



CRESCENT CONFIGURATIONS

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hs002012@mymail.pomona.edu*Received: 10/3/15, Revised: 1/19/16, Accepted: 5/25/16, Published: 6/10/16***Abstract**

In 1989, Erdős conjectured that for a sufficiently large n it is impossible to place n points in general position in a plane such that for every $1 \leq i \leq n - 1$ there is a distance that occurs exactly i times. For small n this is possible and in his paper he provided constructions for $n \leq 8$. The one for $n = 5$ was due to Pomerance while Palásti came up with the constructions for $n = 7, 8$. Constructions for $n = 9$ and above remain undiscovered, and little headway has been made toward a proof that for sufficiently large n no configuration exists. In this paper we consider a natural generalization to higher dimensions and provide a construction which shows that for any given n there exists a sufficiently large dimension d such that there is a configuration in d -dimensional space meeting Erdős' criteria.

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1. Introduction

In 1946, Erdős [4] initiated the first of many problems about distinct distances: what is the minimum number of distinct distances determined by n points in the plane? If we were to randomly place n points in the plane, we would expect all distances between pairs of points to be different. However, as more structure is introduced in the placement of points, some distances may repeat. Erdős conjectured that the minimum number of distances is $\Omega(n/\sqrt{\log n})$, which is attained by the $\sqrt{n} \times \sqrt{n}$ integer lattice. The lower bound has been incrementally improved from Erdős's original $\Omega(n^{1/2})$ to Larry Guth and Nets Katz's $\Omega(n/\log n)$ announced in 2010, which solves the distinct distances problem up to a $\sqrt{\log n}$ factor [6].

Since the introduction of the distinct distances problem, many variants have been tackled. A survey of distance-related problems may be found in [2, 9]. The variant we study in this paper is that of trying to specify the multiplicity with which each distance occurs. In particular we want each distance to appear a different number of times, with each multiplicity 1 through $n - 1$ represented. This condition affects all possible distances among the n points since there are $\binom{n}{2}$ pairs of points and our condition affects $1 + 2 + \dots + (n - 1) = n(n - 1)/2$ distances. On a line this problem is trivial, since such a configuration can be achieved by simply placing all n points in an arithmetic progression. In the plane this problem already becomes interesting, provided we prevent too many points from sitting on a single line. Analogously in higher dimensions we want to exclude lower dimensional constructions which we achieve by insisting that the points lie in *general position*.

Definition 1 (General Position). We say that n points are in *general position* in \mathbb{R}^d if no $d + 1$ points lie on the same hyperplane and no $d + 2$ lie on the same hypersphere.

The multiplicities of the distances are in an increasing order so we introduce the name *crescent configuration* to describe the point configurations we seek.

Definition 2 (Crescent Configuration). We say n points are in *crescent configuration* (in \mathbb{R}^d) if they lie in general position in \mathbb{R}^d and determine $n - 1$ distinct distances, such that for every $1 \leq i \leq n - 1$ there is a distance that occurs exactly i times.

A one-dimensional variant of this problem of finding crescent configurations, known as the “beltway” problem, in which all of the points lie on a wrapped interval, has music theoretic connections. In particular, “deep scales” including commonly used diatonic scales include each interval class a unique number of times [3, 10].

In 1989 Erdős [5] conjectured that for n sufficiently large, it is not possible to place n points in a crescent configuration in the plane. Little progress has been made since on this conjecture. Palásti [7, 8] provided constructions for $n = 7$

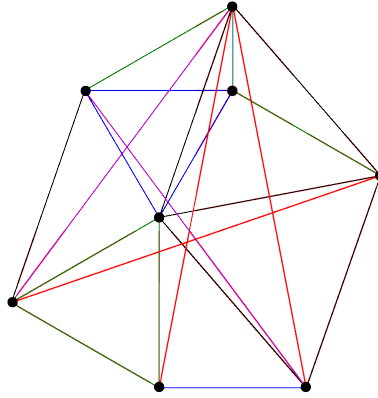


Figure 1: Palasti came up with a crescent configuration for $n = 8$ [8]. Points have cartesian coordinates $(0, 1)$, $(\sqrt{3}, 0)$, $(2\sqrt{3}, 0)$, $(\frac{5\sqrt{3}}{2}, \frac{5}{2})$, $(\frac{3\sqrt{3}}{2}, \frac{9}{2})$, $(\frac{\sqrt{3}}{2}, \frac{7}{2})$, $(\frac{3\sqrt{3}}{2}, \frac{7}{2})$, $(\sqrt{3}, 2)$.

and $n = 8$, both lying in a small portion of the triangular lattice (see Figure 1), but constructions for $n = 9$ and higher remain undiscovered, with no compelling heuristics to suggest a sufficiently large n such that no configuration exists.

We explore a higher dimensional analogue of this problem. Our main result is the following.

Theorem 1. *For all $n \geq 3$, there exists a set of n points in a crescent configuration in \mathbb{R}^{n-2} .*

In particular, this shows that given an n there exists a d sufficiently large such that it is possible to place n points in a crescent configuration in \mathbb{R}^d .

We give a construction proving Theorem 1 in §2, and then conclude with some problems for future work in §3.

2. Construction of a Crescent Configuration

We first show by induction that $n - 1$ points can be arranged in a valid crescent configuration in \mathbb{R}^{n-2} . The base case is clear as an isosceles triangle which is not an equilateral triangle has three points in valid crescent configuration in \mathbb{R}^2 .

For the inductive step, assume we have a set P of $n - 2$ points forming a crescent configuration in \mathbb{R}^{n-3} . Any $n - 2$ points lie on an $(n - 4)$ -dimensional sphere; we choose a sphere S that contains P . We embed P and S naturally in a hyperplane H in \mathbb{R}^{n-2} . There exists a line in \mathbb{R}^{n-2} containing the center of S and perpendicular to

H . We can then place the $(n-1)^{\text{st}}$ point p at any distance on this line not previously determined. This new point is equidistant from every point in P , so it determines a new distance $n-2$ times. Since P is a valid $(n-2)$ -point crescent configuration by the inductive hypothesis, $P \cup \{p\}$ is an $(n-1)$ -point crescent configuration in \mathbb{R}^{n-2} , which concludes the inductive argument.

We complete the proof by adding an additional point. This point is added by including the center of the $(n-3)$ -dimensional sphere T defined by $P \cup \{p\}$. To prevent the addition of the center of T (call it t) from repeating a distance already determined by $P \cup \{p\}$, we control the placement of the point p . As p is placed further and further away from H on the prescribed line, the radius of T increases unboundedly. Place p so far away from H that the radius of T is larger than any distance determined by P . Then add t as the n^{th} point. The new point t is equidistant from all the others without repeating any previously determined distance, adding a distance of multiplicity $n-1$. The resulting configuration satisfies general position: there is no $(n-3)$ -dimensional sphere which contains n points since t is the center of an $(n-3)$ -dimensional sphere which contains the other $n-1$ points. Since p is placed sufficiently far from H , both p and t are not contained in H . Thus no hyperplane contains $n-1$ points. This is a valid n -point crescent configuration in \mathbb{R}^{n-2} and completes the proof. \square

3. Future Work

It seems likely that the actual dimension needed to place points in a valid configuration is significantly lower. There are a number of other related problems of interest. Define $\mathcal{D}(n)$ to be the minimum dimension, greater than 1, such that n points can be placed in $\mathbb{R}^{\mathcal{D}(n)}$ in a crescent configuration. Our construction shows that $\mathcal{D}(n) \leq n-2$ for all $n > 3$.

- Albuje [1] asked: is $\mathcal{D}(n)$ bounded as n goes to infinity?
- Is $\mathcal{D}(n)$ sublinear?
- Is $\mathcal{D}(n)$ monotonically increasing?
- If $\mathcal{D}(n) = k$ is it possible to place n points in a valid configuration in \mathbb{R}^d for all $d > k$? Note that in general we cannot just embed the solution into higher dimensions and maintain general position. For example, if you take the 8 point construction from \mathbb{R}^2 by Palásti and embed it in the most natural way into \mathbb{R}^3 then immediately you have more than 4 points in a plane which violates the condition of general position.
- Do any planar constructions for $n \geq 9$ exist? This has been asked before, see for example [1, 2].

- Can planar constructions for $n \geq 9$ be found on the triangular lattice? It is known that constructions for $n < 9$ exist on the triangular lattice.

Remark 1. With the help of a parallel computing cluster, we have exhaustively searched a 91 point hexagonal region of the triangular lattice for a construction for $n = 9$, but none exist. As the naive implementation took over 900 hours of computation for this size, better (and achievable) techniques are required to search a substantively larger region.

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References

- [1] Alma L. Albuje, Mınimo numero de distancias distintas entre un Numero finito de Puntos, *Tesis de licenciatura*, Departamento de Anlisis Matemtico, Universidad de Alicante, 2004.
- [2] H. Croft, K. Falconer and R. Guy, *Unsolved Problems in Geometry: Unsolved Problems in Intuitive Mathematics*, Springer-Verlag New York, 1991.
- [3] E. Demaine, F. Gomez-Martin, H. Meijer, D. Rappaport, P. Taslakian, G. Toussaint, T. Winograd and D. Wood, The distance geometry of music, *Comput. Geom.* **42** (2009), 429-454.
- [4] P. Erdos, On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248-250.
- [5] P. Erdos, Distances with specified multiplicities, *Amer. Math. Monthly* **96** (1989), 447.
- [6] L. Guth and N. H. Katz, On the Erdos distinct distances problem in the plane, *Ann. of Math.* **181** (2015), 155-190.
- [7] I. Palasti, On the seven points problem of P. Erdos, *Stud. Sci. Math. Hungar.* **22** (1987), 447-448.
- [8] I. Palasti, Lattice point examples for a question of Erdos, *Period. Math. Hungar.* **20** (1989), 231-235.
- [9] A. Sheffer, Distinct Distances: Open Problems and Current Bounds, *arXiv:1406.1949* (2015).
- [10] G. Toussaint, The geometry of musical rhythm, *Proceedings of the Japan Conference on Discrete and Computational Geometry*, (JCDCG 2004), LNCS 3742, Springer-Verlag, Berlin-Heidelberg (2005), 198-212.