ON ASYMPTOTIC FORMULA OF THE PARTITION FUNCTION $p_A(n)$

A. David Christopher
Department of Mathematics, The American College, Tamilnadu, India
davchrame@yahoo.co.in

M. Davamani Christober
Department of Mathematics, The American College, Tamilnadu, India
jothichristopher@yahoo.com

Received: 8/8/13, Revised: 11/11/14, Accepted: 12/18/14, Published: 1/19/15

Abstract

Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k relatively prime positive integers. Let $p_A(n)$ denote the number of partitions of n with parts belonging to A. The aim of this note is to provide a simple proof of the following well-known asymptotic relation of $p_A(n)$:

$$p_A(n) \sim \frac{n^{k-1}}{(a_1 a_2 \cdots a_k)(k-1)!}.$$

1. Introduction and Motivation

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(x_1, x_2, \cdots, x_m)$ such that $x_1 + x_2 + \cdots + x_m = n$. The x_i are called the parts of the partition. Let A be a set of positive integers. The partition function $p_A(n)$ is defined as the number of partitions of n with parts belonging to A.

The generating function of $p_A(n)$ is

$$\sum_{n=0}^{\infty} p_A(n) x^n = \prod_{a \in A} \frac{1}{1 - x^a} \tag{1}$$

with $p_A(0) = 1$; this generating function is valid in the interval $|x| < 1$.

For $\gcd(A) \neq 1$, we have

$$p_A(n) = \begin{cases} p_{\frac{A}{\gcd(A)}}\left(\frac{n}{\gcd(A)}\right) & if\ \gcd(A)|n, \\ 0\ otherwise \end{cases}$$

where the set $\frac{A}{\gcd(A)} = \left\{\frac{a}{\gcd(A)} : a \in A\right\}$. Thus, it is expedient to assume always that $\gcd(A) = 1$.

The function $p_A(n)$ is more appealing when A is a finite set of relatively prime integers. Throughout this note, we assume A to be a finite set of relatively prime positive integers. T. C. Brown, Wun-Seng Chou and Peter J. S. Shiue [3] found exact formulas for $p_A(n)$ when $|A| = 2$ or 3. The exact formula for $p_A(n)$ can also be found by means of partial fraction decomposition of its generating function (see [6]). Gert Almkvist [1] provided the exact formula for $p_A(n)$, without the usage of partial fraction decomposition of its generating function. The following asymptotic relation of $p_A(n)$ is well-known.

Theorem 1. *Let $A = \{a_1, a_2, \cdots, a_k\}$ be a set of k relatively prime positive integers. Then the following asymptotic relation holds true:*

$$p_A(n) \sim \frac{n^{k-1}}{(a_1 a_2 \cdots a_k)\,(k-1)!}. \tag{2}$$

In 1927, E. Netto [8] pioneered in providing a proof of this theorem and subsequently, in 1972, G. Polya and G. Szegö [9] gave another proof; in both the proofs partial fraction decomposition of the generating function was utilized. In 1942, Paul Erdos [5] proved this result for the case: $A = \{1, 2, \cdots, k\}$. In 1991, S. Sertoz and A. E. Ozlük [11] found another proof by wielding the following recurrence relation:

$$1 = \sum_{i=n-k+2}^{n} C_{n-i}[p_A(i) - p_A(i - (a_1 \cdots a_k))]$$

for $n > (a_1 \cdots a_k) - (a_1 + \cdots + a_k) + k - 2$, where

$$C_m = \begin{cases} (-1)^m \binom{k-2}{m} \ \ \mathit{for}\ 0 \le m \le k-2 \\ 0 \ \mathit{otherwise.} \end{cases}$$

In 2000, Melvyn B. Nathanson [7] obtained an arithmetic proof.

The aim of this note is to provide a new proof of this historical result. The proof furnished in this note is based on the fact that: the function $p_A(n)$ is a quasi polynomial.

Definition 2. *An arithmetical function f is said to be a Quasi polynomial if, $f(\alpha l + r)$ is a polynomial in l for each $r = 0, 1, \cdots, \alpha - 1$, where α is a positive integer greater than 1. Each polynomial $f(\alpha l + r)$ is called a constituent polynomial of f and α is called a quasi period of f.*

In 1943, E. T. Bell [2] found that the function $p_A(n)$ is a quasi polynomial by means of partial fraction decomposition of its generating function. In 1961, E. M. Wright [12] reestablished this finding by extracting the term $(1 - x^t)^{-k}$ from the generating function of $p_A(n)$, where $t = lcm(a_1, \cdots, a_k)$. In 2006, using a similar method, O. J. Rødseth and J. A. Sellers [10] obtained the quasi polynomial representation of $p_A(n)$ in binomial coefficients form.

2. Proof of Theorem 1

2.1. A Recurrence Relation Satisfied by $p_A(n)$

Following recurrence relation is crucial to our proof.

Lemma 3. *Let n be a positive integer and let $a \in A$. Then, we have*

$$p_A(n) = p_A(n-a) + p_{A\setminus\{a\}}(n), \tag{3}$$

provided $a \leq n$.

Proof. Let $\pi = (x_1, x_2, \cdots, x_m)$ be a partition of n with parts belonging to A and let $a \in A$.

Case (i) Assume that $x_i = a$ for some i. Then π is of the form: $\pi = (x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_m)$. We enumerate this kind of partitions of n; to that end, we consider the mapping:

$$(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_m) \rightarrow (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m),$$

which clearly establishes a one to one correspondence between the following sets:

- The set of all partitions of n with parts belonging to A and having a as a part;
- The set of all partitions of $n-a$ with parts belonging to A.

By the definition, the cardinality of the latter set is $p_A(n-a)$. Thus, the number of partitions of n of this type is $p_A(n-a)$.

Case(ii) Assume that $x_i \neq a$ $\forall i = 1, 2, \cdots, m$. Then, it is not hard to see that, the enumeration of such partitions is $p_{A\setminus\{a\}}(n)$. Thus, the result follows. □

2.2. Main Part of the Proof

As the consequences of Lemma 3, we will show that:

1. The function $p_A(n)$ is a quasi polynomial with a quasi period $a_1a_2\cdots a_k$.
2. Each constituent polynomial of $p_A(n)$ is of degree $k-1$.
3. The leading coefficient of each constituent polynomial of $p_A(n)$ is $\frac{(a_1a_2\cdots a_k)^{k-2}}{(k-1)!}$.

At this juncture, we note that: establishing the above three statements completes the proof as one can get from these statements that

$$\lim_{l\to\infty} \frac{p_A(a_1a_2\cdots a_kl+r)}{(a_1a_2\cdots a_kl+r)^{k-1}} = \frac{1}{(a_1a_2\cdots a_k)\,(k-1)!}$$

for each $r = 0, 1, \cdots, a_1a_2\cdots a_k - 1$; and the targeted estimate follows readily from this limit.

Now, we prove Statements 1, 2, and 3 simultaneously by using induction on k.

Suppose that $k = 2$. Let $A = \{a_1, a_2\}$ with $\gcd(a_1, a_2) = 1$. Then applying Lemma 3 a_1 times, we get

$$p_A(a_1a_2l + r) - p_A(a_1a_2(l-1) + r) = \sum_{i=0}^{a_1-1} p_{\{a_1\}}(a_1a_2l + r - ia_2) \tag{4}$$

for each $r = 0, 1, \cdots, a_1a_2 - 1$. Since the congruence equation

$$a_2x \equiv r(mod\ a_1)$$

has an unique solution modulo a_1 (see [4] pp. 83-84), the right side of the equation (4) equals 1. Then replacing l by $1, \cdots, l$ in equation (4) and adding, we get that

$$p_A(a_1a_2l + r) = l + p_A(r)$$

for each $r = 0, 1, 2, \cdots, a_1a_2-1$. Thus, the function $p_A(n)$ is a quasi polynomial with each constituent polynomial having degree 1 and leading coefficient $1 = \frac{(a_1a_2)^{2-2}}{(2-1)!}$.

Assume that the result is true when $|A| < k$ for a fixed $k \geq 3$. Consider a set of k positive integers say $A = \{a_1, a_2, \cdots, a_k\}$ with $\gcd(a_1, a_2, \cdots, a_k) = 1$. Let $s = \gcd(a_1, a_2, \cdots, a_{k-1})$. Then applying Lemma 3 $a_1a_2\cdots a_{k-1}$ times again, we get

$$\begin{aligned} & p_A(a_1 \cdots a_kl + r) - p_A(a_1 \cdots a_k(l-1) + r) \\ &= \sum_{0 \leq i \leq a_1\cdots a_{k-1}-1; s|(r-ia_k)} p_{\{a_1,\cdots,a_{k-1}\}}(a_1\cdots a_kl + r - ia_k) \\ &= \sum_{0 \leq i \leq a_1\cdots a_{k-1}-1; s|(r-ia_k)} p_{\{\frac{a_1}{s},\cdots,\frac{a_{k-1}}{s}\}}\left(\frac{a_1}{s}\cdots\frac{a_{k-1}}{s}(a_ks^{k-2}l + q_i) + r_i\right) \end{aligned} \tag{5}$$

for each $r = 0, 1, \cdots, a_1a_2\cdots a_k - 1$, where r_i and q_i were determined from the equality $\frac{r-ia_k}{s} = \frac{a_1\cdots a_{k-1}}{s^{k-1}}q_i + r_i$; here, uniqueness of r_i and q_i and the bound $0 \leq r_i \leq \frac{a_1\cdots a_{k-1}}{s^{k-1}} - 1$ follows from the division algorithm.

It is well-known that the congruence equation

$$a_kx \equiv r(mod\ s) \tag{6}$$

has a solution if and only if $\gcd(a_k, s)|r$ (see [4] pp. 83-84). Furthermore, in such case eqn(6) will have $\gcd(a_k, s)$ number of mutually incongruent solution modulo s. Here $\gcd(a_k, s) = 1$ and hence eqn(6) has an unique solution modulo s.

Since $\gcd(\frac{a_1}{s}, \cdots, \frac{a_{k-1}}{s})$=1, by induction assumption, it follows that the right side of the equation (5) is a sum of $\frac{a_1\cdots a_{k-1}}{s}$ polynomials and each of which is of degree $k-2$ with leading coefficient $\left(\frac{a_1\cdots a_{k-1}}{s^{k-1}}\right)^{k-3} \frac{a_k^{k-2}s^{(k-2)^2}}{(k-2)!}$. Consequently, the right side sum of the equation (5) is a polynomial of degree $k-2$. This implies that $p_A(a_1 \cdots a_k l+r)$ is a polynomial in l of degree $k-1$ for each $r = 0, 1, \cdots, a_1 \cdots a_k - 1$.

Now, we calculate the leading coefficient of $p_A(a_1 \cdots a_k l + r)$. If one denotes the leading coefficient of the polynomial $p_A(a_1 \cdots a_k l + r)$ by c_{k-1}, then by the previous observations it follows that

$$(k-1)c_{k-1} = \frac{(a_1 \cdots a_k)^{k-2}}{(k-2)!},$$

which simplifies to

$$c_{k-1} = \frac{(a_1 \cdots a_k)^{k-2}}{(k-1)!}.$$

The proof is now completed.

References

[1] Gert Almkvist, *Partitions with parts in a finite set and with parts outside a finite set*, Exp. Math, Vol. 11 (2002), No. 4, 449-456.

[2] E. T. Bell, *Interpolated denumerants and Lambert series*, Amer. J. Math. 65 (1943), 382-386.

[3] T. C. Brown, Wun- Seng Chou, Peter J.-S. Shiue, *On the partition function of a finite set*, Australas. J. Comb, 27 (2003), 193-204.

[4] David M. Burton, *Elementary Number theory*, Allyn and Bacon, Inc, 1980.

[5] P. Erdös, *On an elementary proof of some asymptotic formulas in the theory of partitions*, Ann. of Math. (2), 43(1942), 437-450.

[6] Melvyn B. Nathanson, *Elementary methods in Number theory*, Springer-Verlag, New York-Berlin- Heidelberg (2000), 466-467.

[7] Melvyn B. Nathanson, *Partition with parts in a finite set*, Proc. Amer. Math. Soc. 128 (2000), 1269-1273.

[8] E. Netto, *Le hrbuch der Combinatorik*, Teubner, Leipzig, 1927.

[9] G. Polya and G. Szegö, *Aufgaben and Lehrsätze aus der analysis*, Springer- Verlag, Berlin, 1925. English translation: *Problems and Theorems in Analysis*, Springer- verlag, New York, 1972.

[10] Ø. J. Rødseth and J. A. Sellers, *Partition with parts in a finite set*, Int. J. Number Theory 02, 455 (2006).

[11] S. Sertoz and A. E. Ozlük, *On the number of representations of an integer by a linear form*, İstanb. Üniv. Fen Fak. Mat. Fiz. Astron. Derg, 50 (1991).

[12] E. M. Wright, *A simple proof of a known result in partitions*, Amer. Math. Monthly 68 (1961), 144-145.