



PARTITIONS WITH DESIGNATED SUMMANDS IN WHICH ALL PARTS ARE ODD

Nayandeep Deka Baruah

*Department of Mathematical Sciences, Tezpur University, Napaam-784028,
Sonitpur, Assam, India
nayan@tezu.ernet.in*

Kanan Kumari Ojah

*Department of Mathematics, Assam Don Bosco University, Guwahati-781017,
Assam, India
kanan08@tezu.ernet.in*

Received: 2/2/14, Revised: 11/12/14, Accepted: 3/1/15, Published: 3/5/15

Abstract

In 2002, Andrews, Lewis and Lovejoy studied the partition function $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd and found several identities and congruences satisfied by the function. In this paper, we find further identities and congruences satisfied by $PDO(n)$.

1. Introduction

In [1], Andrews, Lewis and Lovejoy introduced and studied a new class of partitions, partitions with designated summands. Partitions with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. For example, there are 10 partitions of 4 with designated summands, namely,

$$4', \quad 3' + 1', \quad 2' + 2, \quad 2 + 2', \quad 2' + 1' + 1, \quad 2' + 1 + 1', \\ 1' + 1 + 1 + 1, \quad 1 + 1' + 1 + 1, \quad 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1'.$$

The total number of partitions of n with designated summands is denoted by $PD(n)$. Hence, $PD(4) = 10$. Further studies on $PD(n)$ were carried out by Chen, Ji, Jin, and Shen [7].

In the same paper [1], Andrews, Lewis and Lovejoy also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd. From the above example, $PDO(4) = 5$. Note that $(PDO(n))$ is sequence A102186 in

the On-line Encyclopedia of Integer Sequences available at “<https://oeis.org/A102186>.”
The generating function found by Andrews, Lewis and Lovejoy for $PDO(n)$ is

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^2}{(q; q)_{\infty} (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}, \quad (1)$$

where, here and the sequel, for $|q| < 1$ and positive integers n , we use the standard notation

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad \text{and} \quad (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$

By using q -series and modular forms, they found (1) as well as the following identities.

Theorem 1.1 [1, Theorem 21 and Theorem 22] *We have*

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \frac{(q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty}^4}{(q; q)_{\infty}^2 (q^3; q^3)_{\infty}^2 (q^{12}; q^{12})_{\infty}^2}, \quad (2)$$

$$\sum_{n=0}^{\infty} PDO(2n+1)q^n = \frac{(q^2; q^2)_{\infty}^6 (q^{12}; q^{12})_{\infty}^2}{(q; q)_{\infty}^4 (q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty}^2}, \quad (3)$$

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{(q^2; q^2)_{\infty}^2 (q^6; q^6)_{\infty}^4}{(q; q)_{\infty}^4 (q^{12}; q^{12})_{\infty}^2}, \quad (4)$$

$$\sum_{n=0}^{\infty} PDO(3n+1)q^n = \frac{(q^3; q^3)_{\infty}^3 (q^2; q^2)_{\infty}^4 (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^5 (q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^2}, \quad (5)$$

$$\sum_{n=0}^{\infty} PDO(3n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^3 (q^6; q^6)_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^4 (q^4; q^4)_{\infty}}. \quad (6)$$

They also deduce the following congruences.

Corollary 1.2 [1, Corollary 19] *We have*

$$PDO(9n+6) \equiv 0 \pmod{3}, \quad (7)$$

$$PDO(12n+6) \equiv 0 \pmod{3}, \quad (8)$$

$$PDO(12n+10) \equiv 0 \pmod{3}, \quad (9)$$

$$PDO(24n) \equiv 0 \pmod{3},$$

$$PDO(24n+16) \equiv 0 \pmod{3}.$$

The aim of this paper is to find proofs of (2)–(6) and the following new identities by using certain dissections of theta functions.

Theorem 1.3 *We have*

$$\sum_{n=0}^{\infty} PDO(4n)q^n = \frac{(q^2; q^2)_{\infty}^2 (q^4; q^4)_{\infty}^5 (q^{12}; q^{12})_{\infty}^5}{(q; q)_{\infty}^5 (q^3; q^3)_{\infty} (q^6; q^6)_{\infty}^2 (q^8; q^8)_{\infty}^2 (q^{24}; q^{24})_{\infty}^2} + 4q^2 \frac{(q^2; q^2)_{\infty}^4 (q^8; q^8)_{\infty}^4 (q^{24}; q^{24})_{\infty}^2}{(q; q)_{\infty}^5 (q^3; q^3)_{\infty} (q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^2 (q^{12}; q^{12})_{\infty}}, \quad (10)$$

$$\sum_{n=0}^{\infty} PDO(4n+1)q^n = \frac{(q^2; q^2)_{\infty}^{12} (q^6; q^6)_{\infty}^2}{(q; q)_{\infty}^8 (q^3; q^3)_{\infty}^2 (q^4; q^4)_{\infty}^4}, \quad (11)$$

$$\sum_{n=0}^{\infty} PDO(4n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^6 (q^6; q^6)_{\infty}^2}{(q; q)_{\infty}^6 (q^3; q^3)_{\infty}^2}, \quad (12)$$

$$\sum_{n=0}^{\infty} PDO(4n+3)q^n = 4 \frac{(q^4; q^4)_{\infty}^4 (q^6; q^6)_{\infty}^2}{(q; q)_{\infty}^4 (q^3; q^3)_{\infty}^2}, \quad (13)$$

$$\sum_{n=0}^{\infty} PDO(6n)q^n = \frac{(q^2; q^2)_{\infty}^{14} (q^3; q^3)_{\infty}^4}{(q; q)_{\infty}^{12} (q^4; q^4)_{\infty}^4 (q^6; q^6)_{\infty}^2}, \quad (14)$$

$$\sum_{n=0}^{\infty} PDO(6n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^{13} (q^3; q^3)_{\infty} (q^6; q^6)_{\infty}}{(q; q)_{\infty}^{11} (q^4; q^4)_{\infty}^4}, \quad (15)$$

$$\sum_{n=0}^{\infty} PDO(6n+3)q^n = 4 \frac{(q^2; q^2)_{\infty}^2 (q^3; q^3)_{\infty}^4 (q^4; q^4)_{\infty}^4}{(q; q)_{\infty}^8 (q^6; q^6)_{\infty}^2}, \quad (16)$$

$$\sum_{n=0}^{\infty} PDO(6n+5)q^n = 8 \frac{(q^2; q^2)_{\infty} (q^3; q^3)_{\infty} (q^4; q^4)_{\infty}^4 (q^6; q^6)_{\infty}}{(q; q)_{\infty}^7}, \quad (17)$$

$$\sum_{n=0}^{\infty} PDO(9n+3)q^n = 4 \left\{ \frac{(q^2; q^2)_{\infty}^{11} (q^3; q^3)_{\infty}^9}{(q; q)_{\infty}^{15} (q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty}^3} + 4q \frac{(q^2; q^2)_{\infty}^8 (q^6; q^6)_{\infty}^6}{(q; q)_{\infty}^{12} (q^4; q^4)_{\infty}^2} \right\}, \quad (18)$$

$$\sum_{n=0}^{\infty} PDO(9n+6)q^n = 12 \frac{(q^2; q^2)_{\infty}^{10} (q^3; q^3)_{\infty}^6}{(q; q)_{\infty}^{14} (q^4; q^4)_{\infty}^2}, \quad (19)$$

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(12n)q^n &= \frac{(q; q)_{\infty}^{14}}{(q^2; q^2)_{\infty}^4 (q^3; q^3)_{\infty}^2} \left\{ \frac{(q^2; q^2)_{\infty}^{24} (q^3; q^3)_{\infty}^{12}}{(q; q)_{\infty}^{36} (q^6; q^6)_{\infty}^8} \right. \\ &\quad + 54q \frac{(q^2; q^2)_{\infty}^{16} (q^3; q^3)_{\infty}^8}{(q; q)_{\infty}^{32}} \\ &\quad \left. + 81q^2 \frac{(q^2; q^2)_{\infty}^8 (q^3; q^3)_{\infty}^4 (q^6; q^6)_{\infty}^8}{(q; q)_{\infty}^{28}} \right\}, \quad (20) \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(12n+2)q^n &= 2 \left\{ \frac{(q^2; q^2)_{\infty}^{12} (q^3; q^3)_{\infty}^{16}}{(q; q)_{\infty}^{20} (q^6; q^6)_{\infty}^8} \right. \\ &\quad \left. + 44q \frac{(q^2; q^2)_{\infty}^9 (q^3; q^3)_{\infty}^7 (q^6; q^6)_{\infty}}{(q; q)_{\infty}^{17}} \right\}, \end{aligned} \quad (21)$$

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(12n+3)q^n &= 4 \left\{ \frac{(q^2; q^2)_{\infty}^{12} (q^4; q^4)_{\infty}^8 (q^6; q^6)_{\infty}^4}{(q; q)_{\infty}^{18} (q^8; q^8)_{\infty}^4 (q^{12}; q^{12})_{\infty}^2} \right. \\ &\quad + 16q \frac{(q^2; q^2)_{\infty}^{11} (q^3; q^3)_{\infty} (q^4; q^4)_{\infty}^4 (q^6; q^6)_{\infty}}{(q; q)_{\infty}^{17}} \\ &\quad + 4q \frac{(q^2; q^2)_{\infty}^{16} (q^6; q^6)_{\infty}^4 (q^8; q^8)_{\infty}^4}{(q; q)_{\infty}^{18} (q^4; q^4)_{\infty}^4 (q^{12}; q^{12})_{\infty}^2} \\ &\quad + 4q \frac{(q^2; q^2)_{\infty}^6 (q^3; q^3)_{\infty}^2 (q^4; q^4)_{\infty}^{12} (q^{12}; q^{12})_{\infty}^2}{(q; q)_{\infty}^{16} (q^6; q^6)_{\infty}^2 (q^8; q^8)_{\infty}^4} \\ &\quad \left. + 16q^2 \frac{(q^2; q^2)_{\infty}^{10} (q^3; q^3)_{\infty}^2 (q^8; q^8)_{\infty}^4 (q^{12}; q^{12})_{\infty}^2}{(q; q)_{\infty}^{16} (q^6; q^6)_{\infty}^2} \right\}, \end{aligned} \quad (22)$$

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(12n+6)q^n &= 12 \left\{ \frac{(q^2; q^2)_{\infty}^{11} (q^3; q^3)_{\infty}^{13}}{(q; q)_{\infty}^{19} (q^6; q^6)_{\infty}^5} \right. \\ &\quad \left. + 10q \frac{(q^2; q^2)_{\infty}^8 (q^3; q^3)_{\infty}^4 (q^6; q^6)_{\infty}^4}{(q; q)_{\infty}^{16}} \right\}, \end{aligned} \quad (23)$$

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(12n+9)q^n &= 16 \left\{ \frac{(q^2; q^2)_{\infty}^{14} (q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty}^4}{(q; q)_{\infty}^{18} (q^{12}; q^{12})_{\infty}^2} \right. \\ &\quad + \frac{(q^2; q^2)_{\infty}^9 (q^3; q^3)_{\infty} (q^4; q^4)_{\infty}^{10} (q^6; q^6)_{\infty}}{(q; q)_{\infty}^{17} (q^8; q^8)_{\infty}^4} \\ &\quad + 4q \frac{(q^2; q^2)_{\infty}^{13} (q^3; q^3)_{\infty} (q^6; q^6)_{\infty} (q^8; q^8)_{\infty}^4}{(q; q)_{\infty}^{17} (q^4; q^4)_{\infty}^2} \\ &\quad \left. + 4q \frac{(q^2; q^2)_{\infty}^8 (q^4; q^4)_{\infty}^6 (q^3; q^3)_{\infty}^2 (q^{12}; q^{12})_{\infty}^2}{(q; q)_{\infty}^{16} (q^6; q^6)_{\infty}^2} \right\}, \end{aligned} \quad (24)$$

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(12n+10)q^n &= 6 \left\{ 7 \frac{(q^2; q^2)_{\infty}^{10} (q^3; q^3)_{\infty}^{10}}{(q; q)_{\infty}^{18} (q^6; q^6)_{\infty}^2} \right. \\ &\quad \left. + 16q \frac{(q^2; q^2)_{\infty}^7 (q^3; q^3)_{\infty} (q^6; q^6)_{\infty}^7}{(q; q)_{\infty}^{15}} \right\}. \end{aligned} \quad (25)$$

From the previous theorem, we easily deduce the following congruences.

Corollary 1.4 *We have*

$$PDO(4n+2) \equiv 0 \pmod{2},$$

$$\begin{aligned}
PDO(4n+3) &\equiv 0 \pmod{4}, \\
PDO(6n+2) &\equiv 0 \pmod{2}, \\
PDO(6n+3) &\equiv 0 \pmod{4}, \\
PDO(6n+5) &\equiv 0 \pmod{8}, \\
PDO(9n+3) &\equiv 0 \pmod{4}, \\
PDO(9n+6) &\equiv 0 \pmod{12}, \tag{26}
\end{aligned}$$

$$PDO(12n+6) \equiv 0 \pmod{12}, \tag{27}$$

$$\begin{aligned}
PDO(12n+9) &\equiv 0 \pmod{16}, \\
PDO(12n+10) &\equiv 0 \pmod{6}. \tag{28}
\end{aligned}$$

Note that, congruences (26), (27) and (28) are improved versions of (7), (8) and (9), respectively.

We also find the following congruences.

Theorem 1.5 *For all nonnegative integers n , we have*

$$PDO(8n+6) \equiv 0 \pmod{4}, \tag{29}$$

$$PDO(8n+7) \equiv 0 \pmod{8}, \tag{30}$$

$$PDO(18n+15) \equiv 0 \pmod{24}, \tag{31}$$

$$PDO(9n+9) \equiv 0 \pmod{4}, \tag{32}$$

$$PDO(24n+9) \equiv 0 \pmod{8}, \tag{33}$$

$$PDO(24n+15) \equiv 0 \pmod{8}, \tag{34}$$

and

$$PDO(24n+21) \equiv 0 \pmod{8}. \tag{35}$$

In the next section, we give some definitions, preliminary results and dissections of some theta functions. In the last section, we prove Theorems 1.1, 1.3, 1.5.

2. Definitions, Preliminary Results, and Proof of Some Dissections

Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

The well-known Jacobi's triple product identity takes the form

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \tag{36}$$

Two special cases of $f(a, b)$ are defined, for $|q| < 1$, by [3, p. 36, Entry 22]

$$\varphi(q) := f(q, q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2} \quad (37)$$

and

$$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}, \quad (38)$$

where the penultimate product representations in (37) and (38) arise from (36).

After Ramanujan, we also define

$$\chi(-q) = (q; q^2)_{\infty} = \frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (39)$$

Next, we recall from [5] that

$$c(q) := \sum_{m, n=-\infty}^{\infty} q^{m^2 + mn + n^2 + m + n} = 3 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}}. \quad (40)$$

Now we state a lemma.

Lemma 2.1 *We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \quad (41)$$

$$f(q, q^2) = \frac{\varphi(-q^3)}{\chi(-q)}, \quad (42)$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \quad (43)$$

Proof. See [3, p. 49, Corollary(ii)] and [3, p. 350, Eq. (2.3)] for the proofs of (41) and (42), respectively. Adding identities (v) and (vi) of [3, Entry 25, p. 40], we can easily derive (43). \square

In the remaining lemmas of this section, we state and prove certain 2- and 3-dissections.

Lemma 2.2 *We have*

$$\begin{aligned} \frac{1}{(q; q)_{\infty} (q^3; q^3)_{\infty}} &= \frac{(q^8; q^8)_{\infty}^2 (q^{12}; q^{12})_{\infty}^5}{(q^2; q^2)_{\infty}^2 (q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^4 (q^{24}; q^{24})_{\infty}^2} \\ &\quad + q \frac{(q^4; q^4)_{\infty}^5 (q^{24}; q^{24})_{\infty}^2}{(q^2; q^2)_{\infty}^4 (q^6; q^6)_{\infty}^2 (q^8; q^8)_{\infty}^2 (q^{12}; q^{12})_{\infty}}, \end{aligned} \quad (44)$$

$$\begin{aligned}
\frac{1}{(q; q)_\infty^2 (q^3; q^3)_\infty^2} &= \frac{(q^8; q^8)_\infty^5 (q^{24}; q^{24})_\infty^5}{(q^2; q^2)_\infty^5 (q^6; q^6)_\infty^5 (q^{16}; q^{16})_\infty^2 (q^{48}; q^{48})_\infty^2} \\
&+ 2q \frac{(q^4; q^4)_\infty^4 (q^{12}; q^{12})_\infty^4}{(q^2; q^2)_\infty^6 (q^6; q^6)_\infty^6} \\
&+ 4q^4 \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2 (q^{16}; q^{16})_\infty^4 (q^{48}; q^{48})_\infty^2}{(q^2; q^2)_\infty^5 (q^6; q^6)_\infty^5 (q^8; q^8)_\infty (q^{24}; q^{24})_\infty}. \tag{45}
\end{aligned}$$

Proof. From [6, Corollary 8], we find

$$\psi(q)\psi(q^3) = \psi(q^4)\varphi(q^6) + q\psi(q^{12})\varphi(q^2). \tag{46}$$

Again, we have from [6, Corollary 4]

$$\varphi(q)\varphi(q^3) = \varphi(q^4)\varphi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24}). \tag{47}$$

Employing (37) and (38) in (46) and (47), we easily derive (44) and (45), respectively. \square

Lemma 2.3 *We have*

$$\frac{1}{(q; q)_\infty^4} = \frac{(q^4; q^4)_\infty^{14}}{(q^2; q^2)_\infty^{14} (q^8; q^8)_\infty^4} + 4q \frac{(q^4; q^4)_\infty^2 (q^8; q^8)_\infty^4}{(q^2; q^2)_\infty^{10}}. \tag{48}$$

Proof. Employing (37) and (38) in (43), we readily arrive at (48). \square

Proofs of the results in the next lemma can be found in [2].

Lemma 2.4 *We have*

$$\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3} = \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^3}{(q^2; q^2)_\infty^9 (q^{12}; q^{12})_\infty^2} + 3q \frac{(q^6; q^6)_\infty (q^{12}; q^{12})_\infty^2 (q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^7}, \tag{49}$$

$$\begin{aligned}
\frac{(q^3; q^3)_\infty^2}{(q; q)_\infty^4} &= \frac{(q^4; q^4)_\infty^4 (q^6; q^6)_\infty (q^8; q^8)_\infty^4 (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty^{10} (q^{16}; q^{16})_\infty^2 (q^{24}; q^{24})_\infty} \\
&+ 2q \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty (q^{12}; q^{12})_\infty^2 (q^{16}; q^{16})_\infty^2}{(q^2; q^2)_\infty^{10} (q^8; q^8)_\infty^2 (q^{24}; q^{24})_\infty} \\
&+ 2q \frac{(q^4; q^4)_\infty (q^6; q^6)_\infty^2 (q^8; q^8)_\infty^6 (q^{24}; q^{24})_\infty}{(q^2; q^2)_\infty^9 (q^{12}; q^{12})_\infty (q^{16}; q^{16})_\infty^2} \\
&+ 4q^2 \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2 (q^{16}; q^{16})_\infty^2 (q^{24}; q^{24})_\infty}{(q^2; q^2)_\infty^9 (q^{12}; q^{12})_\infty}, \tag{50}
\end{aligned}$$

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q \frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)} + 4q^2 w^2(q^3) \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}, \tag{51}$$

where

$$w(q) = \frac{(q; q)_\infty (q^6; q^6)_\infty^3}{(q^2; q^2)_\infty (q^3; q^3)_\infty^3}. \quad (52)$$

Lemma 2.5 *We have*

$$\begin{aligned} & \frac{(q^2; q^2)_\infty^6}{(q; q)_\infty^6} \\ &= \frac{(q^6; q^6)_\infty^{10} (q^9; q^9)_\infty^{16}}{(q^3; q^3)_\infty^{18} (q^{18}; q^{18})_\infty^8} + 6q \frac{(q^6; q^6)_\infty^9 (q^9; q^9)_\infty^{13}}{(q^3; q^3)_\infty^{17} (q^{18}; q^{18})_\infty^5} + 21q^2 \frac{(q^6; q^6)_\infty^8 (q^9; q^9)_\infty^{10}}{(q^3; q^3)_\infty^{16} (q^{18}; q^{18})_\infty^2} \\ &+ 44q^3 \frac{(q^6; q^6)_\infty^7 (q^9; q^9)_\infty^7 (q^{18}; q^{18})_\infty}{(q^3; q^3)_\infty^{15}} + 60q^4 \frac{(q^6; q^6)_\infty^6 (q^9; q^9)_\infty^4 (q^{18}; q^{18})_\infty^4}{(q^3; q^3)_\infty^{14}} \\ &+ 48q^5 \frac{(q^6; q^6)_\infty^5 (q^9; q^9)_\infty (q^{18}; q^{18})_\infty^7}{(q^3; q^3)_\infty^{13}} + 16q^6 \frac{(q^6; q^6)_\infty^4 (q^{18}; q^{18})_\infty^{10}}{(q^9; q^9)_\infty^2 (q^3; q^3)_\infty^{12}}. \end{aligned} \quad (53)$$

Proof. Squaring both sides of (41) and then employing (42), we have

$$\psi^2(q) = \frac{\varphi^2(-q^9)}{\chi^2(-q^3)} + q^2 \psi^2(q^9) + 2q \frac{\varphi(-q^9) \psi(q^9)}{\chi(-q^3)}. \quad (54)$$

Again, squaring both sides of (51), we find that

$$\frac{1}{\varphi^2(-q)} = \frac{\varphi^6(-q^9)}{\varphi^8(-q^3)} \{1 + 4qw(q^3) + 12q^2w^2(q^3) + 16q^3w^3(q^3) + 16q^4w^4(q^3)\}. \quad (55)$$

Now, replacing q by $-q$ in (37), we have

$$\varphi(-q) = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}. \quad (56)$$

Multiplying (54) and (55) and then employing (38), (39), (52) and (56), we easily arrive at (53) to complete the proof. \square

Lemma 2.6 *We have*

$$\begin{aligned} \frac{(q^4; q^4)_\infty}{(q; q)_\infty} &= \frac{(q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty^4}{(q^3; q^3)_\infty^3 (q^{36}; q^{36})_\infty^2} + q \frac{(q^6; q^6)_\infty^2 (q^9; q^9)_\infty^3 (q^{36}; q^{36})_\infty}{(q^3; q^3)_\infty^4 (q^{18}; q^{18})_\infty^2} \\ &+ 2q^2 \frac{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (q^{36}; q^{36})_\infty}{(q^3; q^3)_\infty^3}. \end{aligned} \quad (57)$$

Proof. From [4], we have

$$\frac{c(q)}{c(q^4)} = 1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}. \quad (58)$$

Employing (40) in (58), we find that

$$\frac{(q^4; q^4)_\infty}{(q; q)_\infty} = q \frac{(q^{12}; q^{12})_\infty^3}{(q^3; q^3)_\infty^3} \left\{ 1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)} \right\}. \quad (59)$$

Next, replacing q by q^2 in (54), we have

$$\psi^2(q^2) = \frac{\varphi^2(-q^{18})}{\chi^2(-q^6)} + q^4\psi^2(q^{18}) + 2q^2 \frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}. \quad (60)$$

Using (60) in (59), we obtain

$$\begin{aligned} & \frac{(q^4; q^4)_\infty}{(q; q)_\infty} \\ &= q \frac{(q^{12}; q^{12})_\infty^3}{(q^3; q^3)_\infty^3} \left\{ 1 + \frac{1}{q\psi^2(q^6)} \left(\frac{\varphi^2(-q^{18})}{\chi^2(-q^6)} + q^4\psi^2(q^{18}) + 2q^2 \frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)} \right) \right\} \\ &= q \frac{(q^{12}; q^{12})_\infty^3}{(q^3; q^3)_\infty^3} \left\{ 1 + q^3 \frac{\psi^2(q^{18})}{\psi^2(q^6)} \right\} \\ &+ q \frac{(q^{12}; q^{12})_\infty^3}{(q^3; q^3)_\infty^3} \left\{ \frac{\varphi^2(-q^{18})}{q\psi^2(q^6)\chi^2(-q^6)} + 2q \frac{\varphi(-q^{18})\psi(q^{18})}{\psi^2(q^6)\chi(-q^6)} \right\}. \end{aligned} \quad (61)$$

Employing (38), (39) and (56) in (61), we find that

$$\begin{aligned} \frac{(q^4; q^4)_\infty}{(q; q)_\infty} &= q \frac{(q^{12}; q^{12})_\infty^3}{(q^3; q^3)_\infty^3} \left\{ 1 + q^3 \frac{\psi^2(q^{18})}{\psi^2(q^6)} \right\} + \frac{(q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty^4}{(q^3; q^3)_\infty^3 (q^{36}; q^{36})_\infty^2} \\ &+ 2q^2 \frac{(q^{18}; q^{18})_\infty (q^{36}; q^{36})_\infty (q^6; q^6)_\infty}{(q^3; q^3)_\infty^3}. \end{aligned} \quad (62)$$

Now, multiplying both sides of (59) by $\psi^2(q^6)/\psi^2(q^2)$, replacing q by q^3 , and then employing (38), we deduce that

$$1 + q^3 \frac{\psi^2(q^{18})}{\psi^2(q^6)} = \frac{(q^6; q^6)_\infty^2 (q^9; q^9)_\infty^3 (q^{36}; q^{36})_\infty}{(q^3; q^3)_\infty (q^{18}; q^{18})_\infty^2 (q^{12}; q^{12})_\infty^3}. \quad (63)$$

Employing (63) in (62), we arrive at (62) to finish the proof. \square

3. Proofs of Theorems 1.1, 1.3, and 1.5

Proofs of (2) and (3). Using (44) in (1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(n)q^n &= \frac{(q^8; q^8)_{\infty}^2 (q^{12}; q^{12})_{\infty}^4}{(q^2; q^2)_{\infty}^2 (q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^2 (q^{24}; q^{24})_{\infty}^2} \\ &\quad + q \frac{(q^4; q^4)_{\infty}^6 (q^{24}; q^{24})_{\infty}^2}{(q^2; q^2)_{\infty}^4 (q^8; q^8)_{\infty}^2 (q^{12}; q^{12})_{\infty}^2}. \end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} in the above, we easily arrive at (2) and (3), respectively. \square

Proofs of (4)–(6). Using (57) in (1), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(n)q^n &= \frac{(q^6; q^6)_{\infty}^2 (q^{18}; q^{18})_{\infty}^4}{(q^3; q^3)_{\infty}^4 (q^{36}; q^{36})_{\infty}^2} + q \frac{(q^6; q^6)_{\infty}^4 (q^9; q^9)_{\infty}^3 (q^{36}; q^{36})_{\infty}}{(q^3; q^3)_{\infty}^5 (q^{12}; q^{12})_{\infty} (q^{18}; q^{18})_{\infty}^2} \\ &\quad + 2q^2 \frac{(q^6; q^6)_{\infty}^3 (q^{18}; q^{18})_{\infty} (q^{36}; q^{36})_{\infty}}{(q^3; q^3)_{\infty}^4 (q^{12}; q^{12})_{\infty}}. \end{aligned} \quad (64)$$

Extracting from both sides of (64), the terms involving q^{3n} , q^{3n+1} , and q^{3n+2} , respectively, we arrive at (4)–(6), respectively. \square

Proofs of (10) and (12). Employing (45) in (2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(2n)q^n &= \frac{(q^4; q^4)_{\infty}^2 (q^8; q^8)_{\infty}^5 (q^{24}; q^{24})_{\infty}^5}{(q^2; q^2)_{\infty}^5 (q^6; q^6)_{\infty} (q^{12}; q^{12})_{\infty}^2 (q^{16}; q^{16})_{\infty}^2 (q^{48}; q^{48})_{\infty}^2} \\ &\quad + 2q \frac{(q^4; q^4)_{\infty}^6 (q^{12}; q^{12})_{\infty}^2}{(q^2; q^2)_{\infty}^6 (q^6; q^6)_{\infty}^2} \\ &\quad + 4q^4 \frac{(q^4; q^4)_{\infty}^4 (q^{16}; q^{16})_{\infty}^4 (q^{48}; q^{48})_{\infty}^2}{(q^2; q^2)_{\infty}^5 (q^6; q^6)_{\infty} (q^8; q^8)_{\infty} (q^{12}; q^{12})_{\infty}^2 (q^{24}; q^{24})_{\infty}}. \end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above we obtain (10) and (12), respectively. \square

Proofs of (11) and (13). Employing (48) in (3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(2n+1)q^n &= \frac{(q^2; q^2)_{\infty}^6 (q^{12}; q^{12})_{\infty}^2}{(q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty}^2} \left\{ \frac{(q^4; q^4)_{\infty}^{14}}{(q^2; q^2)_{\infty}^{14} (q^8; q^8)_{\infty}^4} \right. \\ &\quad \left. + 4q \frac{(q^4; q^4)_{\infty}^2 (q^8; q^8)_{\infty}^4}{(q^2; q^2)_{\infty}^{10}} \right\}. \end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above, we easily deduce (11) and (13). \square

Proofs of (14) and (16). Using (48) in (4), we arrive at

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{(q^2; q^2)_{\infty}^2 (q^6; q^6)_{\infty}^4}{(q^{12}; q^{12})_{\infty}^2} \left\{ \frac{(q^4; q^4)_{\infty}^{14}}{(q^2; q^2)_{\infty}^{14} (q^8; q^8)_{\infty}^4} + 4q \frac{(q^4; q^4)_{\infty}^2 (q^8; q^8)_{\infty}^4}{(q^2; q^2)_{\infty}^{10}} \right\}. \quad (65)$$

Now (14) and (16) can be deduced by extracting the terms involving the even and odd powers of q , respectively, of the above. \square

Proofs of (15) and (17). Employing (48) in (6), we arrive at

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(3n+2)q^n &= 2 \frac{(q^2; q^2)_{\infty}^3 (q^6; q^6)_{\infty} (q^{12}; q^{12})_{\infty}}{(q^4; q^4)_{\infty}} \\ &\times \left\{ \frac{(q^4; q^4)_{\infty}^{14}}{(q^2; q^2)_{\infty}^{14} (q^8; q^8)_{\infty}^4} + 4q \frac{(q^4; q^4)_{\infty}^2 (q^8; q^8)_{\infty}^4}{(q^2; q^2)_{\infty}^{10}} \right\}. \quad (66) \end{aligned}$$

Extracting the even and odd powers of q from both sides of (66), we readily deduce (15) and (17), respectively. \square

Proofs of (18) and (19). Squaring both sides of (51) and then employing the resultant identity in (4), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(3n)q^n &= \frac{\varphi^2(-q^6)\varphi^6(-q^9)}{\varphi^8(-q^3)} \{1 + 4q w(q^3) + 12q^2 w^2(q^3) + 16q^3 w^3(q^3) \\ &\quad + 16q^4 w^4(q^3)\}. \quad (67) \end{aligned}$$

Extracting from both sides of (67), those terms involving only q^{3n+1} and q^{3n+2} , respectively, we find that

$$\sum_{n=0}^{\infty} PDO(9n+3)q^n = 4 \frac{w(q)\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)} \{1 + 4q w^3(q)\}$$

and

$$\sum_{n=0}^{\infty} PDO(9n+6)q^n = 12 \frac{w^2(q)\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)},$$

which by (52) and (56) reduce to (18) and (19), respectively. \square

Proofs of (20), (21), (23) and (25). Employing (53) in (12), we find that

$$\begin{aligned}
& \sum_{n=0}^{\infty} PDO(4n+2)q^n \\
&= 2 \left\{ \frac{(q^6; q^6)_{\infty}^{12} (q^9; q^9)_{\infty}^{16}}{(q^3; q^3)_{\infty}^{20} (q^{18}; q^{18})_{\infty}^8} + 6q \frac{(q^6; q^6)_{\infty}^{11} (q^9; q^9)_{\infty}^{13}}{(q^3; q^3)_{\infty}^{19} (q^{18}; q^{18})_{\infty}^5} + 21q^2 \frac{(q^6; q^6)_{\infty}^{10} (q^9; q^9)_{\infty}^{10}}{(q^3; q^3)_{\infty}^{18} (q^{18}; q^{18})_{\infty}^2} \right. \\
&+ 44q^3 \frac{(q^6; q^6)_{\infty}^9 (q^9; q^9)_{\infty}^7 (q^{18}; q^{18})_{\infty}}{(q^3; q^3)_{\infty}^{17}} + 60q^4 \frac{(q^6; q^6)_{\infty}^8 (q^9; q^9)_{\infty}^4 (q^{18}; q^{18})_{\infty}^4}{(q^3; q^3)_{\infty}^{16}} \\
& \left. + 48q^5 \frac{(q^6; q^6)_{\infty}^7 (q^9; q^9)_{\infty} (q^{18}; q^{18})_{\infty}^7}{(q^3; q^3)_{\infty}^{15}} \right\}. \tag{68}
\end{aligned}$$

Extracting from both sides of (68), those terms involving only q^{3n} , q^{3n+1} and q^{3n+2} , respectively, we deduce (21), (23) and (25).

Now we prove (20) and present a second proof of (23).

From (49), we have

$$\begin{aligned}
\frac{(q^3; q^3)_{\infty}^4}{(q; q)_{\infty}^{12}} &= \frac{(q^4; q^4)_{\infty}^{24} (q^6; q^6)_{\infty}^{12}}{(q^2; q^2)_{\infty}^{36} (q^{12}; q^{12})_{\infty}^8} + 12q \frac{(q^4; q^4)_{\infty}^{20} (q^6; q^6)_{\infty}^{10}}{(q^2; q^2)_{\infty}^{34} (q^{12}; q^{12})_{\infty}^4} \\
&+ 54q^2 \frac{(q^4; q^4)_{\infty}^{16} (q^6; q^6)_{\infty}^8}{(q^2; q^2)_{\infty}^{32}} + 108q^3 \frac{(q^4; q^4)_{\infty}^{12} (q^6; q^6)_{\infty}^6 (q^{12}; q^{12})_{\infty}^4}{(q^2; q^2)_{\infty}^{30}} \\
&+ 81q^4 \frac{(q^4; q^4)_{\infty}^8 (q^6; q^6)_{\infty}^4 (q^{12}; q^{12})_{\infty}^8}{(q^2; q^2)_{\infty}^{28}}. \tag{69}
\end{aligned}$$

Employing (69) in (14), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} PDO(6n)q^n &= \frac{(q^2; q^2)_{\infty}^{14}}{(q^4; q^4)_{\infty}^4 (q^6; q^6)_{\infty}^2} \left\{ \frac{(q^4; q^4)_{\infty}^{24} (q^6; q^6)_{\infty}^{12}}{(q^2; q^2)_{\infty}^{36} (q^{12}; q^{12})_{\infty}^8} \right. \\
&+ 12q \frac{(q^4; q^4)_{\infty}^{20} (q^6; q^6)_{\infty}^{10}}{(q^2; q^2)_{\infty}^{34} (q^{12}; q^{12})_{\infty}^4} + 54q^2 \frac{(q^4; q^4)_{\infty}^{16} (q^6; q^6)_{\infty}^8}{(q^2; q^2)_{\infty}^{32}} \\
&+ 108q^3 \frac{(q^4; q^4)_{\infty}^{12} (q^6; q^6)_{\infty}^6 (q^{12}; q^{12})_{\infty}^4}{(q^2; q^2)_{\infty}^{30}} \\
& \left. + 81q^4 \frac{(q^4; q^4)_{\infty}^8 (q^6; q^6)_{\infty}^4 (q^{12}; q^{12})_{\infty}^8}{(q^2; q^2)_{\infty}^{28}} \right\}. \tag{70}
\end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of (70), we readily arrive at (20) and (23) to finish the proof. \square

Proofs of (22) and (24). Squaring both sides of (50), we have

$$\begin{aligned}
\frac{(q^3; q^3)_\infty^4}{(q; q)_\infty^8} &= \frac{(q^4; q^4)_\infty^8 (q^6; q^6)_\infty^2 (q^8; q^8)_\infty^8 (q^{12}; q^{12})_\infty^4}{(q^2; q^2)_\infty^{20} (q^{16}; q^{16})_\infty^4 (q^{24}; q^{24})_\infty^2} \\
&+ 4q \frac{(q^4; q^4)_\infty^{10} (q^6; q^6)_\infty^2 (q^8; q^8)_\infty^2 (q^{12}; q^{12})_\infty^4}{(q^2; q^2)_\infty^{20} (q^{24}; q^{24})_\infty^2} \\
&+ 4q \frac{(q^4; q^4)_\infty^5 (q^6; q^6)_\infty^3 (q^8; q^8)_\infty^{10} (q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty^{19} (q^{16}; q^{16})_\infty^4} \\
&+ 16q^2 \frac{(q^4; q^4)_\infty^7 (q^6; q^6)_\infty^3 (q^8; q^8)_\infty^4 (q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty^{19}} \\
&+ 4q^2 \frac{(q^4; q^4)_\infty^{12} (q^6; q^6)_\infty^2 (q^{12}; q^{12})_\infty^4 (q^{16}; q^{16})_\infty^4}{(q^2; q^2)_\infty^{20} (q^8; q^8)_\infty^4 (q^{24}; q^{24})_\infty^2} \\
&+ 4q^2 \frac{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^4 (q^8; q^8)_\infty^{12} (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^{18} (q^{12}; q^{12})_\infty^2 (q^{16}; q^{16})_\infty^4} \\
&+ 16q^3 \frac{(q^4; q^4)_\infty^9 (q^6; q^6)_\infty^3 (q^{12}; q^{12})_\infty (q^{16}; q^{16})_\infty^4}{(q^2; q^2)_\infty^{19} (q^8; q^8)_\infty^2} \\
&+ 16q^3 \frac{(q^4; q^4)_\infty^4 (q^8; q^8)_\infty^6 (q^6; q^6)_\infty^4 (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^{18} (q^{12}; q^{12})_\infty^2} \\
&+ 16q^4 \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^4 (q^{16}; q^{16})_\infty^4 (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^{18} (q^{12}; q^{12})_\infty^2}. \tag{71}
\end{aligned}$$

Now, using (71) in (16), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} PDO(6n+3)q^n &= 4 \left\{ \frac{(q^4; q^4)_\infty^{12} (q^8; q^8)_\infty^8 (q^{12}; q^{12})_\infty^4}{(q^2; q^2)_\infty^{18} (q^{16}; q^{16})_\infty^4 (q^{24}; q^{24})_\infty^2} \right. \\
&+ 4q \frac{(q^4; q^4)_\infty^{14} (q^8; q^8)_\infty^2 (q^{12}; q^{12})_\infty^4}{(q^2; q^2)_\infty^{18} (q^{24}; q^{24})_\infty^2} \\
&+ 4q \frac{(q^4; q^4)_\infty^9 (q^6; q^6)_\infty (q^8; q^8)_\infty^{10} (q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty^{17} (q^{16}; q^{16})_\infty^4} \\
&+ 16q^2 \frac{(q^4; q^4)_\infty^{11} (q^6; q^6)_\infty (q^8; q^8)_\infty^4 (q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty^{17}} \\
&+ 4q^2 \frac{(q^4; q^4)_\infty^{16} (q^{12}; q^{12})_\infty^4 (q^{16}; q^{16})_\infty^4}{(q^2; q^2)_\infty^{18} (q^8; q^8)_\infty^4 (q^{24}; q^{24})_\infty^2} \\
&+ 4q^2 \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^2 (q^8; q^8)_\infty^{12} (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^{16} (q^{12}; q^{12})_\infty^2 (q^{16}; q^{16})_\infty^4} \\
&\left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + 16q^3 \frac{(q^4; q^4)_\infty^{13} (q^6; q^6)_\infty (q^{12}; q^{12})_\infty (q^{16}; q^{16})_\infty^4}{(q^2; q^2)_\infty^{17} (q^8; q^8)_\infty^2} \\
& + 16q^3 \frac{(q^4; q^4)_\infty^8 (q^8; q^8)_\infty^6 (q^6; q^6)_\infty^2 (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^{16} (q^{12}; q^{12})_\infty^2} \\
& + 16q^4 \frac{(q^4; q^4)_\infty^{10} (q^6; q^6)_\infty^2 (q^{16}; q^{16})_\infty^4 (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^{16} (q^{12}; q^{12})_\infty^2} \}. \tag{72}
\end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above, we easily deduce (22) and (24), respectively. \square

Proofs of (29)–(32). By the binomial theorem, it is easy to deduce that

$$(q; q)_\infty^2 \equiv (q^2; q^2)_\infty \pmod{2} \tag{73}$$

and

$$(q; q)_\infty^4 \equiv (q^2; q^2)_\infty^2 \pmod{4}. \tag{74}$$

Employing (73) in (12), (13), and (19), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{PDO(4n+2)}{2} q^n &\equiv (q^2; q^2)_\infty^3 (q^6; q^6)_\infty \pmod{2}, \\
\sum_{n=0}^{\infty} \frac{PDO(4n+3)}{4} q^n &\equiv (q^2; q^2)_\infty^6 (q^6; q^6)_\infty \pmod{2},
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} \frac{PDO(9n+6)}{12} q^n \equiv \frac{(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty} \pmod{2},$$

respectively. Now (29)–(31) are apparent from the above.

Now, from (67), we find that

$$\sum_{n=0}^{\infty} PDO(3n) q^n \equiv \frac{\varphi^2(-q^6) \varphi^6(-q^9)}{\varphi^8(-q^3)} \pmod{4}.$$

Thus,

$$\sum_{n=0}^{\infty} PDO(9n) q^n \equiv \frac{\varphi^2(-q^2) \varphi^6(-q^3)}{\varphi^8(-q)} \pmod{4},$$

which can be rewritten with the help of (56) as

$$\sum_{n=0}^{\infty} PDO(9n)q^n \equiv \frac{(q^2; q^2)_{\infty}^{12} (q^3; q^3)_{\infty}^{12}}{(q; q)_{\infty}^{16} (q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty}^6} \pmod{4}.$$

Employing (74) in the above, we obtain

$$\sum_{n=0}^{\infty} PDO(9n)q^n \equiv 1 \pmod{4},$$

from which (32) follows readily. \square

Proofs of (33)–(35). From (72), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} PDO(6n+3)q^n &\equiv 4 \frac{(q^4; q^4)_{\infty}^{12} (q^8; q^8)_{\infty}^8 (q^{12}; q^{12})_{\infty}^4}{(q^2; q^2)_{\infty}^{18} (q^{16}; q^{16})_{\infty}^4 (q^{24}; q^{24})_{\infty}^2} \pmod{16}. \\ &\equiv 4 (q^4; q^4)_{\infty}^3 \pmod{16}, \end{aligned} \tag{75}$$

which immediately yields (33)–(35). \square

4. Concluding Remarks

1. It is not known whether the congruences in Corollary 1.2, Corollary 1.4, or Theorem 1.5 are part of a family of congruences modulo prime powers or not. It would also be interesting to find proofs of the congruences in this paper by appealing to a rank/crank-type statistic.

2. Identities (1) and (2) imply that

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \left(\sum_{n=0}^{\infty} PDO(n)q^n \right)^2.$$

It would be interesting to find a combinatorial proof of this identity.

Acknowledgment. The authors are extremely grateful to the anonymous referee who read our manuscript with great care, uncovered several errors and offered useful suggestions.

References

- [1] G. E. Andrews, R. P. Lewis, J. Lovejoy, *Partitions with designated summands*, Acta Arith. 105 (2002), 51–66.
- [2] N. D. Baruah, K. K. Ojah, *Some congruences deducible from Ramanujan's cubic continued fraction*, Int. J. Number Theory 7 (2011), 1331–1343.
- [3] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.
- [4] B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer, 1998.
- [5] J. M. Borwein, P. B. Borwein, F. G. Garvan, *Some cubic modular identities of Ramanujan*, Trans. Amer. Math. Soc. 343 (1994), 35–47.
- [6] S. -L. Chen, S. -S. Huang, *Identities for certain products of theta functions*, Ramanujan J. 8 (2004), 5–12.
- [7] W. Y. C. Chen, K. Q. Ji, H. -T. Jin, E. Y. Y. Shen, *On the number of partitions with designated summands*, J. Number Theory 133 (2013), 2929–2938.