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ON THE LEAST POSITIVE SOLUTION TO A PROPORTIONALLY MODULAR DIOPHANTINE INEQUALITY

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Abstract

Given three positive integers a, b, c, a proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \leq cx$. Our aim is to give a recursive formula for the least solution to such an inequality. We then use the formula to derive an algorithm. Finally, we apply our results to a question of Rosales and García-Sánchez.

1. Introduction

A *proportionally modular Diophantine inequality* is an expression of the form

$$(ax \bmod b) \leq cx,$$

where the positive integers a, b, c are called respectively the *factor*, *modulus* and *proportion*. It is well-known that the set of the non-negative integer solutions of this inequality is a *numerical semigroup* (cf. [8], [9]), i.e. a submonoid S of $(\mathbb{N}, +)$ with finite complement in it. Denoting by $\mathrm{S}(a, b, c)$ the set of solutions, the structure of this set (called a *proportionally modular semigroup*) has been widely studied, but is not completely understood yet. In particular, it is an open problem (cf. [8]) to find explicit formulas for several classical invariants of these numerical semigroups. Several works in literature focused on the *multiplicity* of these numerical semigroup, which is the smallest positive solution of the inequality $(ax \bmod b) \leq cx$. Although some partial results are known (cf. [9], [11], [12]) as of today the main problem of finding a formula for this invariant still remains unsolved. Notably, this particular invariant pops up in other problems: it has been proved (cf. [8]) that each proportionally modular numerical semigroups is exactly the set of numerators of fractions belonging to a certain bounded rational interval. Thus, another formulation for this problem asks for the least possible numerator of a rational number in a given interval, or , equivalently, for the least possible denominator of such rational numbers.

This formulation also highlights a connection with continued fractions and Farey sequences (cf. [2], [6]). Moreover, Bullejos and Rosales showed that this problem is strictly related to that of finding the common ancestor of two rational numbers in the Stern-Brocot tree (cf. [4]). These equivalences lead to different approaches and formulas, based on the context in which the problem is studied. Using elementary number theory we will provide a recursive formula for the smallest positive solution of the inequality $(ax \bmod b) \leq cx \;\; a, b \in \mathbb{Z}^+, \; c \in \mathbb{Q}^+$, and thus an algorithm for its computation (with similar complexity to the Euclidean algorithm).

Our work is structured as follows: in the first section we prove our main theorem, and provide the recursive formula for the computation of the multiplicity of S. In Section 2 we describe the algorithm that can be derived from our main theorem. In the final section we explain how our result can be applied to a question of Rosales and García-Sánchez ([8, Problem 5.20]).

2. Main Result

Given two integers m and n with $n > 0$ we define the *remainder operator* $[m]_n$ as follows

$$[m]_n = \min\{i \in \mathbb{N} \;\mid i \equiv m \pmod{n}\}.$$

Notice that, if m and n are positive integers such that $m < n$, then $m = [m]_n$. The following properties follow from the definition of floor and ceiling function, and we will use them extensively.

Proposition 1. *Let $a, b \in \mathbb{Z}^+$. Then:*

1. $\left\lfloor \dfrac{b}{a} \right\rfloor a + [b]_a = b,$

2. $\left\lceil \dfrac{b}{a} \right\rceil a - [-b]_a = b.$

Let $a, b \in \mathbb{Z}^+$, and let $c \in \mathbb{Q}^+$. Consider the inequality $(ax \bmod b) = [ax]_b \leq cx$, and define

$$\mathrm{L}(a, b, c) = \min\{x \in \mathbb{Z}^+ \mid [ax]_b \leq cx\} = \min\{\mathrm{S}(a, b, c) \setminus \{0\}\}.$$

Clearly, if $a \geq b$, then $[ax]_b = [[a]_b x]_b$, and hence $\mathrm{L}(a, b, c) = \mathrm{L}([a]_b, b, c)$, so the condition $a < b$ that we will impose in the next results is not restrictive. Moreover, if $d = \gcd(a, b)$ and $a = da'$ and $b = db'$, we have $[a]_b = d[a']_{b'}$; therefore $[ax]_b \leq cx$ if and only if $[a'x]_{b'} \leq \dfrac{c}{d}x$, which implies $\mathrm{S}(a, b, c) = \mathrm{S}(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$. Conversely, if d is a positive integer, then $\mathrm{S}(a, b, c) = \mathrm{S}(ad, bd, cd)$. Furthermore, if $c = \frac{m}{n}$ is a positive rational number, then $\mathrm{S}(a, b, c) = \mathrm{S}(an, bn, cn)$ is a proportionally modular numerical semigroup: thus the set of numerical semigroups $\mathrm{S}(a, b, c)$, with c a positive rational number, equals the set of proportionally modular numerical semigroups.

Proposition 2. *Let $a, b \in \mathbb{Z}^+$ be such that $a < b$, and let $c \in \mathbb{Q}^+$ be a positive rational number. Then:*

1. *If $c \geq a$, then $\mathrm{L}(a, b, c) = 1$,*
2. *If $c < a$ and $a \mid b$, then $\mathrm{L}(a, b, c) = \frac{b}{a}$.*

Proof. The first part is obvious. If $x < \frac{b}{a}$, then $ax < b$ and $[ax]_b = ax > cx$; hence the inequality is false for $x < \frac{b}{a}$. Since for $x = \frac{b}{a}$ we have $ax = b$ and $[ax]_b = 0 \leq cx$, we conclude that $\mathrm{L}(a, b, c) = \frac{b}{a}$. □

With these premises we can reduce our problem to the case $c < a < b$, $a \nmid b$.

Proposition 3. *Let $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{Q}^+$ be such that $c < a < b$ and $a \nmid b$. Then there exists $\mu \in \mathbb{Z}^+$ such that*

$$\mathrm{L}(a, b, c) = \left\lceil \frac{\mu b}{a} \right\rceil.$$

Proof. If $x < \left\lceil \frac{b}{a} \right\rceil$ is a positive integer, then $ax < b$ and $[ax]_b = ax > cx$, so $\mathrm{L}(a, b, c) \geq \left\lceil \frac{b}{a} \right\rceil$. From this bound it follows that there exists $\mu \in \mathbb{Z}^+$ such that

$$\left\lceil \frac{\mu b}{a} \right\rceil \leq \mathrm{L}(a, b, c) < \left\lceil \frac{(\mu + 1)b}{a} \right\rceil.$$

Suppose now that $\mathrm{L}(a, b, c) \neq \left\lceil \frac{\mu b}{a} \right\rceil$; this is equivalent to saying that there exists $r \in \mathbb{N}$, $r \neq 0$ such that

$$\mathrm{L}(a, b, c) = \left\lceil \frac{\mu b}{a} \right\rceil + r \quad \text{where } r < \left\lceil \frac{(\mu + 1)b}{a} \right\rceil - \left\lceil \frac{\mu b}{a} \right\rceil.$$

Therefore $a\mathrm{L}(a, b, c) = a \left\lceil \frac{\mu b}{a} \right\rceil + ar \leq a \left\lceil \frac{(\mu+1)b}{a} \right\rceil - a$, and by Proposition 1

$$a \left\lceil \frac{(\mu + 1)b}{a} \right\rceil = (\mu + 1)b + [-(\mu + 1)b]_a.$$

Hence, if $r \neq 0$ we have

$$\mu b \leq a \left\lceil \frac{\mu b}{a} \right\rceil < a\mathrm{L}(a, b, c) = (\mu + 1)b + [-(\mu + 1)b]_a - a < (\mu + 1)b.$$

By definition of remainder, we have $\mu b < a\mathrm{L}(a, b, c) < (\mu + 1)b$, implying

$$b > [a\mathrm{L}(a, b, c)]_b = a\mathrm{L}(a, b, c) - \mu b = \left(a \left\lceil \frac{\mu b}{a} \right\rceil - \mu b \right) + ar \geq ar \geq a.$$

Thus $b > [a\mathrm{L}(a,b,c)]_b \geq a$, and we obtain that $[a\mathrm{L}(a,b,c)]_b - a = [a\mathrm{L}(a,b,c) - a]_b$. Now consider $x = \mathrm{L}(a,b,c) - 1$. We get

$$[ax]_b = [a(\mathrm{L}(a,b,c) - 1)]_b = [a\mathrm{L}(a,b,c) - a]_b = [a\mathrm{L}(a,b,c)]_b - a$$

and $cx = c\mathrm{L}(a,b,c) - c$. Hence we have

$$[ax]_b = [a\mathrm{L}(a,b,c)]_b - a < [a\mathrm{L}(a,b,c)]_b - c \leq c\mathrm{L}(a,b,c) - c = cx,$$

leading to $x = \mathrm{L}(a,b,c) - 1 \in \mathrm{S}(a,b,c)$, which is a contradiction. $\square$

Note that by definition it is clear that $\mathrm{L}(a,b,c) \leq b$, and hence $1 \leq \mu \leq a$. Define, for every $\mu = 1, \ldots, a$, R_μ as the unique positive integer satisfying

$$\frac{(R_\mu - 1)a}{[b]_a} < \mu \leq \frac{R_\mu a}{[b]_a}.$$

Lemma 4. *Let $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{Q}^+$ be such that $c < a < b$ and $a \nmid b$. Let $\mu \in \mathbb{Z}^+$. Then we have:*

1. $\left\lceil \frac{\mu b}{a} \right\rceil = \mu \left\lfloor \frac{b}{a} \right\rfloor + R_\mu$,

2. $[a \left\lceil \frac{\mu b}{a} \right\rceil]_b = R_\mu a - \mu [b]_a$.

Proof.

1. By using Proposition 1 we have that $b = \left\lfloor \frac{b}{a} \right\rfloor a + [b]_a$, and then

$$\left\lceil \frac{\mu b}{a} \right\rceil = \left\lceil \frac{\mu \left(\left\lfloor \frac{b}{a} \right\rfloor a + [b]_a \right)}{a} \right\rceil = \left\lceil \mu \left\lfloor \frac{b}{a} \right\rfloor + \frac{\mu [b]_a}{a} \right\rceil.$$

Since $\mu \left\lfloor \frac{b}{a} \right\rfloor \in \mathbb{Z}^+$, we can deduce easily from the definition of R_μ that $R_\mu = \left\lceil \frac{\mu [b]_a}{a} \right\rceil$. Then it follows that:

$$\left\lceil \frac{\mu b}{a} \right\rceil = \left\lceil \mu \left\lfloor \frac{b}{a} \right\rfloor + \frac{\mu [b]_a}{a} \right\rceil = \mu \left\lfloor \frac{b}{a} \right\rfloor + R_\mu.$$

2. From the definition of R_μ we know that $\frac{(R_\mu - 1)a}{[b]_a} < \mu$. This implies $R_\mu a - \mu [b]_a < a < b$, and consequently $\left[R_\mu a - \mu [b]_a\right]_b = R_\mu a - \mu [b]_a$, which is our thesis. $\square$

In order to find a recursion, we will prove that R_μ itself is the smallest solution of another proportionally modular Diophantine inequality with smaller values of factor, modulus and proportion, and then we will compute μ from R_μ.

Theorem 5. *Let $a, b \in \mathbb{Z}^+$, $c \in \mathbb{Q}^+$ be such that $c < a < b$ and $a \nmid b$. Let $\mu \in \mathbb{Z}^+$ be such that $\mathrm{L}(a,b,c) = \left\lceil \frac{\mu b}{a} \right\rceil$. Then*

$$R_\mu = \mathrm{L}\left([a]_{[b]_a}, [b]_a, \frac{cb}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}\right), \qquad \mu = \left\lceil \frac{R_\mu(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right\rceil.$$

Proof. Using Lemma 4 we have that $c\mathrm{L}(a,b,c) = c\mu\left\lfloor \frac{b}{a} \right\rfloor + R_\mu c$ and $[a\mathrm{L}(a,b,c)]_b = R_\mu a - \mu[b]_a$. Then, from $c\mathrm{L}(a,b,c) \geq [a\mathrm{L}(a,b,c)]_b$ it follows that $c\mathrm{L}(a,b,c) \geq [a\mathrm{L}(a,b,c)]_b$, which leads, by substitution, to

$$c\mu\left\lfloor \frac{b}{a} \right\rfloor + R_\mu c \geq R_\mu a - \mu[b]_a.$$

Solving the inequality in μ we have

$$\mu \geq \frac{R_\mu(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}.$$

However, by definition of R_μ, we also have $\mu \leq \frac{R_\mu a}{[b]_a}$. Therefore, we proved that

$$\frac{R_\mu(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \leq \mu \leq \frac{R_\mu a}{[b]_a}. \tag{1}$$

Then, the interval $\left[\frac{R_\mu(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}, \frac{R_\mu a}{[b]_a}\right]$ contains at least one integer. Let N be the smallest positive integer such that

$$\left[\frac{N(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}, \frac{Na}{[b]_a}\right] \cap \mathbb{Z} \neq \emptyset,$$

let $\sigma < \mu$ be the smallest integer in this interval, and assume that $N < R_\mu$. From the definition of R_σ, $\sigma \leq \frac{Na}{[b]_a}$ implies that $R_\sigma \leq N$ and $\frac{R_\sigma(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \leq \frac{N(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}$. However, the last inequality affirms that σ is actually contained in the interval $\left[\frac{R_\sigma(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}, \frac{R_\sigma a}{[b]_a}\right]$; hence $R_\sigma = N$.

By Lemma 4, we have

$$\left\lceil \frac{\sigma b}{a} \right\rceil = \sigma\left\lfloor \frac{b}{a} \right\rfloor + R_\sigma\ , \ \left[a\left\lceil \frac{\sigma b}{a} \right\rceil\right]_b = R_\sigma a - \sigma[b]_a.$$

Moreover $\frac{R_\sigma(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \leq \frac{N(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \leq \sigma$, and hence $R_\sigma a - \sigma[b]_a \leq c\left(\sigma\left\lfloor \frac{b}{a} \right\rfloor + R_\sigma\right)$. Thus we obtain the inequality

$$\left[a\left\lceil \frac{\sigma b}{a} \right\rceil\right]_b \leq c\left\lceil \frac{\sigma b}{a} \right\rceil,$$

which implies that $\left\lceil \frac{\sigma b}{a} \right\rceil \in \mathrm{S}(a,b,c)$. However, since $\sigma < \mu$ and $a < b$, we have $\left\lceil \frac{\sigma b}{a} \right\rceil < \left\lceil \frac{\mu b}{a} \right\rceil = \mathrm{L}(a,b,c)$, which is a contradiction. Therefore, we deduce that

$$R_\mu = \min\left\{ z \in \mathbb{Z}^+ \mid \left[\frac{z(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}, \frac{za}{[b]_a} \right] \cap \mathbb{N} \neq \emptyset \right\}. \tag{2}$$

From the definition of R_μ, we further deduce that

$$\mu = \min\left\{ \left[\frac{R_\mu(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}, \frac{R_\mu a}{[b]_a} \right] \cap \mathbb{N} \right\} = \left\lceil \frac{R_\mu(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right\rceil, \tag{3}$$

which proves the second part of our thesis. In order to prove the first part, by simple calculations we see that

$$\left[\frac{z(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}, \frac{za}{[b]_a} \right] \cap \mathbb{N} \neq \emptyset \quad \text{if and only if} \quad \left\lfloor \frac{za}{[b]_a} \right\rfloor \geq \frac{z(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a}.$$

By recalling Proposition 1, we get the two identities $\left\lfloor \frac{za}{[b]_a} \right\rfloor = \frac{za - [za]_{[b]_a}}{[b]_a}$ and $\left\lfloor \frac{b}{a} \right\rfloor = \frac{b - [b]_a}{a}$. Plugging these equations in our last inequality we obtain that

$$\frac{za - [za]_{[b]_a}}{[b]_a} \geq z\frac{a-c}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \quad \text{if and only if} \quad z\left(\frac{cb}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right) \geq [za]_{[b]_a}.$$

Finally, plugging this condition in Eq. (2), we obtain

$$R_\mu = \min\left\{ z \in \mathbb{Z}^+ \mid z\left(\frac{cb}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right) \geq [za]_{[b]_a} \right\} = \mathrm{L}\left([a]_{[b]_a}, [b]_a, \frac{cb}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right),$$

which proves our thesis. □

Combining Proposition 3 and Theorem 5, we obtain the promised recursive formula for $\mathrm{L}(a,b,c)$.

Corollary 6. *Let $a, b \in \mathbb{Z}^+$, $c \in \mathbb{Q}^+$ be such that $c < a < b$ and $a \nmid b$. Then*

$$\mathrm{L}(a,b,c) = \left\lceil \left\lceil \frac{L_1(a-c)}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right\rceil \frac{b}{a} \right\rceil, \qquad \textit{where} \quad L_1 = \mathrm{L}\left([a]_{[b]_a}, [b]_a, \frac{cb}{c\left\lfloor \frac{b}{a} \right\rfloor + [b]_a} \right).$$

3. The Algorithm

The main result of the previous section gives rise to the following algorithm, which computes $\mathrm{L}(a,b,c)$ for any given triple (a,b,c) such that $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{Q}^+$.

Algorithm 1 Algorithm for L(a, b, c)

```
1: if c ≥ a then return 1;
2: if a|b then return b/a;
3: den := c*Floor(b/a)+(b mod a);
4: L1 := L(a mod (b mod a),b mod a,c*b/den);
5: return Ceiling(b/a*Ceiling(L1*(a-c)/den));
```

Proposition 7. *Algorithm 1 stops after a finite number of steps.*

Proof. Consider the three sequences of integers a_i, b_i and c_i defined recursively as

$$a_i = \begin{cases} a_0 = a \\ a_i = [a_{i-1}]_{b_i} & \text{if } i > 0, \end{cases}$$

$$b_i = \begin{cases} b_0 = b \\ b_i = [b_{i-1}]_{a_{i-1}} & \text{if } i > 0, \end{cases}$$

$$c_i = \begin{cases} c_0 = c \\ c_i = \dfrac{c_{i-1}b_{i-1}}{c_{i-1}\left\lfloor \frac{b_{i-1}}{a_{i-1}} \right\rfloor + [b_{i-1}]_{a_{i-1}}} & \text{if } i > 0. \end{cases}$$

It is obvious that $a_{i+1} < a_i$ if $a_i \geq 2$ and that $c_i \geq 1$ for any $i \geq 1$. Therefore, after a finite number of steps we will have $a_i \leq 1$ and $c_i \geq a_i$, thus meeting the condition for termination. □

4. Applications

The given algorithm has an application in the context of numerical semigroups. Given two coprime integers a_1 and a_2, consider the numerical semigroup

$$S = \langle a_1, a_2 \rangle = \{\lambda_1 a_1 + \lambda_2 a_2 \mid \lambda_1, \lambda_2 \in \mathbb{N}\}.$$

We define the *quotient* of a numerical semigroup S by a positive integer d as follows:

$$\frac{S}{d} := \{x \in \mathbb{N} \mid xd \in S\}.$$

The quotient $\frac{S}{d}$ is a numerical semigroup, but it does not have necessarily the same structure as S; actually, little is known about the existence of a relation between the invariants of S and $\frac{S}{d}$. In particular, given three positive integers a_1, a_2, d, it is an open problem (cf. [8, Problem 5.20]) to find a formula for the smallest multiple of d that belongs to $\langle a_1, a_2 \rangle$ and for the largest multiple of d that does not belong to $\langle a_1, a_2\rangle$; these problems ask for invariants of the quotient semigroup $\frac{\langle a_1,a_2\rangle}{d}$. Moreover, this class of quotients of numerical semigroups is tightly related to the Diophantine inequalities we have studied, as it has been proved that a numerical semigroup is proportionally modular if and only if it is the quotient of an embedding dimension two numerical semigroup. In particular, the numerical semigroup $\langle a_1, a_2\rangle$ is proportionally modular, and the next result provides its related proportionally modular Diophantine inequality.

Lemma 8 ([12, Lemma 18]). *Let a_1, a_2 be relatively prime positive integers and let u be a positive integer such that $ua_2 \equiv 1 \pmod{a_1}$. Then*

$$\langle a_1, a_2\rangle = \{x \in \mathbb{N} \mid [ua_2x]_{(a_1a_2)} \leq x\}.$$

This lemma directly implies that

$$\langle a_1, a_2\rangle = \left\{x \in \mathbb{N} \mid [ux]_{a_1} \leq \frac{x}{a_2}\right\}.$$

Consider now the quotient

$$\frac{\langle a_1, a_2\rangle}{d} = \{x \in \mathbb{N} \mid xd \in \langle a_1, a_2\rangle\} = \left\{x \in \mathbb{N} \mid [uxd]_{a_1} \leq \frac{xd}{a_2}\right\}.$$

Its multiplicity is

$$\mathrm{m}\left(\frac{\langle a_1, a_2\rangle}{d}\right) = \min\left\{x \in \mathbb{N} \mid [uxd]_{a_1} \leq \frac{xd}{a_2}\right\} = \mathrm{L}\left([ud]_{a_1}, a_1, \frac{d}{a_2}\right),$$

and therefore it can be obtained by applying Algorithm 1.

The second application regards the set $\mathrm{S}(a, b, c)$ itself. Since this set is a numerical semigroup, it has finite complement in $\mathbb{N}$; the greatest integer not belonging to $\mathrm{S}(a, b, c)$ is called the *Frobenius number* of $\mathrm{S}(a, b, c)$, which we will denote here with $F(a, b, c)$. In [13] the authors give a relation between $F(a, b, 1)$ and the multiplicity of a particular proportionally modular numerical semigroup. Given $p, q \in \mathbb{Q}^+$ such that $p < q$, denote by $[p, q]$ and $\langle [p, q]\rangle$ the sets

$$[p, q] = \{x \in \mathbb{Q} \mid p \leq x \leq q\} \quad \text{and}$$

$$\langle [p, q]\rangle = \{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N},\ \ a_1, \ldots, a_n \in [p, q],\ \ n \in \mathbb{N}\setminus\{0\}\},$$

respectively. It is known that, for any $p, q \in \mathbb{Q}^+$ such that $p < q$, the set $\mathrm{S}([p, q]) = \langle [p, q]\rangle \cap \mathbb{N}$ is a proportionally modular numerical semigroup, as the next proposition shows:

Proposition 9 ([13, Proposition 1]). *Let $a_1, b_1, a_2, b_2 \in \mathbb{Z}^+$ be such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$. Then $\mathrm{S}([\frac{b_1}{a_1}, \frac{b_2}{a_2}]) = \mathrm{S}(a_1b_2, b_1b_2, a_1b_2 - a_2b_1)$.*

A direct consequence of Proposition 9 is that $\mathrm{m}(\mathrm{S}([\frac{b_1}{a_1}, \frac{b_2}{a_2}])) = \mathrm{L}(a_1b_2, b_1b_2, a_1b_2 - a_2b_1)$. Furthermore, we can divide each term by b_2, obtaining

$$\mathrm{m}\left(\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right)\right) = \mathrm{L}\left(a_1, b_1, \frac{a_1b_2 - a_2b_1}{b_1}\right). \tag{4}$$

Theorem 10 ([13, Theorem 18]). *Let $a, b \in \mathbb{Z}^+$ be such that $2 \leq a < b$ and $S = \mathrm{S}([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$. Then $F(a, b, 1) = b - \mathrm{m}(S)$.*

By Theorem 10 and Eq. (4) we have

$$F(a, b, 1) = b - \mathrm{m}(S) = b - \mathrm{L}\left(2b, 2b^2 + 1, \frac{4b^3 - 4ab + 2b}{2b^2 + 1}\right),$$

and hence we can apply Algorithm 1.

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