ON SUBSETS OF ORDERED TREES ENUMERATED BY A SUBSEQUENCE OF FIBONACCI NUMBERS

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Abstract

In this paper, we provide a bijection between subsets of ordered trees with n edges where no two vertices at the same level have different parents and those with height at most three. We show that the number of these subsets of ordered trees corresponds to every other Fibonacci number, and provide a combinatorial interpretation of Chen and Shapiro's generalization of this sequence using k-trees. We also prove Shapiro's identity involving the generating function of this sequence and Riordan arrays.

1. Introduction

Herbert S. Wilf defines a *fountain of coins* as an arrangement of n coins in rows such that the coins in the first row form a single contiguous block, and that in all higher rows each coin is tangent to exactly two coins from the row beneath it [17]. There is an obvious bijection between these objects and combinatorial objects enumerated by the Catalan numbers [15]. If we require that every row in our fountain of coins consists of a single contiguous block (see Figure 1), then the number $f(n)$ of such contiguous arrangements with exactly n coins in the first row satisfies the recurrence relation

$$f(n) = \sum_{j=1}^{n}(n-j)f(j) + 1 \ (n = 2, 3, \cdots), \text{ where } f(1) = 1.$$

Using the above recurrence relation, one can easily show that the generating function is

$$F(z) = \sum_{n=1}^{\infty} f(n)z^n = \frac{z - z^2}{1 - 3z + z^2}.$$

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If we look at the first few terms of the series expansion of this generating function, we notice that $\{f(n)\}_{n=1}^{\infty}$ is a subsequence of the Fibonacci Numbers and, in fact,

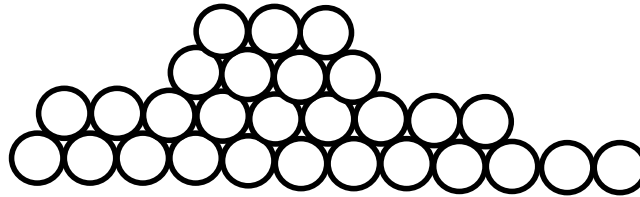$f(n) = F_{2n-1}, n = 1, 2, 3, \ldots$ – odd terms of the Fibonacci numbers with $F_1 = F_2 = 1$.

Figure 1: An example of a contiguous fountain of coins.

On the other hand, Emeric Deutsch and Helmut Prodinger [2] studied polyominoes built by starting with a single cell and adding new cells on the right or on the top of an existing cell in which every column is formed by contiguous cells.
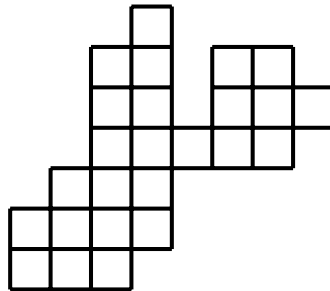
Figure 2: A directed column-convex polyomino with 25 cells.

They exhibit a bijection between these polyominoes and ordered trees with height at most three and show that the number of such directed column-convex polyominoes of area n (area = number of cells) is given by every other Fibonacci number F_{2n-1} with $F_1 = F_2 = 1$.

It is well-known that *ordered trees* (often referred to as rooted plane trees or simply plane trees) are trees with a distinguished vertex called the root where the children of each internal vertex are linearly ordered [15]. The *level number* of an ordered tree is the length of the path from the root to a given vertex. The *height* of a tree is the largest level number in the tree. If we denote the number of nonempty ordered trees with n edges where *no two vertices at the same level have different parents* (we will refer to these subsets of ordered trees as Skinny Trees) by $st(n)$,

then from the decomposition rule [5, 6] shown in Figure 3 we see that its generating function $ST(z) = \sum_{n=1}^{\infty} st(n)z^n$
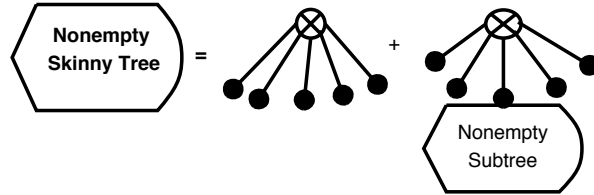

Figure 3: Decomposition of a non-empty skinny tree.

satisfies the equation

$$ST(z) = \frac{z}{1-z} + \frac{1}{1-z}zST(z)\frac{1}{1-z}.$$

Solving this equation for $ST(z)$, we get

$$ST(z) = \frac{z - z^2}{1 - 3z + z^2}.$$

Hence, the number of nonempty skinny trees with n edges is also enumerated by every other Fibonacci number like the contiguous fountains of coins, directed column-convex polyominoes, and ordered trees of height at most three.

2. Contiguous Fountain of Coins, Skinny Trees, and Ordered Trees of Height at Most Three

In this section, we provide bijections between a collection of contiguous fountains of coins with n coins in the first row, ordered trees with n edges where no two vertices at the same level have different parents (skinny trees), and ordered trees of height at most three.

2.1. From Contiguous Fountain of Coins to Skinny Trees

Given a contiguous fountain of coins with n coins at the first level, draw n diagonals with slope $m = -\sqrt{3}$ through each of the n coins at the first level. These diagonals represent the non-root vertices in the ordered tree to be constructed.

For $i = 1, 2, \cdots, n$, if diagonal i has j coins, then non-root vertex i of the corresponding tree will be at level j. Whenever we have more than one vertex at the same level (this happens if two or more diagonals contain the same number of coins), we let the parent be the vertex with the largest label less than the labels

of the diagonals at this level. Hence, no two vertices at the same level will have different parents and the ordered tree constructed in this way is indeed a skinny tree.

For example, in the contiguous fountain of coins shown in Figure 1, $n = 12$ and the corresponding tree will have twelve non-root vertices. Vertices 1, 11, and 12 will be at level 1 connected to the root. Vertices 2, 3, 4, 9 and 10 are at level 2, and they are all children of vertex 1. Vertex 5 is the only vertex at level 3 and is the child of vertex 4. Finally, we see that vertices 6, 7, and 8 are all at level 4, and are connected to vertex 5. The skinny tree which corresponds to the contiguous fountain of coins in Figure 1 is shown below.
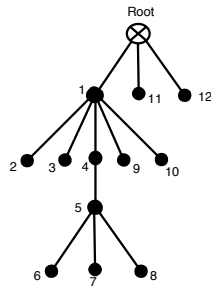

Figure 4: A skinny tree corresponding to the contiguous fountain in Figure 1.

2.2. From Skinny Trees to Contiguous Fountain of Coins

Given a skinny tree on n edges, label the non-root vertices using a *pre-order-traversal* from left to right with labels $\{1, 2, \cdots, n\}$. Since the tree has n edges or non-root vertices, we start with a contiguous fountain consisting of n coins in the first row. Draw n diagonals labeled $\{1, 2, \cdots, n\}$ through each of these n coins. Next, look at the vertices at levels greater or equal to 2 in the given tree and identify the smallest and largest vertices. Add a contiguous block of coins between the smallest and the largest labels on the second level. Repeat the above until you reach the largest level in the given tree. Hence, the number of contiguous fountains of coins with n coins in the first row is equal to the number of ordered trees with n edges in which no two vertices at the same level have different parents.

2.3. Remark on the Bijection

Wilf and Odlyzko [10] gave a similar bijection between fountains of coins and partitions of integers studied by Szekeres [12] in connection with a combinatorial interpretation of Ramanujan's continued fraction. The bijection we have introduced

in the previous section extends to non-contiguous fountains of coins as well, and provides a direct bijection between fountains of coins consisting of a contiguous block of n coins in the bottom row and ordered trees with n edges. Figure 5 is a demonstration of this bijection (the diagonal and vertex labels are included to simply show how the mapping works).
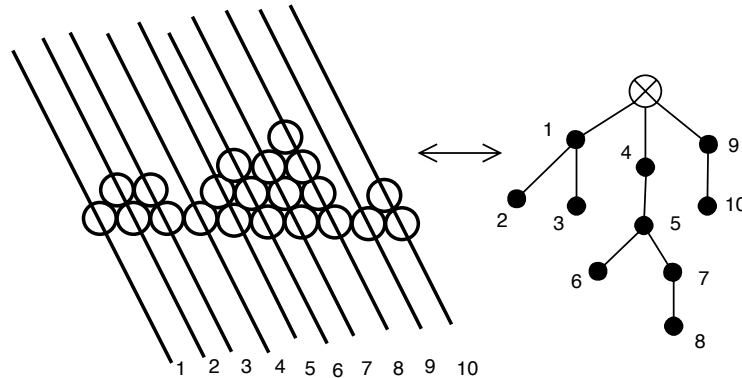

Figure 5: An example of a mapping from fountains of coins to ordered trees.

2.4. Ordered Trees of Height at Most Three and Skinny Trees

Emeric Deutsch and Helmut Prodinger provide two different bijections between ordered trees of height at most three and directed column-convex polyominoes [2]. Using the symbolic method, one can easily show that the generating function of the number of nonempty ordered trees of height at most three is the same as that of nonempty skinny trees [2, 8].

We now give a bijection between ordered trees in which no two vertices at the same level have different parents (skinny trees) and ordered trees of height at most three. Start with any ordered tree of height at most three on n edges, and convert it to a Dyck path of length $2n$. This is done very easily using a pre-order-traversal of the tree from left to right, and by associating each move away from the root vertex in the traversal to an up step (U) and each move towards the root in the traversal to a down step (D). Represent the Dyck path by a $\{U, D\}$-word of length $2n$.

In this word, scan from left to right and look for the first UU occurring after DD. Then form a subword of the form $DD \cdots UU$ (which represents a valley of depth 2 in the Dyck path) taking the first DD to the *left* of the UU located in the previous step. Now, interchange U and D throughout the subword, and repeat this process until the $\{U, D\}$-word is free of a subword of the form $DD \cdots UU$. Then draw an ordered tree with n edges corresponding to this final word. This tree is clearly a

skinny tree for we have eliminated all possibilities for two vertices at the same level to have different parents.
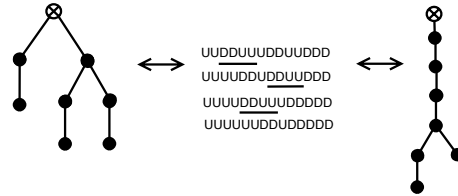

Figure 6: A mapping of an ordered tree of height at most three to a skinny tree.

To go from skinny trees to ordered trees of height at most three, start with a skinny tree on n edges and convert it to a Dyck path of length $2n$ using the simple process described above. Scan the corresponding $\{U, D\}$-word from *right to left* and look for the first DD occurring after UU. Check if the difference between the number of up steps and down steps is greater than two, and if so, interchange U and D throughout the $UU \cdots DD$ subword. If the difference between the number of up steps and down steps is less than or equal to two, interchanging U and D will force the Dyck path to cross the x-axis (results in more down steps than up steps in the Dyck path), and in this case we look for the next DD occuring after UU which gives a *permissible* $UU \cdots DD$ subword. Repeat this process until the resulting $\{U, D\}$-word is free of a subword of the form $UU \cdots DD$, and then draw the ordered tree corresponding to this final $\{U, D\}$-word. This ordered tree is clearly of height at most three for we have eliminated all possibilities for the tree to have height more than three.
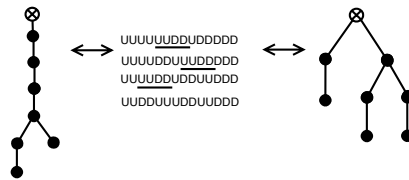

Figure 7: A mapping of a skinny tree to an ordered tree of height at most three.

3. Shapiro's Identity and Its Generalization

Let $g(z) = 1 + \sum_{k=1}^{\infty} g_k z^k$ and $f(z) = \sum_{k=1}^{\infty} f_k z^k$, where $f_1 \neq 0$. A Riordan Array $D = (g(z), f(z))$ is an infinite lower triangular matrix whose **column generating**

functions are
$$g(z)(f(z))^k, \text{ where } k = 0, 1, 2, 3, \cdots.$$

A typical element $d_{n,k}$ of the Riordan Array $D = (g(z), f(z))$ is given by

$$d_{n,k} = [z^n]g(z)(f(z))^k, \text{ where } n, k \geq 0.$$

The Fundamental Theorem of Riordan Arrays [13] states that if $A(z)$ and $B(z)$ are the generating functions of the column vectors $A = (a_0, a_1, a_2, \cdots)^T$ and $B = (b_0, b_1, b_2, \cdots)^T$, then

$$(g, f) \cdot A = B \text{ if and only if } B(z) = g(z)A(f(z)).$$

Multiplying the Riordan Array $D = (1, zC(z)^2)$ with a periodic column vector $A = (0, 1, 0, 0, -1, 0, 1, 0, 0, -1, \cdots)^T$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 5 & 4 & 1 & 0 & \cdots \\ 0 & 14 & 14 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 13 \\ \vdots \end{bmatrix},$$

Lou Shapiro noticed that the first few terms of the column vector on the right-hand side of the above equation are the same as the numbers of contiguous fountains of coins or skinny trees. Since the generating function of the periodic column vector is clearly $A(z) = \frac{z - z^4}{1 - z^5}$, we obtain, by applying the Fundamental Theorem of Riordan Arrays, that

$$\frac{z - z^2}{1 - 3z + z^2} = \frac{zC(z)^2 - \left(zC(z)^2\right)^4}{1 - \left(zC(z)^2\right)^5}.$$

We can actually obtain the above identity directly from the generating function $F(z) = \frac{z - z^2}{1 - 3z + z^2}$ of skinny trees, by replacing z with $\frac{C(z) - 1}{C(z)^2}$ (It is well known [9, 11, 18] that the Catalan generating function satisfies a functional equation $C(z) = 1 + zC(z)^2$).

$$\frac{z - z^2}{1 - 3z + z^2} = \frac{\left(\frac{C-1}{C^2}\right) - \left(\frac{C-1}{C^2}\right)^2}{1 - 3\left(\frac{C-1}{C^2}\right) + \left(\frac{C-1}{C^2}\right)^2} = \frac{C^2(C-1) - (C-1)^2}{C^4 - 3C^2(C-1) + (C-1)^2}$$

$$= \frac{(C-1)\left((C-1)^2 + (C-1) + 1\right)}{(C-1)^4 + (C-1)^3 + (C-1)^2 + (C-1) + 1}$$

$$= \frac{(C-1)\left(1 - (C-1)^3\right)}{1 - (C-1)^5}$$

Replacing $C - 1$ in the above equation with zC^2, we obtain Shapiro's identity

$$\frac{z - z^2}{1 - 3z + z^2} = \frac{zC(z)^2 - \left(zC(z)^2\right)^4}{1 - \left(zC(z)^2\right)^5}.$$

Since the left-hand side of the above equation is the generating function of non-empty ordered trees of height at most three, it is natural to ask how the exponents of $zC(z)^2$ in the right-hand side of the equation are going to change if we have considered non-empty ordered trees of height at most m in general.

Theorem 1. *The generating function of non-empty ordered trees of height at most m is*

$$f_m(z) = \frac{zC(z)^2 - \left(zC(z)^2\right)^{m+1}}{1 - \left(zC(z)^2\right)^{m+2}}.$$

Proof. Let $T_m(z)$ be the generating function of ordered trees of height at most m. Then from the decomposition rule shown in Figure 8, we see that $T_m(z)$ satisfies the recurrence relation $T_m(z) = 1 + zT_{m-1}(z)T_m(z)$.
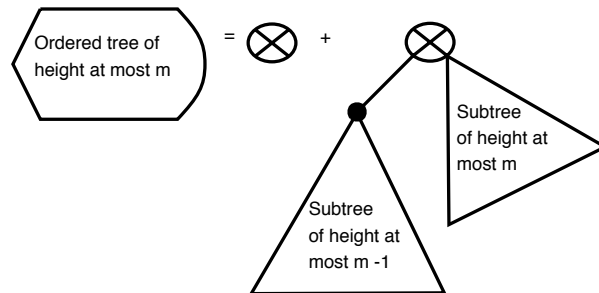

Figure 8: Decomposition of Ordered Trees of Height at most m.

Using the fact that the theorem is true for $m = 3$ (Shapiro's observation) with the recurrence relation $T_m(z) = 1 + zT_{m-1}(z)T_m(z)$, one can easily show by induction that

$$T_m(z) = C(z) \left(\frac{zC(z)^2 - \left(zC(z)^2\right)^{m+1}}{1 - \left(zC(z)^2\right)^{m+2}} \right).$$

Thus,

$$
\begin{aligned}
f_m(z) = T_m(z) - 1 &= C(z)\left(\frac{zC(z)^2 - \left(zC(z)^2\right)^{m+1}}{1 - \left(zC(z)^2\right)^{m+2}}\right) - 1 \\[2mm]
&= \frac{C(z) - C(z)\left(zC(z)^2\right)^{m+1} - 1 + \left(zC(z)^2\right)^{m+2}}{1 - \left(zC(z)^2\right)^{m+2}} \\[2mm]
&= \frac{zC(z)^2\left(1 - \left(zC(z)^2\right)^m\left[C(z) - zC(z)^2\right]\right)}{1 - \left(zC(z)^2\right)^{m+2}} \\[2mm]
&= \frac{zC(z)^2\left(1 - \left(zC(z)^2\right)^m\right)}{1 - \left(zC(z)^2\right)^{m+2}}.
\end{aligned}
$$

$\square$

3.1. Remarks

1. Repeated use of the recurrence relation

$$
T_m(z) = 1 + zT_{m-1}(z)T_m(z) \Leftrightarrow T_m(z) = \frac{1}{1 - zT_{m-1}(z)}
$$

shows that $T_m(z)$ is clearly a rational function in z. Hence,

$$
f_m(z) = T_m(z) - 1 = \frac{zC(z)^2 - \left(zC(z)^2\right)^{m+1}}{1 - \left(zC(z)^2\right)^{m+2}}
$$

is also a rational function in z. In fact, Doron Zeilberger, with the help of his computer Shalosh B. Ekhad, obtained that [4]

$$
f_m(z) = \frac{zC(z)^2 - \left(zC(z)^2\right)^{m+1}}{1 - \left(zC(z)^2\right)^{m+2}} = \frac{N_{m-1}(z)}{D_{m-1}(z)}
$$

where

$$
N_m(z) = [X^m]\left(\frac{-z - zX + z^2X^2}{1 + (1 - 2z)X^2 + z^2X^4}\right)
$$

and

$$
D_m(z) = [X^m]\left(\frac{-1 + z + (2z - 1)X - z^2X^2 - z^2X^3}{1 + (1 - 2z)X^2 + z^2X^4}\right).
$$

2. Ira Gessel and Gouce Xin obtained a similar expression for the generating function $G_m(z)$ of Dyck paths of height at most m in [7] and showed that

$$G_m(z) = \frac{p_m(z)}{p_{m+1}(z)}, \text{ where } p_m(z) = \sum_{0 \le k \le \frac{m}{2}} (-1)^k \binom{m-k}{k} z^k.$$

4. A Generalization of Skinny Trees Using k-Trees

In sections one and two, we have seen that contiguous fountains of coins, directed column-convex polyominoes, ordered trees of height at most three, and skinny trees are all enumerated by every other Fibonacci number. Chen and Shapiro looked at sequences satisfying the recurrence relation

$$G_{d,n} = (d+2)G_{d,n-1} - G_{d,n-2} \ (d \ge 1)$$

and provided a combinatorial interpretation for this class of sequences in terms of skinny ordered trees with dn edges in which the outdegree of each vertex is a multiple of $d \ge 1$ [1]. They refer to these subsets of ordered trees as $STd's$. It is not difficult to see, choosing suitable initial conditions, that

$$G_{1,n} = F_{2n-1}, n \ge 1 \text{ occur when } d = 1.$$

In our previous works [9, 18], we have introduced k-trees as a generalization of ordered trees. A k-tree is constructed from a single distinguished k-cycle, an elementary cycle with k-sides, by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to a non-terminal or internal edge. We define skinny k-trees to be subsets of k-trees where no two cycles at the same level have different parent-edges. For example, among the twelve 3-trees consisting of three 3-cycles shown in Figure 9, all except the third 3-tree in the second row are skinny trees.
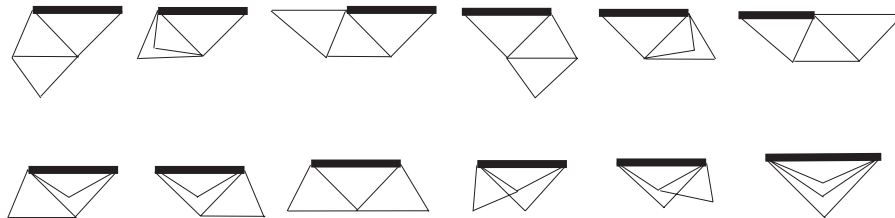

Figure 9: The twelve 3-trees on three cycles.

Let $S_{k,n}$ be the number of skinny k-trees with n k-cycles and $S_k(z) = \sum_n S_{k,n} z^n$ be the generating function of $\{S_{k,n}\}_{n=0}^\infty$. From the decomposition rule shown in Figure 10,
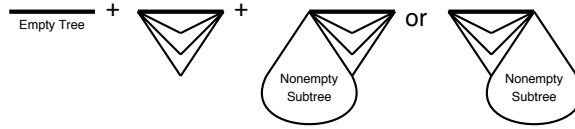
Figure 10: Decomposition of a skinny k-tree.

we obtain the functional equation

$$S_k(z) = 1 + \frac{z}{1-z} + (k-1)\left(\frac{1}{1-z}z(S_k(z)-1)\frac{1}{1-z}\right).$$

and solving this equation for $S_k(z)$, we get

$$S_k(z) = \frac{1-kz}{1-(k+1)z+z^2}.$$

Applying results from the theory of rational generating functions [14] to the above generating function, or counting the number of skinny k-trees directly, we obtain

$$S_{k,n} = (k+1)S_{k,n-1} - S_{k,n-2}, \text{ for } n \geq 2 \ (k \geq 2).$$

The initial conditions are obviously $S_{k,0} = 1$ and $S_{k,1} = 1$. Hence, $G_{d,n} = S_{d+1,n}$ for $d \geq 1$, and this provides another combinatorial interpretation in terms of k-trees for the generalized sequence considered by Chen and Shapiro in [1].

There is a natural correspondence between Chen and Shapiro's STd's and our skinny k-trees, where $k = d+1$, and all the results obtained in [1] including the one-to-one correspondence between ST2's with $2n$ edges and tilings of a $3 \times 2(n-1)$ board with tricolor dominoes can be obtained from skinny k-trees.
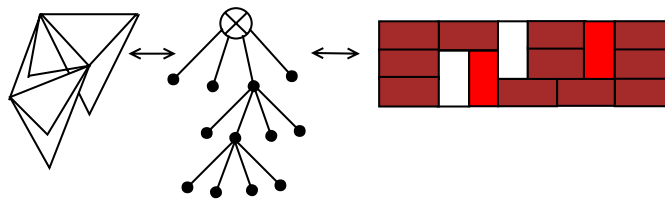

Figure 11: An example of a mapping of a 3-tree to a tricolor domino tiling.

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