r-COMPLETENESS OF SEQUENCES OF POSITIVE INTEGERS

William Linz[1]
Department of Mathematics, Texas A&M University, College Station, Texas
willdomath@tamu.edu

Edna Jones[2]
Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, Indiana
jonesel@rose-hulman.edu

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Abstract

A complete sequence (a_n) is a strictly increasing sequence of positive integers such that every sufficiently large positive integer is representable as the sum of one or more distinct terms from (a_n). In this paper, we consider the more general notion of an r-complete sequence where every sufficiently large positive integer is now representable as the sum of r or more distinct terms from (a_n). In particular, for all r we construct an example of a sequence which is r-complete but not $(r+1)$-complete.

1. Introduction

Let $(a_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers. We say that (a_k) is *complete* if there exists an integer N, $N > 0$, such that if $q \geq N$, q an integer, then there exist coefficients b_k such that $b_k \in \{0, 1\}$ for all k, and $q = \sum_{k=1}^{\infty} b_k a_k$ (that is, q is representable as a sum of distinct elements of the sequence (a_k)). We refer to the minimal such N as the *threshold of completeness*. As an example, the sequence defined by $a_k = k + 1$ (i.e., $(2, 3, 4, 5, \ldots)$) is complete (one may take $N = 2$), but the sequence defined by $a_k = 2k$ is not complete, as no odd integer is representable as a sum of elements of (a_k), let alone as a sum of distinct elements of (a_k).

A brief note on terminology: some authors (such as Honsberger [5]) choose to call such sequences as we have defined above *weakly complete*, and reserve the term complete for the case when $N = 1$. Brown [1] has extensively studied and characterized these sequences. Many authors also allow complete sequences to be merely

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non-decreasing: we choose to define a complete sequence to be strictly increasing so as to give a clear meaning to the word "distinct" in the above definition.

There are many specific known examples of complete sequences. Sprague [10] proved that the infinite sequence of kth powers, $(n^k)_{n\geq 1}$, for a fixed integer k is complete. Graham [4] gave conditions for completeness of sequences defined by polynomials. Porubský [9] used a result of Cassels [2] which gave a sufficient condition for a sequence to be complete to show that the sequence consisting of the kth powers of prime numbers for a fixed integer k is complete. All of these results demonstrated the completeness of certain classes of sequences, and some computationally derived a threshold of completeness (for instance, Sprague [11] showed that the threshold of completeness for the sequence of squares is 129).

Our aim in this paper is to consider a generalization of complete sequences that as far as we know Johnson and Laughlin [6] first described, which we will call r-*completeness*. We define a sequence (a_k) to be r-complete if there exists a positive integer N_r such that if $q \geq N_r$, q an integer, then q may be represented as the sum of r or more distinct terms from the sequence (a_k). Equivalently, q has a representation of the form $\sum_{k=1}^{\infty} b_k a_k$, where $b_k \in \{0,1\}$ for all k, and $\sum_{k=1}^{\infty} b_k \geq r$. By analogy with the definition of completeness, we call the minimal such N_r the *threshold of* r*-completeness*.

If (a_k) is r-complete for all positive integers r, then we will say that (a_k) is *infinitely complete*. For example, it is clear that that the sequence of positive integers is infinitely complete, and Looper and Saritzky [8] showed that in fact the sequence of kth power of integers, $(n^k)_{n\geq 1}$, for a fixed positive integer k is infinitely complete. It is also immediate that if (a_n) is r-complete, then (a_n) is k-complete for all positive integers k such that $k < r$.

At the end of their paper, Johnson and Laughlin [6] posed two questions which we can rephrase in our terminology. For $b > 2$, is the increasing sequence $B(b) = \{ab^{m-1} | a \in \{1, \ldots, b-1\}, m = 1, 2, \ldots\}$ infinitely complete? For any given positive integer r, is there a sequence (a_k) which is r-complete but not $(r+1)$-complete? We will provide affirmative answers to both of these questions and also give a complete characterization of sequences which are complete but not 2-complete.

2. Confirming that the Set $B(b)$ is Infinitely Complete

Before we dispense with the general set of base b representations, it will be enlightening to consider the specific case $b = 2$. In this case, we have $a_n = 2^{n-1}$, and it is well known that every positive integer has a unique binary representation. Therefore, since each power of two is representable as a sum of only one term from (a_n), it is readily seen that (2^{n-1}) is complete but not 2-complete.

As previously mentioned, Johnson and Laughlin conjectured that if $b > 2$, then

$B(b)$ is infinitely complete. We first recall the following well-known lemma regarding base b representations of positive integers:

Lemma 1. *Let $b > 1$ be a positive integer. Every positive integer a has a unique representation of the form*

$$a = \sum_{i=0}^{k} c_i b^i$$

subject to the following conditions:

1. $c_i \in \{0, 1, \ldots, b-1\}$ *for all* $i \in \{0, 1, \ldots, k\}$
2. $c_k \neq 0$
3. $b^k \leq a < b^{k+1}$

Lemma 1 proves immediately that $B(b)$ is complete. However, as the coefficients c_i may be 0 in the standard basis representation (for example, every integer of the form cb^k has a 0 everywhere in the standard basis representation except for the b^k term), this lemma is not sufficient to prove that $B(b)$ is infinitely complete.

We now prove that $B(b)$ is infinitely complete by making a suitable modification to the standard basis representation:

Theorem 1. *If $b > 2$ is an integer, then the increasing sequence $B(b) = \{ab^{m-1} | a \in \{1, \ldots, b-1\}, m = 1, 2, \ldots\}$ is infinitely complete.*

Proof. Let k be a positive integer. Let a be a positive integer such that $b^k \leq a < b^{k+1}$. We will show that a can be represented as a sum of $k+1$ or more distinct terms from $B(b)$. This will be sufficient to prove that for each positive integer r, $B(b)$ is r-complete. By Lemma 1 we know that a has a representation

$$a = c_k b^k + c_{k-1} b^{k-1} + \cdots + c_0 b^0$$

where $c_k \neq 0$. If each coefficient c_i is nonzero, then a has a representation as a sum of $k+1$ terms of $B(b)$. If not, then there is a coefficient c_m such that $c_m = 0$ and $c_j \neq 0$ for all $j > m$ (note that $m < k$). In the standard basis representation of a, we then replace $c_{m+1}b^{m+1}$ by $(c_{m+1} - 1)b^{m+1} + (b-1)b^m + b^m$. We now therefore have either two or three terms of $B(b)$ (depending on whether or not $c_{m+1} = 1$) in place of the one original $c_{m+1}b^{m+1}$. We can now proceed inductively by finding the next zero coefficient of largest index, and performing the same operation as follows:

Suppose that the representation of a has been modified to

$$a = \sum_{j=0}^{t} c_j b^j + S$$

for some $t \in \{0, 1, \ldots, k-2\}$, in which S is (inductively) a sum of at least $k-t$ distinct elements from the set $\{cb^j \mid c \in \{1, \ldots, b-1\}, j \in \{t+1, \ldots, k\}\}$, and that the only parts of S involving terms of the form cb^{t+1}, $c \in \{1, \ldots, b-1\}$ will either be of the form $c_{t+1}b^{t+1}$ (with $c_{t+1} \neq 0$) or $(b-1)b^{t+1} + b^{t+1}$. The algorithm now proceeds:

1. If $c_t > 0$ and $t = 0$, then a has a representation as a sum of at least $k+1$ terms of $B(b)$ and we are done. If $c_t > 0$ and $t > 0$, replace t by $t-1$ and repeat the process.

2. If $c_t = 0$ and there is a term of the form $c_{t+1}b^{t+1}$ in the representation of S, replace $c_{t+1}b^{t+1}$ by $(c_{t+1}-1)b^{t+1} + (b-1)b^t + b^t$ (as before). If $t = 0$, we are now done. Otherwise, replace t by $t-1$ and repeat the process.

3. If $c_t = 0$ and there is a term of the form $(b-1)b^{t+1} + b^{t+1}$ in S, replace $(b-1)b^{t+1} + b^{t+1}$ by $(b-1)b^{t+1} + (b-1)b^t + b^t$. If $t = 0$, we are now done. Otherwise, replace t by $t-1$ and repeat the process.

At each coefficient c_j, the process either leaves c_j fixed or replaces it by an expression involving two or three terms in the case that $c_j = 0$ or $c_{j-1} = 0$. Since $b - 1 > 1$, the terms in this representation are also distinct. Therefore, the process will leave a representation in which there are at least as many terms as in the typical base b representation of a, with each term drawn from $B(b)$. Hence, a has a representation as a sum of $k+1$ or more distinct terms of $B(b)$. Since this holds for any $a \in \{b^k, \ldots, b^{k+1}-1\}$, $B(b)$ is $(r+1)$-complete for all r. Therefore, $B(b)$ is infinitely complete. $\square$

3. Sequences which are r-complete but not $(r+1)$-complete

We have seen an example of a sequence ($a_n = 2^{n-1}$) which is complete but not 2-complete. In a later section we will characterize all sequences which are complete but not 2-complete. But first we would like to show that our definition of an r-complete sequence is interesting and worthwhile to study. To that end, we now state the theorem which is the subject of this section.

Theorem 2. *For any positive integer r, there is a sequence which is r-complete but not $(r+1)$-complete.*

We need a lemma before proceeding to the proof of this theorem.

Lemma 2. *If p is a positive integer, and c is a positive integer such that $\frac{p(p+1)}{2} \leq c < \frac{(p+1)(p+2)}{2}$ then c can be represented as a sum of p distinct positive integers, but not of q distinct positive integers for any $q > p$.*

Proof. By hypothesis there is a positive integer a such that $c + a = \frac{(p+1)(p+2)}{2} = 1 + 2 + \cdots + (p + 1)$. Since $c \geq \frac{p(p+1)}{2} = 1 + 2 + \cdots + p$, it follows that $a \leq p + 1$. Hence

$$c = \sum_{n=1}^{p+1} n - a$$

is a representation of c as a sum of p distinct positive integers. Since $1+2+\cdots+q = \frac{q(q+1)}{2}$ is the minimal sum that can be created using q distinct positive integers, it is clear that c has no representation as a sum of q distinct positive integers whenever $q > p$. $\square$

We now give an explicit construction of a sequence which is r-complete but not $(r + 1)$-complete, which will prove Theorem 2.

Proof of Theorem 2. We claim that the increasing sequence $(b_n) = (1, 2, \dots, r, r + 1, 2(r+1), 2^2(r+1), 2^3(r+1), \dots)$ is r-complete but not $(r+1)$-complete. As it has already been verified that when $r = 1$, the sequence is complete but not 2-complete, we may assume that $r \geq 2$. By Lemma 2 and its proof, every integer from $\frac{r(r-1)}{2}$ to $\frac{(r+1)(r+2)}{2}$ is representable as the sum of at least $r-1$ distinct positive integers from (b_n) and one need only draw from the first $r+1$ terms of (b_n). In particular, $2(r+1)$ does not appear in any integer's representation. Therefore, if a is a positive integer such that $2(r + 1) + \frac{r(r-1)}{2} \leq a \leq 2(r + 1) + \frac{(r+1)(r+2)}{2}$, then a has a representation with at least r distinct terms from (b_n).

We now proceed by strong induction and assume that for some positive integer k, every integer from $\frac{r(r-1)}{2}$ to $2^k(r+1)+\cdots+2(r+1)+\frac{(r+1)(r+2)}{2} = 2^{k+1}(r+1)+\frac{r(r-1)}{2}-1$ is representable as a sum of at least $r-1$ distinct terms from (b_n). As part of the inductive hypothesis, we assume that each such representation can be chosen to contain only terms drawn from the subset $\{1, 2, \dots, r, r+1, 2(r+1), \dots, 2^k(r+1)\}$. If b is a positive integer such that $2^{k+1}(r+1)+\frac{r(r-1)}{2} \leq b \leq 2^{k+2}(r+1)+\frac{r(r-1)}{2}-1$, then $b = 2^{k+1}(r + 1) + c$, where c is a positive integer which by the inductive hypothesis is representable as a sum of $r-1$ or more distinct terms drawn from the subset $\{1, 2, \dots, r, r + 1, 2(r + 1), \dots, 2^k(r + 1)\}$. We conclude that every integer from $2(r + 1) + \frac{r(r+1)}{2}$ on is representable as the sum of r or more distinct terms from (b_n), so this sequence is r-complete.

On the other hand, we claim that the integers of the form $2^k(r + 1) + \frac{r(r-1)}{2}$ are only representable as a sum of at most r distinct terms from the given sequence. Since

$$2^{k-1}(r + 1) + \cdots + 2(r + 1) + \frac{(r + 1)(r + 2)}{2} = 2^k(r + 1) + \frac{r(r - 1)}{2} - 1,$$

it is clear that any representation of $2^k(r + 1) + \frac{r(r-1)}{2}$ must contain a term of the form $2^d(r+1)$, where $d \geq k$. We may assume that $2^k(r+1)+\frac{r(r-1)}{2} = 2^d(r+1)+e$, where e is a positive integer. Then,

$$\frac{r(r-1)}{2} - e = 2^k(r+1)(2^{d-k}-1) \geq 0$$

which implies that $e \leq \frac{(r-1)(r)}{2}$. Hence, e can be represented as the sum of at most $r-1$ distinct terms from (b_n). Therefore, $2^k(r+1) + \frac{r(r-1)}{2}$ can be represented as the sum of at most r distinct terms drawn from $(1, 2, \ldots, r, r+1, 2(r+1), 2^2(r+1), 2^3(r+1), \ldots)$, so (b_n) is not $(r+1)$-complete. □

We remark that if one sets $r = 1$ in the above sequence, then the sequence $a_n = 2^{n-1}$ is the sequence the theorem gives as an example of a sequence which is complete but not 2-complete. (Note that it is easy to modify the proof above to include the case $r = 1$.)

The sequence (b_n) we defined above also has a nice recursive form that may be of some interest:

$$b_n = \begin{cases} n, & \text{if } n \leq r; \\ 1 + \sum_{i=r}^{n-1} b_i, & \text{if } n > r. \end{cases}$$

4. A Characterization of Sequences which are Complete but not 2-complete

The sequence (2^{n-1}) is not the only example of a sequence which is complete but not 2-complete. Knapp et al. [7, 3] have studied complete sequences that are in some sense minimal and constructible via a greedy algorithm. Namely, one arbitrarily selects two positive integers a_1 and a_2 to serve as seeds for a complete sequence, and then for $j > 2$, a_j is defined to be the smallest integer greater than a_{j-1} which is not representable as a sum of distinct terms from the set $G_j = \{a_1, \ldots, a_{j-1}\}$. For our purposes, we will label the smallest such integer ψ_j. The choice $a_1 = 1$, $a_2 = 2$ gives the sequence $a_n = 2^{n-1}$.

We observe that a sequence (a_n) defined by this method is not merely complete but it is also in fact not 2-complete, since no term of (a_n) is representable as a sum of lesser, distinct terms of the sequence. We readily derive the following more general construction for a complete but not 2-complete sequence:

Proposition 1. *Let (a_n) be an increasing, complete sequence. Then (a_n) is not 2-complete if and only if there is an infinite subsequence $a_{i_1} < a_{i_2} < \cdots$ such that a_{i_k} is not representable as a sum of lesser, distinct terms of (a_n) for all $k \geq 1$.*

Proof. Any positive integer larger than the threshold of completeness of (a_n) which is not a term of (a_n) must be representable as a sum of two or more terms from (a_n). Hence, if (a_n) is not 2-complete, then there must be an infinite number of terms of (a_n) which are not representable as a sum of distinct, lesser terms of (a_n), and these terms can be arranged as an infinite subsequence of (a_n). Conversely, if such an infinite subsequence exists, then there are an infinite number of positive integers which cannot be represented as a sum of two or more terms of (a_n), so (a_n) is not 2-complete. □

We can also describe sequences (a_n) which are complete but not 2-complete based on the parameter ψ_j.

Proposition 2. *Let (a_n), G_j and ψ_j be defined as above. Then we can characterize (a_n) as follows:*

1. *If $a_j > \psi_j$ for an infinite number of positive integers j, then (a_n) is not complete.*
2. *If $a_j \leq \psi_j$ for all but a finite number of positive integers j and $a_j = \psi_j$ for an infinite number of positive integers j, then (a_n) is complete but not 2-complete.*
3. *If $a_j < \psi_j$ for all but a finite number of positive integers j, then (a_n) is 2-complete.*

Proof. First, if $a_j > \psi_j$ for an infinite number of positive integers j, then since ψ_j is not representable as a sum of the terms of G_j, there are an infinite number of positive integers not representable as a sum of distinct terms of (a_n), so (a_n) is not complete.

Assume that $a_j \leq \psi_j$ for all but a finite number of positive integers j. There exists a positive integer N such that if $i > N$, then $a_i \leq \psi_i$. We will show that every positive integer b such that $b > a_N$ has a representation as a sum of distinct terms of (a_n). Note that $G_m \subseteq G_n$ if and only if $m \leq n$, which implies that (ψ_n) is a non-decreasing sequence. Hence, if $i > N$, we have that $a_{i-1} < a_i \leq \psi_i \leq \psi_{i+1}$. In particular, we have that

$$\{x \in \mathbb{Z} |\ a_{i-1} \leq x < \psi_i\} \cup \{x \in \mathbb{Z} |\ a_i \leq x < \psi_{i+1}\} = \{x \in \mathbb{Z} |\ a_{i-1} \leq x < \psi_{i+1}\}.$$

On the other hand, since (a_n) and (ψ_n) are both unbounded sequences, it follows that the set of sets of integers $\{\{x \in \mathbb{Z} |\ a_{i-1} \leq x < \psi_i\} |\ i \geq N+1\}$ completely covers the set $\{b \in \mathbb{Z} |\ b \geq a_N\}$. Hence, if $b \geq a_N$, then b is an element of some set of integers $\{x \in \mathbb{Z} |\ a_{m-1} \leq x < \psi_m\}$, and therefore b has a representation as a sum of distinct elements of (a_n), by the definition of ψ_m.

If $a_j = \psi_j$ for an infinite number of values of j, then we can extract an infinite subsequence from (a_n) whose terms are not representable as the sum of lesser, distinct terms of (a_n). Proposition 1 then implies that (a_n) is complete but not 2-complete.

Finally, if $a_j < \psi_j$ for all but a finite number of values of j, then there exists a positive integer N such that if $i > N$, then a_i is representable as a sum of lesser, distinct terms of (a_n). Because the sequence (a_n) is complete, every sufficiently large integer not in the sequence (a_n) is also representable as a sum of distinct terms of (a_n), so it follows that (a_n) is 2-complete. $\square$

The idea behind Proposition 2 can be used to construct a complete but not 2-complete sequence from any given finite increasing sequence of positive integers $(b_i)_{i=1}^n$. After choosing such a sequence, to define b_j for $j > n$, compute ψ_j and pick $b_j = \psi_j$ an infinite number of times (and after some finite point, require that $b_j \leq \psi_j$).

We have only so far been able to give a characterization of sequences which are complete but not 2-complete. We would be interested in future work to see if similar characterizations could be given for sequences which are r-complete but not $(r+1)$-complete.

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