). *Let G be a finite and nontrivial abelian group and let $S \in \mathcal{F}(G)$ be a sequence.*

(1) If $|S| \geq \log_2 |G| + 1$ and G is not an elementary 2-group then S contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence.

(2) If $|S| \geq \log_2 |G| + 2$ and G is not an elementary 2-group of even rank then S contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence of even length.

(3) If $|S| > \log_2 |G|$, then S contains a nontrivial $\{\pm 1\}$-weighted zero-sum subsequence, and if $|S| > \log_2 |G| + 1$, then such a subsequence may be found with even length.

We also need to state the following theorem.

Theorem 2 (Adhikari et al. [5]). *For a positive odd integer n, we have*

$$s_{\pm}(C_n \oplus C_n) = 2n - 1.$$

Observation 1. Trivially, for any finite abelian group G, we have

$$D_{\pm}(G) \leq \eta_{\pm}(G) \leq s_{\pm}(G).$$

From Theorem 1.3 of [6], if $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$, with $n_1 | n_2 | \cdots | n_r$, we have

$$\sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor + 1 \leq D_{\pm}(G) \leq \lfloor \log_2 |G| \rfloor + 1.$$

If $|G|$ is a power of 2 (or even at most one of the n_i's is not a power of 2), the upper bound and the lower bound for $D_{\pm}(G)$ in the above are the same and we get the precise value for $D_{\pm}(G)$.

Therefore, if $|G|$ is a power of 2, then $D_{\pm}(G) = \lfloor \log_2 |G| \rfloor + 1 = \log_2 |G| + 1$.

Now, if $\lfloor \log_2 |G| \rfloor + 1 \leq n_r$, then $D_{\pm}(G) = \eta_{\pm}(G)$ and if $|G|$ is a power of 2 and G is not an elementary 2-group then by Part (1) of Theorem 1, a sequence over G of length $\log_2 |G| + 1$ contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence.

To sum up, if $|G|$ is a power of 2, G is not an elementary 2-group and $\log_2 |G| + 1 \leq n_r + 1$, then we have $D_{\pm}(G) = \eta_{\pm}(G) = \log_2 |G| + 1$ in this case.

In particular, if $n_i = 2^l$ with $l > 1$, for $i = 1, \ldots, r$, that is, $G = C_{2^l}^r$, then for l satisfying $rl \leq 2^l$,

$$\eta_{\pm}(G) = D_{\pm}(G) = rl + 1.$$

For instance, taking $r = 3$, for $l \geq 4$, we have

$$\eta_{\pm}(C_{2^l} \oplus C_{2^l} \oplus C_{2^l}) = D_{\pm}(C_{2^l} \oplus C_{2^l} \oplus C_{2^l}) = 3l + 1.$$

Some more consequences of the above observation are stated in the following theorem.

Theorem 3. *Let l, n be positive integers. We have the following results.*

(1) If $2^l n \geq 4$ then $\eta_{\pm}(C_{2^l} \oplus C_{2^l n}) = D_{\pm}(C_{2^l} \oplus C_{2^l n}) = \lfloor \log_2 2^l n \rfloor + l + 1$.

(2) If $n \geq 2$ then $\eta_{\pm}(C_{2^l} \oplus C_{2^l} \oplus C_{2^l n}) = D_{\pm}(C_{2^l} \oplus C_{2^l} \oplus C_{2^l n}) = \lfloor \log_2 2^l n \rfloor + 2l + 1$.

Proof. We start with part (1). For $n = 1, 2$, the order of the group being a power of 2, the result follows from Observation 1. Suppose $n > 2$. From Observation 1,

$$D_{\pm}(C_{2^l} \oplus C_{2^l n}) = \lfloor \log_2 2^l n \rfloor + l + 1.$$

Here trivially, $l + 1 \leq 2^{l-1} n$. Now, writing $k = \lfloor \log_2 2^l n \rfloor$, since $2k \leq 2^k$ for each positive integer k, we get $2^l n \leq 2^{2^{l-1} n}$ and hence, $k \leq 2^{l-1} n$. Therefore, $k + l + 1 \leq 2^{l-1} n + 2^{l-1} n = 2^l n$ and once again, the result follows from Observation 1.

Moving on to part (2), here too, for $n = 2$, the order of the group being a power of 2, the result follows from Observation 1. Suppose $n > 2$. Writing $k = \lfloor \log_2 2^l n \rfloor$, we have only to show that $k + 2l + 1 \leq 2^l n$. Arguing as in Part (1) above, $k \leq 2^{l-1} n$. Since for $n > 2$, we have $2l + 1 \leq 2^{l-1} n$, it follows that $k + 2l + 1 \leq 2^{l-1} n + 2^{l-1} n = 2^l n$ and the result follows from Observation 1. $\square$

Next, we shall state Warning's Theorem, which will be used in the proof of Theorem 5.

Theorem 4 (Warning's Theorem (see [1], for instance)). *Let $\mathbb{F}_q$ be the finite field with q-elements, where $q = p^r$ for a prime number p, and an integer $r \geq 1$. Let $f_1(x_1, x_2, \cdots, x_n)$, $f_2(x_1, x_2, \cdots, x_n)$, $\cdots$, $f_k(x_1, x_2, \cdots, x_n) \in \mathbb{F}_q[x_1, x_2, \cdots, x_n]$ be set of polynomials in n variables such that $\sum_{i=1}^{k} \deg(f_i) \leq n - 1$. Then the number of simultaneous solutions of the system $f_i(x_1, x_2, \cdots, x_n) = 0$, $i = 1, 2, \ldots k$, in $\mathbb{F}_q^n$ is divisible by p.*

Theorem 5. *Let n be an odd integer. Then $\eta_{\pm}(C_n \oplus C_n) \leq n$.*

Proof. Let $S = (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)$ be a sequence of length n over a group $C_n \oplus C_n$ and $A = \{\pm 1\}$. First, we shall prove the theorem for $n = p$, p an odd prime and the general result will then follow by a standard argument (as employed in the proof of Theorem 3 in [5]) involving induction on the number of prime factors of n.

Consider the following system of equations:

$$\sum_{i=1}^{p} a_i x_i^{\frac{p-1}{2}} = 0 \qquad \text{and} \qquad \sum_{i=1}^{p} b_i x_i^{\frac{p-1}{2}} = 0.$$

Clearly, $(0,0,\ldots,0) \in C_p^p$ is a solution of the given system and so by Theorem 4 we have a nonzero solution of the system. This solution gives us a nonempty zero-sum subsequence of length $\leq p$ with respect to weight $\{\pm 1\}$. $\square$

Observation 2. One observes that the sequence $(1,0)(0,1)$ over $C_3 \oplus C_3$ does not have $\{\pm 1\}$-weighted zero-sum subsequence. Therefore, in view of the above theorem, $\eta_{\pm}(C_3 \oplus C_3) = 3$. Similarly, since the sequence $(1,0)(2,0)(0,1)(0,2)$ over $C_5 \oplus C_5$ does not have any $\{\pm 1\}$-weighted zero-sum subsequence, we have $\eta_{\pm}(C_5 \oplus C_5) = 5$.

In 2007, Gao et al. [10] conjectured that $s(G) = \eta(G) + \exp(G) - 1$ for any finite abelian group G. The corresponding weighted analog $s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$, for weight $\{\pm 1\}$, is thus satisfied for the groups $C_3 \oplus C_3$ and $C_5 \oplus C_5$, by Theorem 2.

By Part (1) of Theorem 1, any sequence of length $\geq \log_2 n^2 + 1$ over $C_n \oplus C_n$ contains a proper non-trivial zero-sum subsequence with respect to weight $\{\pm 1\}$. Since for any positive integer $n > 7$, $\log_2 n^2 + 1 < n$, for such an n, we have $\eta_{\pm}(C_n \oplus C_n) < n$ and hence $\eta_{\pm}(C_n \oplus C_n) + \exp(C_n \oplus C_n) - 1 < 2n - 1$. However, for a positive odd integer n, we have $s_{\pm}(C_n \oplus C_n) = 2n-1$, by Theorem 2. Therefore, for an odd integer $n > 7$, $\eta_{\pm}(C_n \oplus C_n) + \exp(C_n \oplus C_n) - 1 < s_{\pm}(C_n \oplus C_n)$. That is, the weighted analog does not hold in these cases.

The following theorem gives some instances where the analog holds. In fact, it will be easy to observe that by our method, one can prove the analog for some more abelian groups of small order when the order of the group is a power of 2.

Theorem 6. *The following statements hold true.*

(1) $\eta_{\pm}(C_2 \oplus C_4) = 4$ *and* $s_{\pm}(C_2 \oplus C_4) = \eta_{\pm}(C_2 \oplus C_4) + 4 - 1 = 7$. *We have,* $s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$, *for any abelian group* G *of order 8.*

(2) Let G *be a finite abelian group of order 16 with* $\exp(G) \geq 4$. *We have* $\eta_{\pm}(G) = 5$. *Further,* $s_{\pm}(G) = 12$ *or* 8 *depending on* $\exp(G) = 8$ *or* 4 *respectively. We conclude that* $s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$, *for any abelian group* G *of order 16.*

Proof. We first prove (1). It follows from Part (1) of Theorem 1 that $\eta_{\pm}(C_2 \oplus C_4) \leq 4$ and the example of the length 3 sequence $(0,1)(0,2)(1,0)$ shows that $\eta_{\pm}(C_2 \oplus C_4) \geq 4$ and hence, $\eta_{\pm}(C_2 \oplus C_4) = 4$.

We proceed to prove that $s_{\pm}(C_2 \oplus C_4) = 7$. Let $S = \prod_{i=1}^{7} x_i$ be a sequence over G of length 7. If $\mathsf{h}(S) = 1$, that is, no element appears more than once in S, we are done, as all but one element of the group appear in S and the zero-sum sequences $(1,3)(1,1)(0,3)(0,1)$ and $(0,0)(0,2)(1,0)(1,2)$ have no element in common. Therefore, we may assume that $\mathsf{h}(S) \geq 2$. Let us write $S = x_1^2 x_2 x_3 x_4 x_5 x_6$. The element x_1 repeated twice gives a $\{\pm 1\}$-weighted zero-sum subsequence of length 2.

Since the number of 2-subsets of $\{2,3,4,5,6\}$ is $10 > 8$, either $x_i + x_j = x_j + x_k$ or $x_i + x_j = x_k + x_l$ for distinct elements i, j, k, l of the set $\{2,3,4,5,6\}$. In the first case, we have $x_i = x_k$ giving us another $\{\pm 1\}$-weighted zero-sum subsequence of length 2. In the second case, $x_i x_j x_k x_l$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 4 and so in any case there is a $\{\pm 1\}$-weighted zero-sum subsequence of length 4 and we have $s_{\pm}(C_2 \oplus C_4) \leq 7$. Since trivially, for any finite abelian group G, $s_{\pm}(G) \geq \eta_{\pm}(G) + \exp(G) - 1$, we have $7 \geq s_{\pm}(C_2 \oplus C_4) \geq \eta_{\pm}(C_2 \oplus C_4) + \exp(C_2 \oplus C_4) - 1 = 4 + 4 - 1 = 7$, and we are through.

If $G = C_8$, the result $s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$ follows from [3] and for the elementary 2 group (here working with the weight set $\{\pm 1\}$ is nothing but working with the weight set $\{1\}$, that is, $s_{\pm}(G)$ is the classical constant $s(G)$) of order 8, it is trivial. This completes the proof of Part (1) of the theorem.

We now prove part (2). When $G = C_4 \oplus C_4$ or $C_2 \oplus C_8$, the result $\eta_{\pm}(G) = 5$ follows from Part (1) of Theorem 3; when $G = C_2 \oplus C_2 \oplus C_4$, it follows from Part (2) of Theorem 3 and finally, for $G = C_{16}$, it is a particular case of a result in [3].

Given a sequence $S = \prod_{i=1}^{12} x_i$ over G in this case, if it does not have a $\{\pm 1\}$-weighted zero-sum subsequence of length 2, then considering

$$\mathcal{A} = \left\{ A_I = \sum_{i \in I} x_i : I \subset \{1, 2, \cdots, 12\}, |I| = 2 \right\},$$

since the number of 2-subsets of $\{1, 2, \cdots, 12\}$ is 66 and $|G| = 16$, by repeated application of the pigeonhole principle there exist distinct 2-subsets I_1, I_2, I_3, I_4 such that $A_{I_1} = A_{I_2} = A_{I_3} = A_{I_4}$. Clearly, I_i and I_j will have to be disjoint for $i \neq j$, as otherwise we would have a $\{\pm 1\}$ -weighted zero-sum subsequence of length 2, contrary to our assumption. We therefore have $A_D = 0$ where $D = I_1 \cup I_2 \cup I_3 \cup I_4$ giving us a $\{\pm 1\}$-weighted zero-sum subsequence of length 8.

Let us first assume that $\exp(G) = 8$. Given a sequence S of length 12 over G, if it does not have a $\{\pm 1\}$-weighted zero-sum subsequence of length 2, then as observed above, it has a $\{\pm 1\}$-weighted zero-sum subsequence of length 8. We now assume that S has a $\{\pm 1\}$-weighted zero-sum subsequence S_1 of length 2. Now, $|SS_1^{-1}| = 10$ and by Part (2) of Theorem 1, a 6 length subsequence of SS_1^{-1} has a $\{\pm 1\}$-weighted zero-sum subsequence S_2 of length 2 or 4. By the same argument, $SS_1^{-1}S_2^{-1}$ will have a $\{\pm 1\}$-weighted zero-sum subsequence S_3 of length 2 or 4. If one of S_2 or S_3 is of length 4, we already have a $\{\pm 1\}$-weighted zero-sum subsequence of length 8.

If, both S_2 and S_3 are of length 2, then $SS_1^{-1}S_2^{-1}S_3^{-1}$ is of length 6 and will have a $\{\pm 1\}$-weighted zero-sum subsequence S_3 of length 2 or 4 and hence there is a $\{\pm 1\}$ -weighted zero-sum subsequence of length 8. Thus, if $\exp(G) = 8$, $s_\pm(G) \leq 12$. On the other hand $s_\pm(G) \geq \eta_\pm(G) + \exp(G) - 1 = 5 + 8 - 1 = 12$.

Now, we assume that $\exp(G) = 4$. If $G = C_4 \oplus C_4$, then from Part (i) of Theorem 4.3 in [6], $s_\pm(G) = 8$. If $G = C_2 \oplus C_2 \oplus C_4$, then $s_\pm(G) \geq \eta_\pm(G) + \exp(G) - 1 = 5 + 4 - 1 = 8$. Here, given a sequence S of length 8 over G, by Part (2) of Theorem 1, a 6 length subsequence of S has a a $\{\pm 1\}$-weighted zero-sum subsequence S_1 of length 2 or 4. If S_1 is of length 2, then SS_1^{-1} is of length 6 and will have a $\{\pm 1\}$-weighted zero-sum subsequence S_2 of length 2 or 4. Thus, there must be a $\{\pm 1\}$-weighted zero-sum subsequence of length 4 and hence $s_\pm(G) \leq 8$.

With these and the remark at the end of the proof of Part (1) above, for the elementary 2 group case, we get $s_\pm(G) = \eta_\pm(G) + \exp(G) - 1$ for any abelian group G with $|G| = 16$. □

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