ON RECIPROCAL SUMS FORMED BY SOLUTIONS OF PELL'S EQUATION

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Abstract

Let $(X_n, Y_n)_{n\geq 1}$ denote the positive integer solutions of Pell's equation $X^2 - DY^2 = 1$ or $X^2 - DY^2 = -1$. We introduce the Dirichlet series $\zeta_X(s) = \sum_{n=1}^{\infty} 1/X_n^s$ and $\zeta_Y(s) = \sum_{n=1}^{\infty} 1/Y_n^s$ for $\Re(s) > 0$ and prove that both functions do not satisfy any nontrivial algebraic differential equation. For any positive integers s_1 and s_2 the two numbers $\zeta_X(2s_1)$ and $\zeta_Y(2s_2)$ are algebraically independent over a transcendental field extension of $\mathbb{Q}$, whereas the three numbers $\zeta_X(2)$, $\zeta_Y(2)$, and $\sum_{n=1}^{\infty} 1/(X_nY_n)^2$ are linearly dependent over $\mathbb{Q}$. From the transcendence of $\zeta_Y(2)$ and the corresponding alternating series we obtain an application to the Archimedean cattle problem. Irrationality results for series of the form $\sum_{n=1}^{\infty} (-1)^{n+1}/X_n$, $\sum_{n=1}^{\infty} (-1)^{n+1}/Y_n$, and $\sum_{n=1}^{\infty} (-1)^{n+1}/X_nY_n$ are obtained by a theorem of R. André-Jeannin.

1. A Summary on the Solutions of Pell's Equation

Let D denote a positive integer which is not a perfect square. In this paper we investigate positive integer solutions (X, Y) of Pell's equations $X^2 - DY^2 = 1$ and $X^2 - DY^2 = -1$, provided that the Pell-Minus-equation $X^2 - DY^2 = -1$ is solvable in integers.

Let $\left\{(X_n, Y_n) \in \mathbb{N}^2 : X_n^2 - DY_n^2 = \pm 1 \wedge n = 1, 2, \dots\right\}$ be the set of all integer solutions, which are given recursively by

$$X_n + Y_n\sqrt{D} = \left(X_1 + Y_1\sqrt{D}\right)^n \qquad (n = 1, 2, \dots) \quad \text{if } X^2 - DY^2 = 1\,, \qquad (1)$$

and

$$X_n + Y_n\sqrt{D} = \left(X_1 + Y_1\sqrt{D}\right)^{2n-1} \qquad (n = 1, 2, \dots) \quad \text{if } X^2 - DY^2 = -1\,, \qquad (2)$$

where (X_1, Y_1) is the *primitive solution* of the corresponding equation with smallest coordinates X and Y. Let k be the length of the period of the continued fraction expansion of $\sqrt{D}$, and let p_m/q_m denote the m-th convergent of $\sqrt{D}$. First we treat the integer solutions of $X^2 - DY^2 = 1$. It is well-known [8, § 27] that

$$(X_n, Y_n) \;=\; (p_{mk-1}, q_{mk-1}) \tag{3}$$

holds for $n = 1, 2, 3 \ldots$ with $m = n$, when k is even; in particular we have $(X_1, Y_1) = (p_{k-1}, q_{k-1})$. When k is odd, all solutions X_n and Y_n are given by (3) for $m = 2n$. Then, $(X_1, Y_1) = (p_{2k-1}, q_{2k-1})$.

Pell's equation $X^2 - DY^2 = -1$ is solvable by integers if and only if k is odd. Then we obtain all solutions X_n and Y_n from (3) for $m = 2n - 1$, where $(X_1, Y_1) = (p_{k-1}, q_{k-1})$.
Expanding the right-hand sides of (1) and (2) using the Binomial theorem and rearranging the terms, we find the representations

$$X_n \;=\; \sum_{\substack{0 \le \nu \le m \\ \nu \equiv 0 \pmod 2}} \binom{m}{\nu} D^{\nu/2} Y_1^{\nu} X_1^{m-\nu}\,, \tag{4}$$

$$Y_n \;=\; \sum_{\substack{1 \le \nu \le m \\ \nu \equiv 1 \pmod 2}} \binom{m}{\nu} D^{(\nu-1)/2} Y_1^{\nu} X_1^{m-\nu}\,, \tag{5}$$

where $m \in \{n, 2n-1\}$, depending on whether the solutions of $X^2 - DY^2 = 1$ or $X^2 - DY^2 = -1$ are under consideration. This gives the formulas

$$X_n \;=\; \frac{1}{2}\big(\alpha^m + \beta^m\big)\,, \tag{6}$$

$$Y_n \;=\; \frac{1}{2\sqrt{D}}\big(\alpha^m - \beta^m\big)\,, \tag{7}$$

where $m \in \{n, 2n-1\}$, and $\alpha := X_1 + Y_1\sqrt{D}$, $\beta := X_1 - Y_1\sqrt{D}$. Then we obtain

$$\alpha\beta \;=\; X_1^2 - DY_1^2 \;=\; \pm 1\,.$$

Since $\alpha > 1$ it is clear that $|\beta| < 1$. We have $0 < \beta < 1$ and $\alpha\beta = 1$ (if $X^2 - DY^2 = 1$), and $-1 < \beta < 0$ and $\alpha\beta = -1$ (if $X^2 - DY^2 = -1$). Let s_1, s_2 be integers. In this paper we investigate the series

$$\sum_{n=1}^{\infty} \frac{1}{X_n^{s_1}}\,, \qquad \sum_{n=1}^{\infty} \frac{1}{Y_n^{s_2}}\,, \qquad \sum_{n=1}^{\infty} \frac{1}{X_n Y_n}\,, \qquad \sum_{n=1}^{\infty} \frac{1}{(X_n Y_n)^2}$$

and the corresponding alternating sums. The first and second sum can be considered as Dirichlet series which code the algebraic structure of Pell's equations in analytic terms. We shall show in Section 2 that these series do not satisfy any nontrivial algebraic differential equation similar to the Riemann Zeta function or the Fibonacci

Zeta function. Discrete values of these series can be investigated by deep methods of transcendence theory. In particular, we are interested in algebraic independence (and dependence) results between two or three numbers represented by these series (Sections 4 to 6).

2. Hypertranscendental Series

In this paper we investigate the two Zeta functions

$$\begin{aligned}\zeta_X(s) &:= \sum_{n=1}^{\infty} \frac{1}{X_n^s} \qquad \bigl(\Re(s) > 0\bigr)\,,\\ \zeta_Y(s) &:= \sum_{n=1}^{\infty} \frac{1}{Y_n^s} \qquad \bigl(\Re(s) > 0\bigr)\end{aligned}$$

associated with all positive integer solutions $(X_n, Y_n)_{n\geq 1}$ of Pell's equation $X^2 - DY^2 = \pm 1$. Both functions, $\zeta_X(s)$ and $\zeta_Y(s)$, can be extended meromorphically to the complex plane by the method described in [6] for the Fibonacci Zeta function.

Let $D \in \mathbb{N}$ be not a perfect square, and let $(X_n^{(D)}, Y_n^{(D)})$ with

$$1 < X_1^{(D)} < X_2^{(D)} < \dots \qquad \text{and} \qquad 1 \leq Y_1^{(D)} < Y_2^{(D)} < \dots$$

denote all integer solutions of Pell's equation $X^2 - DY^2 = 1$. Moreover, we introduce the sets

$$M_X^{(D)} := \bigl\{X_1^{(D)}, X_2^{(D)}, \dots\bigr\} \qquad \text{and} \qquad M_Y^{(D)} := \bigl\{Y_1^{(D)}, Y_2^{(D)}, \dots\bigr\}\,.$$

Lemma 1. *1. Every positive integer k divides infinitely many elements of the set $M_Y^{(D)}$.*

2. There is a sequence $(p_m)_{m\geq 1}$ of distinct primes such that every prime p_m divides at least one element from the set $M_X^{(D)}$.

Proof. 1. For every given $k \in \mathbb{N}$, Pell's equation $X^2 - Dk^2Y^2 = 1$ has infinitely many solutions $\bigl(X_n^{(Dk^2)}, Y_n^{(Dk^2)}\bigr) \in \mathbb{N}^2$, because $Dk^2 > 1$ is not a perfect square by the condition on D. Therefore, for $n = 1, 2, \dots$ we have

$$\bigl(X_n^{(Dk^2)}\bigr)^2 - D\bigl(kY_n^{(Dk^2)}\bigr)^2 = 1\,,$$

which shows that $kY_n^{(Dk^2)} \in M_Y^{(D)}$ $(n = 1, 2, \dots)$. With $kY_n^{(Dk^2)} \equiv 0 \pmod{k}$ for $n \geq 1$ the first part of the lemma is proven.
2. We construct the primes p_m of the sequence $(p_m)_{m\geq 1}$ recursively.

Construction of p_1: Let $D_0 := D$. The equation $X^2 - D_0Y^2 = 1$ has infinitely many positive integer solutions. We know from $X_1^{(D_0)} > 1$ that there is a (smallest) prime divisor p_1 of $X_1^{(D_0)} \in M_X^{(D)}$.
Construction of p_{m+1}: Let us assume that we have already defined distinct primes $p_1, \ldots, p_m$ such that every p_μ $(1 \le \mu \le m)$ divides at least one element from the set $M_X^{(D)}$. Set

$$D_m := D(p_1p_2\cdots p_m)^2 .$$

D is not a perfect square. Therefore, Pell's equation $X^2 - D_mY^2 = 1$ can be solved by positive integers. We consider the particular solution $\big(X_1^{(D_m)}, Y_1^{(D_m)}\big)$, i.e.,

$$\big(X_1^{(D_m)}\big)^2 - D_m\big(Y_1^{(D_m)}\big)^2 = 1 . \tag{8}$$

Assume that there is a prime p_μ from the set $\{p_1, p_2, \ldots, p_m\}$ dividing $X_1^{(D_m)}$. With $D_m \equiv 0 \pmod{p_\mu}$ it follows from (8) that

$$0^2 - 0\big(Y_1^{(D_m)}\big)^2 \equiv 1 \pmod{p_\mu} ,$$

a contradiction. This shows that no prime from the set $\{p_1, p_2, \ldots, p_m\}$ divides $X_1^{(D_m)}$. Thus, from $X_1^{(D_m)} > 1$ we conclude that there is a (smallest) prime divisor p_{m+1} of $X_1^{(D_m)}$ satisfying

$$p_{m+1} \notin \big\{p_1, p_2, \ldots, p_m\big\} . \tag{9}$$

Equation (8) can be transformed into

$$\big(X_1^{(D_m)}\big)^2 - D\big(p_1p_2\cdots p_mY_1^{(D_m)}\big)^2 = 1 ,$$

which proves that $X_1^{(D_m)}$ belongs to the set $M_X^{(D)}$. By $X_1^{(D_m)} \equiv 0 \pmod{p_{m+1}}$ and (9) we complete the proof of the lemma. □

Remark 1. In general, the sequence $(p_m)_{m\ge1}$ does not increase monotonously.

In what follows we apply a theorem due to A. Reich [9]. In order to state this result we firstly introduce the set $\mathcal{D}$ of ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with integer coefficients a_n satisfying two conditions:

1.) The abscissa of absolute convergence of $\sum_{n=1}^{\infty} a_n n^{-s}$ is finite. We denote this abscissa by $\sigma_a(f)$.

2.) The set of all divisors of indices n corresponding to nonvanishing coefficients a_n contains infinitely many prime numbers.

Next, let ν be a nonnegative integer. For a meromorphic function f set

$$\underline{f}^{[\nu]}(s) := \big(f(s), f'(s), \ldots, f^{(\nu)}(s)\big) .$$

Lemma 2 ([9], [11], A. Reich **).** *Let $f \in \mathcal{D}$, $h_0 < h_1 < \cdots < h_m$ be any real numbers, $\nu_0, \nu_1, \ldots, \nu_m$ be any nonnegative integers, and let $\sigma_0 > \sigma_a(f) - h_0$. Put $k := \sum_{j=0}^{m}(\nu_j + 1)$. If $\Phi : \mathbb{C}^k \to \mathbb{C}$ is a continuous function such that the difference-differential equation*

$$\Phi\big(\underline{f}^{[\nu_0]}(s+h_0), \underline{f}^{[\nu_1]}(s+h_1), \ldots, \underline{f}^{[\nu_m]}(s+h_m)\big) \;=\; 0$$

holds for all s with $\Re(s) > \sigma_0$, then Φ vanishes identically.

A function is said to be *hypertranscendental* if it satisfies no nontrivial algebraic differential equation. We may consider the functions $\zeta_X(s)$ and $\zeta_Y(s)$ as ordinary Dirichlet series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ having the (finite) abscissas $\sigma_0(\zeta_X(s)) = \sigma_0(\zeta_Y(s)) = 0$ of absolute convergence, i.e., they converge absolutely for $\sigma = \Re(s) > 0$. Moreover, by Lemma 1, we know that for both functions the set of all divisors of indices n with $a_n \neq 0$ contains infinitely many prime numbers, since $a_n = 1$ for $n \in M_X^{(D)}$ or $n \in M_Y^{(D)}$, respectively, and $a_n = 0$ otherwise. On this situation we apply Reich's theorem (Lemma 2).

Theorem 1. *Let $h_0 < h_1 < \cdots < h_m$ be any real numbers, $\nu_0, \nu_1, \ldots, \nu_m$ be any nonnegative integers, and let $\sigma_0 > -h_0$. Put $k := \sum_{j=0}^{m}(\nu_j + 1)$. If $\Phi : \mathbb{C}^k \to \mathbb{C}$ is a continuous function satisfying*

$$\Phi\big(\underline{\xi}_X^{[\nu_0]}(s+h_0), \underline{\xi}_X^{[\nu_1]}(s+h_1), \ldots, \underline{\xi}_X^{[\nu_m]}(s+h_m)\big) \;=\; 0$$

or

$$\Phi\big(\underline{\xi}_Y^{[\nu_0]}(s+h_0), \underline{\xi}_Y^{[\nu_1]}(s+h_1), \ldots, \underline{\xi}_Y^{[\nu_m]}(s+h_m)\big) \;=\; 0\,,$$

for all s with $\Re(s) > \sigma_0$, then Φ vanishes identically.

Corollary 1. *Let $(X_n, Y_n)_{n\geq 1}$ be the sequence of all positive integer solutions of Pell's equation $X^2 - DY^2 = 1$. Then, both the functions $\zeta_X(s)$ and $\zeta_Y(s)$ are hypertranscendental.*

3. On the Irrationality of Some Reciprocal Sums

It follows from (6) and (7) that every sequence $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$, and $(X_nY_n)_{n\geq 1}$ satisfies a linear three-term recurrence formula of the form

$$Z_{n+2} + aZ_{n+1} + bZ_n \;=\; 0\,. \tag{10}$$

In particular, both sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$, satisfy the same formula. In the sequel we compute the coefficients a, b in (10) for every reciprocal sum in the set

$$\Big\{\,\sum_{n=1}^{\infty}\frac{1}{X_n}\,,\;\sum_{n=1}^{\infty}\frac{1}{Y_n}\,,\;\sum_{n=1}^{\infty}\frac{1}{X_nY_n}\Big\}$$

and show that a, b are integers with $b = \pm 1$.

Case 1. (X_n, Y_n) satisfies $X^2 - DY^2 = 1$.
We study the recurrence for $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ simultaneously. The characteristic polynomial of the recurrence is

$$P(\lambda) \,=\, \lambda^2 + a\lambda + b \,=\, (\lambda - \alpha)(\lambda - \beta) \,=\, \lambda^2 - (\alpha + \beta)\lambda + \alpha\beta \,=\, \lambda^2 - 2X_1\lambda + 1\,.$$

Thus, for both sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$, (10) can be written as

$$Z_{n+2} \,=\, 2X_1 Z_{n+1} - Z_n\,. \tag{11}$$

The sequence $(X_n Y_n)_{n\geq 1}$ is given by

$$X_n Y_n \,=\, \frac{1}{4\sqrt{D}}\big(\alpha^n + \beta^n\big)\big(\alpha^n - \beta^n\big) \,=\, \frac{1}{4\sqrt{D}}\big(\alpha^2\big)^n - \frac{1}{4\sqrt{D}}\big(\beta^2\big)^n\,.$$

Hence we have

$$P(\lambda) \,=\, (\lambda - \alpha^2)(\lambda - \beta^2) \,=\, \lambda^2 - (\alpha^2 + \beta^2)\lambda + \alpha^2\beta^2 \,=\, \lambda^2 - 2\big(X_1^2 + DY_1^2\big)\lambda + 1$$

and

$$Z_{n+2} \,=\, 2\big(X_1^2 + DY_1^2\big)Z_{n+1} - Z_n\,. \tag{12}$$

Case 2. (X_n, Y_n) satisfies $X^2 - DY^2 = -1$.
The characteristic polynomial of the recurrence for the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ is

$$P(\lambda) \,=\, (\lambda - \alpha^2)(\lambda - \beta^2) \,=\, \lambda^2 - 2\big(X_1^2 + DY_1^2\big)\lambda + 1\,,$$

since

$$\begin{aligned} X_n &= \frac{1}{2}\big(\alpha^{2n-1} + \beta^{2n-1}\big) \,=\, \frac{1}{2\alpha}(\alpha^2)^n + \frac{1}{2\beta}(\beta^2)^n\,, \\ Y_n &= \frac{1}{2\sqrt{D}}\big(\alpha^{2n-1} - \beta^{2n-1}\big) \,=\, \frac{1}{2\alpha\sqrt{D}}(\alpha^2)^n - \frac{1}{2\beta\sqrt{D}}(\beta^2)^n\,. \end{aligned}$$

Therefore, (10) reduces to

$$Z_{n+2} \,=\, 2\big(X_1^2 + DY_1^2\big)Z_{n+1} - Z_n\,. \tag{13}$$

Moreover, for the sequence $(X_n Y_n)_{n\geq 1}$ we obtain the formula

$$\begin{aligned} X_n Y_n &= \frac{1}{4\sqrt{D}}\big(\alpha^{2n-1} + \beta^{2n-1}\big)\big(\alpha^{2n-1} - \beta^{2n-1}\big) \\ &= \frac{1}{4\alpha^2\sqrt{D}}\big(\alpha^4\big)^n - \frac{1}{4\beta^2\sqrt{D}}\big(\beta^4\big)^n\,. \end{aligned}$$

Hence we have

$$\begin{aligned} P(\lambda) &= (\lambda-\alpha^4)(\lambda-\beta^4) \;=\; \lambda^2-(\alpha^4+\beta^4)\lambda+\alpha^4\beta^4 \\ &= \lambda^2-2\big(X_1^4+6DX_1^2Y_1^2+D^2Y_1^4\big)\lambda+1 \end{aligned}$$

and

$$Z_{n+2} \;=\; 2\big(X_1^4+6DX_1^2Y_1^2+D^2Y_1^4\big)Z_{n+1}-Z_n\,. \tag{14}$$

On the above sequences and their recurrences (11), (12), (13), and (14) we may apply a result of André-Jeannin [5, Théorème], from which we derive that the reciprocal sums of these sequences as well as the corresponding alternating sums are irrational. In order to state André-Jeannin's result we consider a sequence $(w_n)_{n\in\mathbb{Z}}$ of integers satisfying a linear three-term recurrence formula,

$$w_n \;=\; rw_{n-1}-sw_{n-2} \qquad (n\in\mathbb{Z})$$

with nonvanishing constant integer coefficients r, s and arbitrary initial values $w_0, w_1 \in \mathbb{Z}$. We assume that $w_n \neq 0$ for $n \geq 1$, and that the discriminant $\delta := r^2-4s$ is positive. The characteristic equation $X^2-rX+s=0$ of the recurrence has two real zeros α, β such that $|\alpha|>|\beta|$ and $|\alpha|>1$. Then there are two real constants C_1, C_2 with

$$w_n \;=\; C_1\alpha^n+C_2\beta^n$$

for $n \geq 0$.

Lemma 3 ([5], Théorème, R. André-Jeannin**).** *Let x be a nonvanishing integer, and let the following conditions be satisfied;*

(i) $s=\pm 1,\ \ C_1C_2\neq 0\,,$

(ii) $|x|<|\alpha|,\ \ |C_1C_2x^2|<|\alpha|\ \ .$

Then the series $\sum_{n=1}^{\infty} x^n/w_n$ has a real irrational limit.

We apply this lemma with $s=1$, $r\in\{2X_1,\, 2(X_1^2+DY_1^2),\, 2(X_1^4+6DX_1^2Y_1^2+D^2Y_1^4)\}$, $\delta>0$, $x\in\{-1,+1\}$, $|C_1C_2x^2|\leq 1/4D<1<|\alpha|$.

Theorem 2. *Let (X_n, Y_n) denote positive integer solutions of either Pell's equation $X^2-DY^2=1$ or of Pell's equation $X^2-DY^2=-1$. Then any number in the set*

$$\Big\{\sum_{n=1}^{\infty}\frac{1}{X_n}\,,\ \sum_{n=1}^{\infty}\frac{1}{Y_n}\,,\ \sum_{n=1}^{\infty}\frac{1}{X_nY_n}\,,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{X_n}\,,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{Y_n}\,,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{X_nY_n}\Big\}$$

is irrational.

4. An Algebraic Independence Result for Two Reciprocal Sums Formed by Powers of Solutions of Pell's Equation

Let K and E be the complete elliptic integrals of the first and the second kind defined by

$$\begin{aligned} K &= K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\,, \\ E &= E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt \end{aligned}$$

with $k^2 \in \mathbb{C} \setminus \big(\{0\} \cup [1,\infty)\big)$, where the branch of each integrand is chosen so that it tends to 1 as $t \to 0$. Furthermore, let

$$K' = K'(k) := K(k')\,,$$

where $k^2 + (k')^2 = 1$.
In this section we shall prove the following theorem.

Theorem 3. *Let $(X_n, Y_n)_{n\geq 1}$ be the sequence of all positive integer solutions either of Pell's equation $X^2 - DY^2 = 1$ or of Pell's equation $X^2 - DY^2 = -1$. Let the modulus k be given by*

$$X_1 - Y_1\sqrt{D} = e^{-\pi K(k')/K(k)} \qquad \textit{if} \quad X_n^2 - DY_n^2 = 1\,, \tag{15}$$

and

$$\big(X_1 - Y_1\sqrt{D}\big)^2 = e^{-\pi K(k')/K(k)} \qquad \textit{if} \quad X_n^2 - DY_n^2 = -1\,. \tag{16}$$

Then, for any positive integers s_1 and s_2, the two numbers

$$\zeta_X(2s_1) = \sum_{n=1}^{\infty} \frac{1}{X_n^{2s_1}} \qquad \textit{and} \qquad \zeta_Y(2s_2) = \sum_{n=1}^{\infty} \frac{1}{Y_n^{2s_2}}$$

are algebraically independent over $\mathbb{Q}(E/\pi)$.

Remark 2. It can be shown that the modulus k with $0 < k < 1$ is uniquely determined either by (15) or by (16).

Remark 3. It follows from [3, Table 1] for $q = X_1 - Y_1\sqrt{D}$ or $q = \big(X_1 - Y_1\sqrt{D}\big)^2$ according to (15) or (16) that

$$\frac{E}{\pi} = \frac{1}{2} - 2q + 10q^2 - 32q^3 + 82q^4 - 196q^5 + 448q^6 - 960q^7 + 1954q^8 + \mathcal{O}(q^9)\,.$$

4.1. Preparation of the Proof

Here we introduce several quantities which will be used for the proof of Theorem 3. The Jacobian elliptic function $w = \operatorname{sn} z = \operatorname{sn}(z,k)$ with modulus k may be defined by

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\,,$$

and

$$\operatorname{cn} z = \sqrt{1-\operatorname{sn}^2 z}\,, \qquad \operatorname{dn} z = \sqrt{1-k^2\operatorname{sn}^2 z}\,,$$
$$\operatorname{ns} z = \frac{1}{\operatorname{sn} z}\,, \quad \operatorname{nc} z = \frac{1}{\operatorname{cn} z}\,, \quad \operatorname{nd} z = \frac{1}{\operatorname{dn} z}\,.$$

We cite the subsequent lemmas 4-7 from [2]. For some slightly modified proofs see [10, ch. 3.1].

Lemma 4. *The coefficients c_j of the expansion*

$$\operatorname{ns}^2 z = \frac{1}{z^2} + \sum_{j=0}^{\infty} c_j z^{2j}$$

are given by

$$c_0 = \frac{1}{3}(1+k^2)\,, \qquad c_1 = \frac{1}{15}(1-k^2+k^4)\,, \qquad c_2 = \frac{1}{189}(1+k^2)(1-2k^2)(2-k^2)\,,$$

$$(j-2)(2j+3)c_j = 3\sum_{i=1}^{j-2} c_i c_{j-i-1} \qquad (j \geq 3)\,.$$

Lemma 5. *The coefficients d_j of the expansion*

$$(1-k^2)\operatorname{nd}^2 z = 1 - k^2 + \sum_{j=1}^{\infty} d_j z^{2j}$$

are given by

$$d_1 = k^2(1-k^2)\,, \qquad d_2 = -\frac{1}{3}k^2(1-k^2)(1-2k^2)\,,$$

$$j(2j-1)d_1 d_j = 6d_2 d_{j-1} - 3d_1 \sum_{i=1}^{j-2} d_i d_{j-i-1} \qquad (j \geq 3)\,.$$

Lemma 6. *The coefficients* e_j *of the expansion*

$$(1-k^2)(\mathrm{nc}^2\, z - 1) \;=\; \sum_{j=1}^{\infty} e_j z^{2j}$$

are given by

$$e_1 \;=\; 1-k^2\,, \qquad e_2 \;=\; \frac{1}{3}(1-k^2)(2-k^2)\,,$$

$$j(2j-1)e_1 e_j \;=\; 6e_2 e_{j-1} + 3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1} \qquad (j \geq 3)\,.$$

Lemma 7. *The coefficients* f_j *of the expansion*

$$\mathrm{dn}^2\, z \;=\; 1 + \sum_{j=1}^{\infty} f_j z^{2j}$$

are given by

$$f_1 \;=\; -k^2\,, \qquad f_2 \;=\; \frac{1}{3}k^2(1+k^2)\,,$$

$$j(2j-1)f_1 f_j \;=\; 6f_2 f_{j-1} - 3f_1 \sum_{i=1}^{j-2} f_i f_{j-i-1} \qquad (j \geq 3)\,.$$

All the coefficients c_j, d_j, e_j, f_j introduced above are polynomials in k with rational coefficients. The recurrence formulas in the preceding lemmas allow us to compute the degree of each such polynomial.

Lemma 8. *We have for* $j \geq 1$ *that*

$$\begin{aligned} \deg c_j &= \deg d_j \;=\; 2j+2\,,\\ \deg e_j &= \deg f_j \;=\; 2j\,. \end{aligned}$$

Proof. We restrict the arguments on the degree of e_j. The proofs are similar for the remaining polynomials.

Let $\mu(e_j)$ be the sign of the leading coefficient of e_j. We show the simultaneous identities $\deg e_j = 2j$ and $\mu(e_j) = (-1)^j$ by induction on $j \geq 1$. We have $\deg e_1 = 2$, $\mu(e_1) = -1$, $\deg e_2 = 4$, and $\mu(e_2) = 1$ by Lemma 6. Now we fix some $j \geq 3$ and assume the identities to be true for $1, 2, \ldots, j-1$. When i runs from 1 to $j-2$, we obtain step by step that

$$\begin{aligned}
\mu(e_i e_{j-i-1}) &= (-1)^i (-1)^{j-i-1} \;=\; (-1)^{j-1}\,, \\
\deg(e_i e_{j-i-1}) &= 2i + 2(j-i-1) \;=\; 2j-2\,, \\
\mu\Big(3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1}\Big) &= -(-1)^{j-1} \;=\; (-1)^j\,, \\
\deg\Big(3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1}\Big) &= 2 + (2j-2) \;=\; 2j\,, \\
\mu(6e_2 e_{j-1}) &= (-1)^{j-1}\,, \\
\deg(6e_2 e_{j-1}) &= 4 + 2(j-1) \;=\; 2j+2\,, \\
\mu\Big(6e_2 e_{j-1} + 3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1}\Big) &= \mu(6e_2 e_{j-1}) \;=\; (-1)^{j-1}\,, \qquad (17) \\
\deg\Big(6e_2 e_{j-1} + 3e_1 \sum_{i=1}^{j-2} e_i e_{j-i-1}\Big) &= \deg(6e_2 e_{j-1}) \;=\; 2j+2\,. \qquad (18)
\end{aligned}$$

For the left-hand side of the recurrence formula in Lemma 6 we know that $\mu\big(j(2j-1)e_1\big) = -1$ and $\deg\big(j(2j-1)e_1\big) = 2$. Therefore, we obtain by (17) and (18) the desired identities

$$\begin{aligned}
\mu(e_j) &= -(-1)^{j-1} \;=\; (-1)^j\,, \\
\deg e_j &= (2j+2) - 2 \;=\; 2j\,.
\end{aligned}$$

□

The q-series

$$\begin{aligned}
A_{2j+1}(q) &= \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{2n}}{1-q^{2n}}\,, \qquad & B_{2j+1}(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^{2j+1} q^{2n}}{1-q^{2n}}\,, \\
C_{2j+1}(q) &= \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{n}}{1-q^{2n}}\,, \qquad & D_{2j+1}(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^{2j+1} q^{n}}{1-q^{2n}}
\end{aligned}$$

are generated from the Fourier expansions of the Jacobian elliptic functions; see the formulas (i), (ii), (iii), and (iv) in [3, Table 5]. Combining each Fourier series with the power series expansion of the corresponding elliptic function given by Lemmas 4-7, we obtain in Lemma 9 below the following closed form expressions of these q-series in terms of K/π, E/π, and k (cf. [3, Table 1] and [14]). To state these expressions we additionally need the series expansions of the circular functions $\sec^2 z$ and $\csc^2 z$

given by

$$\csc^2 z = \frac{1}{z^2} + \sum_{j=0}^{\infty} a_j z^{2j}\,, \qquad a_j = \frac{(-1)^j (2j+1) 2^{2j+2} B_{2j+2}}{(2j+2)!}\,, \tag{19}$$

$$\sec^2 z = \sum_{j=0}^{\infty} b_j z^{2j}\,, \qquad b_j = \frac{(-1)^j (2j+1) 2^{2j+2} (2^{2j+2}-1) B_{2j+2}}{(2j+2)!}\,. \tag{20}$$

Here, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$ are the Bernoulli numbers.

Lemma 9. *Let $q = e^{-\pi K'/K}$. Then we have*

$$3\Big(\frac{2K}{\pi}\Big)\Big(\frac{2E}{\pi}\Big) + \Big(\frac{2K}{\pi}\Big)^2 (k^2 - 2) = 1 - 24A_1(q)\,,$$

$$\Big(\frac{2K}{\pi}\Big)^{2j+2} c_j = a_j - (-1)^j \frac{2^{2j+3}}{(2j)!} A_{2j+1}(q) \qquad (j \geq 1)\,,$$

$$\Big(\frac{2K}{\pi}\Big)\Big(\frac{2E}{\pi}\Big) = 1 + 8B_1(q)\,,$$

$$\Big(\frac{2K}{\pi}\Big)^{2j+2} e_j = b_j + (-1)^j \frac{2^{2j+3}}{(2j)!} B_{2j+1}(q) \qquad (j \geq 1)\,,$$

$$\Big(\frac{2K}{\pi}\Big)^2 - \Big(\frac{2K}{\pi}\Big)\Big(\frac{2E}{\pi}\Big) = 8C_1(q)\,,$$

$$\Big(\frac{2K}{\pi}\Big)^{2j+2} f_j = (-1)^j \frac{2^{2j+3}}{(2j)!} C_{2j+1}(q) \qquad (j \geq 1)\,,$$

$$\Big(\frac{2K}{\pi}\Big)\Big(\frac{2E}{\pi}\Big) + \Big(\frac{2K}{\pi}\Big)^2 (k^2 - 1) = 8D_1(q)\,,$$

$$\Big(\frac{2K}{\pi}\Big)^{2j+2} d_j = (-1)^{j-1} \frac{2^{2j+3}}{(2j)!} D_{2j+1}(q) \qquad (j \geq 1)\,.$$

As already mentioned above, for each $q \in \mathbb{C}$ with $0 < |q| < 1$ we can choose the modulus k such that $q = e^{-\pi K'/K}$. The following Lemma 10 can be obtained from Nesterenko's theorem [7, Theorem 4.2] on Ramanujan's functions

$$\begin{aligned} P(q^2) &:= 1 - 24A_1(q)\,, \\ Q(q^2) &:= 1 + 240A_3(q)\,, \\ R(q^2) &:= 1 - 504A_5(q)\,, \end{aligned}$$

which can be expressed in terms of K/π, E/π, and k using the identities for A_1, A_3, and A_5 in Lemma 9. By $\overline{\mathbb{Q}}$ we denote the set of all complex algebraic numbers over $\mathbb{Q}$.

Lemma 10 ([3], Lemma 1). *If $q = e^{-\pi K'/K} \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$, then K/π, E/π, and k are algebraically independent over $\mathbb{Q}$.*

Lemma 11 ([4], Satz 7, Algebraic independence criterion**).** *Let $\mathbb{K}$ be a field satisfying $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. Let $x_1, \dots, x_n \in \mathbb{C}$ be algebraically independent over $\mathbb{K}$ and let $y_1, \dots, y_k \in \mathbb{C}$ with $1 \le k < n$ satisfy the systems of equations*

$$f_j(x_1, \dots, x_n, y_1, \dots, y_k) \,=\, 0 \qquad (1 \le j \le k)\,,$$

where $f_j(X_1, \dots, X_n, Y_1, \dots, Y_k) \in \mathbb{K}[X_1, \dots, X_n, Y_1, \dots, Y_k]$ for $1 \le j \le k$. Assume that

$$\det\Big(\frac{\partial f_j}{\partial X_i}(x_1, \dots, x_n, y_1, \dots, y_k) \Big)_{1 \le i,j \le k} \neq 0\,.$$

Then the numbers $y_1, \dots, y_k$ are algebraically independent over $\mathbb{K}(x_{k+1}, \dots, x_n)$.

This independence criterion is a slightly generalized version of Lemma 2 in [3]: *Let $x_1, \dots, x_n \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$ and let $y_1, \dots, y_n \in \mathbb{C}$ satisfy the system of equations $f_j(x_1, \dots, x_n, y_1, \dots, y_n) = 0$ for $j = 1, \dots, n$, where $f_j(X_1, \dots, X_n, Y_1, \dots, Y_n)$ belongs to $\mathbb{Q}[X_1, \dots, X_n, Y_1, \dots, Y_n]$ for $1 \le j \le n$. If the Jacobian determinant of $f_1, \dots, f_n$ with respect to $X_1, \dots, X_n$ does not vanish for $X_i = x_i$, $Y_i = y_i$ $(i = 1, \dots, n)$, then the numbers $y_1, \dots, y_n$ are algebraically independent over $\mathbb{Q}$.*
For Theorem 3 the system of equations in Lemma 11 will take the form

$$f_1 \,=\, \zeta_X(2s_1) - y_1 \quad \text{and} \quad f_2 \,=\, \zeta_Y(2s_2) - y_2\,,$$

where $\zeta_X(2s_1), \zeta_Y(2s_2) \in \mathbb{Q}[x_1, x_2, x_3]$ (cf. Section 4.2). Replacing x_i by variables X_i $(i = 1, 2, 3)$, we obtain

$$\frac{\partial f_1}{\partial X_i} \,=\, \frac{\partial \xi_X(2s_1)}{\partial X_i} \qquad \text{and} \qquad \frac{\partial f_2}{\partial X_i} \,=\, \frac{\partial \xi_Y(2s_2)}{\partial X_i}\,.$$

Finally, by $\sigma_1(s), \dots, \sigma_{s-1}(s)$ we denote the elementary symmetric functions of $-1^2, -2^2, -3^2, \dots, -(s-1)^2$ defined by

$$\sigma_i(s) = (-1)^i \sum_{1 \le r_1 < \cdots < r_i \le s-1} r_1^2 \cdots r_i^2 \qquad (1 \le i \le s-1)\,,$$

and let $\sigma_0(s) = 1$ for every $s \ge 1$. Let

$$\begin{aligned} U_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \\ V_n &= \alpha^n + \beta^n \end{aligned}$$

for $n = 1, 2, \dots$. From [1] we get the following identities.

Case 1. Let $s \in \mathbb{N}$, and $(X_n, Y_n)_{n\geq 1}$ be all positive integer solutions of $X^2 - DY^2 = 1$. Then, with $A_{2j+1} = A_{2j+1}(\beta)$ and $B_{2j+1} = B_{2j+1}(\beta)$ for $\beta = X_1 - Y_1\sqrt{D}$, we have

$$\begin{aligned}\zeta_X(2s) &= 2^{2s}\sum_{n=1}^{\infty}\frac{1}{V_n^{2s}}\\ &= \frac{(-1)^{s-1}2^{2s}}{(2s-1)!}\Big(\sigma_{s-1}(s)B_1 + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)B_{2j+1}\Big), \qquad (21)\\ \zeta_Y(2s) &= \frac{2^{2s}D^s}{(\alpha-\beta)^{2s}}\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}}\\ &= \frac{2^{2s}D^s}{(2s-1)!}\Big(\sigma_{s-1}(s)A_1 + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)A_{2j+1}\Big). \qquad (22)\end{aligned}$$

Case 2. Let $s \in \mathbb{N}$, and $(X_n, Y_n)_{n\geq 1}$ be all positive integer solutions of $X^2 - DY^2 = -1$. Then, with $C_{2j+1} = C_{2j+1}(\beta^2)$ and $D_{2j+1} = D_{2j+1}(\beta^2)$, we have

$$\begin{aligned}\zeta_X(2s) &= 2^{2s}\sum_{n=1}^{\infty}\frac{1}{V_{2n-1}^{2s}}\\ &= \frac{2^{2s}}{(2s-1)!}\Big(\sigma_{s-1}(s)C_1 + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)C_{2j+1}\Big), \qquad (23)\end{aligned}$$

$$\begin{aligned}\zeta_Y(2s) &= \frac{2^{2s}D^s}{(\alpha-\beta)^{2s}}\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2s}}\\ &= \frac{(-1)^{s-1}2^{2s}D^s}{(2s-1)!}\Big(\sigma_{s-1}(s)D_1 + \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)D_{2j+1}\Big). \qquad (24)\end{aligned}$$

The formulas (21-24) rely on identities for series on hyperbolic functions found by I.J. Zucker [14].

4.2. Proof of Theorem 3

In (21-24) we express the q-series A_{2j+1}, B_{2j+1}, C_{2j+1}, and D_{2j+1} in terms of $x_1 = k$, $x_2 = K/\pi$, and $x_3 = E/\pi$ using the identities in Lemma 9.

`Case 1.` $X^2 - DY^2 = 1$. In what follows the triple (x_1, x_2, x_3) indicates that the variables X_i are finally replaced by the values x_i $(i = 1, 2, 3)$.

$$
\begin{aligned}
\zeta_X(2s) &= \frac{2^{2s}}{(2s-1)!}\Big[\frac{(s-1)!^2}{8}(4x_2x_3-1) \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{s-j-1}(2j)!}{2^{2j+3}}\big((2x_2)^{2j+2}e_j-b_j\big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\zeta_X(2s)}{\partial X_1}(x_1,x_2,x_3) &= \frac{2^{2s}}{(2s-1)!} \\
&\times \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{s-j-1}(2j)!}{2^{2j+3}}(2x_2)^{2j+2}e_j', \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\zeta_X(2s)}{\partial X_2}(x_1,x_2,x_3) &= \frac{2^{2s}}{(2s-1)!}\Big[\frac{(s-1)!^2}{2}x_3 \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{s-j-1}(2j)!(j+1)}{2^{2j+1}}(2x_2)^{2j+1}e_j\Big], \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_Y(2s) &= \frac{2^{2s}D^s}{(2s-1)!}\Big[\frac{(s-1)!^2(-1)^{s-1}}{24}\big(1-12x_2x_3-(x_1^2-2)(2x_2)^2\big) \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\big(a_j-(2x_2)^{2j+2}c_j\big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\zeta_Y(2s)}{\partial X_1}(x_1,x_2,x_3) &= \frac{2^{2s}D^s}{(2s-1)!}\Big[\frac{(s-1)!^2(-1)^s}{12}x_1(2x_2)^2 \\
&- \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}(2x_2)^{2j+2}c_j'\Big], \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\zeta_Y(2s)}{\partial X_2}(x_1,x_2,x_3) &= \frac{2^{2s}D^s}{(2s-1)!}\Big[\frac{(s-1)!^2(-1)^s}{6}\Big(3x_3+(x_1^2-2)(2x_2)\Big) \\
&- \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!(j+1)}{2^{2j+1}}(2x_2)^{2j+1}c_j\Big]. \qquad (28)
\end{aligned}
$$

Here, c_j' and e_j' are the derivatives of the polynomials $c_j = c_j(k)$ and $e_j = e_j(k)$, respectively, with respect to their variable $x_1 = k$.

Case 2. $X^2 - DY^2 = -1$.

$$\begin{aligned}
\zeta_X(2s) &= \frac{2^{2s}}{(2s-1)!}\Big[\frac{(s-1)!^2(-1)^{s-1}}{2}(x_2^2 - x_2x_3) \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}(2x_2)^{2j+2}f_j\Big],
\end{aligned}$$

$$\frac{\partial\zeta_X(2s)}{\partial X_1}(x_1,x_2,x_3) = \frac{2^{2s}}{(2s-1)!}\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}(2x_2)^{2j+2}f_j', \quad (29)$$

$$\begin{aligned}
\frac{\partial\zeta_X(2s)}{\partial X_2}(x_1,x_2,x_3) &= \frac{2^{2s}}{(2s-1)!}\Big[\frac{(s-1)!^2(-1)^{s-1}}{2}(2x_2 - x_3) \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!(j+1)}{2^{2j+1}}(2x_2)^{2j+1}f_j\Big], \quad (30)
\end{aligned}$$

$$\begin{aligned}
\zeta_Y(2s) &= \frac{2^{2s}D^s}{(2s-1)!}\Big[\frac{(s-1)!^2}{2}\big(x_2x_3 + (x_1^2-1)x_2^2\big) \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{s-j}(2j)!}{2^{2j+3}}(2x_2)^{2j+2}d_j\Big],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\zeta_Y(2s)}{\partial X_1}(x_1,x_2,x_3) &= \frac{2^{2s}D^s}{(2s-1)!}\Big[\frac{(s-1)!^2}{4}x_1(2x_2)^2 \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{s-j}(2j)!}{2^{2j+3}}(2x_2)^{2j+2}d_j'\Big], \quad (31)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\zeta_Y(2s)}{\partial X_2}(x_1,x_2,x_3) &= \frac{2^{2s}D^s}{(2s-1)!}\Big[\frac{(s-1)!^2}{2}\big(x_3 + (x_1^2-2)(2x_2)\big) \\
&+ \sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^{s-j}(2j)!(j+1)}{2^{2j+1}}(2x_2)^{2j+1}d_j\Big]. \quad (32)
\end{aligned}$$

d_j' and f_j' are the derivatives of the polynomials $d_j = d_j(k)$ and $f_j = f_j(k)$, respectively, with respect to their variable $x_1 = k$.
Since β and β^2 are algebraic numbers of degree two, we know by Lemma 10 that the three numbers x_1, x_2, x_3 are algebraically independent over $\mathbb{Q}$. For Theorem 3 it is enough to show by Lemma 11 with $y_1 = \zeta_X(2s_1)$ and $y_2 = \zeta_Y(2s_2)$ that the

Jacobian determinant

$$\begin{aligned}\Delta &= \Delta(x_1,X_2,x_3) = \begin{vmatrix} \dfrac{\partial\zeta_X(2s_1)}{\partial X_1} & \dfrac{\partial\zeta_X(2s_1)}{\partial X_2} \\ \dfrac{\partial\zeta_Y(2s_2)}{\partial X_1} & \dfrac{\partial\zeta_Y(2s_2)}{\partial X_2} \end{vmatrix}(x_1,X_2,x_3) \\ &= \frac{\partial\zeta_X(2s_1)}{\partial X_1}\frac{\partial\zeta_Y(2s_2)}{\partial X_2} - \frac{\partial\zeta_X(2s_1)}{\partial X_2}\frac{\partial\zeta_Y(2s_2)}{\partial X_1} \in \mathbb{Q}(x_1,x_3)[2X_2]\end{aligned}$$

does not vanish identically. In what follows let $\lambda(\Delta)$ denote the leading coefficient of the polynomial $\Delta(x_1,X_2,x_3)$ with respect to its variable $2X_2$. We are forced to treat the cases for $s_1=1$ or $s_2=1$ separately.

`Case 1.` $X^2-DY^2=1$

`Case 1.1.` $s_1>1$, $s_2>1$. We have from (25-28) that $\deg_{2X_2}(\Lambda)=2s_1+2s_2-1$, and

$$\lambda(\Delta) = \frac{D^{s_2}(-1)^{s_2}}{(2s_1-1)(2s_2-1)}\bigl(s_2c_{s_2-1}e'_{s_1-1}-s_1c'_{s_2-1}e_{s_1-1}\bigr)\,. \tag{33}$$

Let us assume that

$$s_2c_{s_2-1}e'_{s_1-1}-s_1c'_{s_2-1}e_{s_1-1} \equiv 0\,, \tag{34}$$

where the left-hand side is a polynomial in k with rational coefficients. The polynomials $c_j(k)$ and $e_j(k)$ never vanish identically. Therefore, there is an interval $[k_0,k_1]\subseteq[0,1]$ with $c_{s_2-1}(t)e_{s_1-1}(t)\neq 0$ for $k_0\le t\le k_1$. Then, by (34), we have

$$s_2\frac{e'_{s_1-1}(t)}{e_{s_1-1}(t)} = s_1\frac{c'_{s_2-1}(t)}{c_{s_2-1}(t)} \qquad (k_0\le t\le k_1)\,.$$

Integrating both sides with respect to t from k_0 to k we find that

$$\frac{e^{s_2}_{s_1-1}(k)}{c^{s_1}_{s_2-1}(k)} = \mathrm{const}_k \tag{35}$$

is a constant function for $k_0\le k\le k_1$. Consequently this identity holds for $0\le k\le 1$. But from Lemma 8 we obtain

$$\deg_k\bigl(e^{s_2}_{s_1-1}(k)\bigr)=2s_1s_2-2s_2 \qquad \text{and} \qquad \deg_k\bigl(c^{s_1}_{s_2-1}(k)\bigr)=2s_1s_2\,,$$

which is inconsistent with (35). Hence, $s_2c_{s_2-1}(k)e'_{s_1-1}(k)-s_1c'_{s_2-1}(k)e_{s_1-1}(k)$ does not vanish identically. Since k is a transcendental number, it follows from (33) that $\lambda(\Delta)\neq 0$. The three numbers x_1,x_2,x_3 are algebraically independent over $\mathbb{Q}$. Thus, the polynomial $\Delta(X_1,X_2,X_3)$ does not vanish at $X_i=x_i$ $(i=1,2,3)$, because its leading coefficient with respect to $2X_2$ does not vanish.

`Case 1.2.` $s_1>1$, $s_2=1$. We have

$$\zeta_Y(2s_2) = \zeta_Y(2) = \frac{D}{6}\bigl(1-12x_2x_3-(x_1^2-2)(2x_2)^2\bigr)\,, \tag{36}$$

which yields

$$\Delta(x_1, X_2, x_3) \;=\; \Big(-2Dx_3 - \frac{2}{3}D(x_1^2-2)(2X_2)\Big)\frac{\partial\zeta_X(2s_1)}{\partial X_1} + \frac{D}{3}x_1(2X_2)^2\frac{\partial\zeta_X(2s_1)}{\partial X_2}\,.$$

Here, $\deg_{2X_2}(\Delta) = 1+2s_1$, and with $x_1 = k$ we obtain

$$\lambda(\Delta) \;=\; \frac{D}{3(2s_1-1)}\big(2ks_1e_{s_1-1} - (k^2-2)e'_{s_1-1}\big)\,.$$

From $2ks_1e_{s_1-1}(k) - (k^2-2)e'_{s_1-1}(k) \equiv 0$ we obtain as in Case 1.1 that

$$\frac{e_{s_1-1}(k)}{(2-k^2)^{s_1}}$$

is a constant function for $0 \le k \le 1$. But again by using Lemma 8 we have

$$\deg_k\big(e_{s_1-1}(k)\big) = 2s_1-2 \qquad\text{and}\qquad \deg_k\big((2-k^2)^{s_1}\big) = 2s_1\,,$$

a contradiction. Thus we find $\Delta(x_1, x_2, x_3) \neq 0$ by the same arguments as in Case 1.1.

`Case 1.3.` $s_1 = 1$, $s_2 > 1$. We have

$$\zeta_X(2s_1) \;=\; \zeta_X(2) \;=\; 2x_2x_3 - \frac{1}{2}\,, \tag{37}$$

and therefore

$$\Delta(x_1, x_2, x_3) \;=\; -2x_3\frac{\partial\zeta_Y(2s_2)}{\partial X_1}\,.$$

We conclude that $\deg_{2X_2}(\Delta) = 2s_2 \ge 4$ and

$$\lambda(\Delta) \;=\; \frac{x_3D^{s_2}(-1)^{s_2-1}}{2s_2-1}c'_{s_2-1} \;\neq\; 0\,.$$

The nonvanishing of $\lambda(\Delta)$ follows from $\deg_k(c'_{s_2-1}(k)) = 2s_2-1 \ge 3$ and from the algebraic independence of x_1 and x_3 over $\mathbb{Q}$. As in Case 1.1 it turns out that $\Delta(x_1, x_2, x_3) \neq 0$.

`Case 1.4.` $s_1 = 1$, $s_2 = 1$. From (36) and (37) we obtain

$$\Delta(x_1, x_2, x_3) \;=\; \frac{8}{3}Dx_1x_2^2x_3 \;\neq\; 0\,.$$

`Case 2.` $X^2 - DY^2 = -1$. The arguments are essentially the same as in Case 1 using the formulas (29-32). Therefore we omit the details and confine ourselves to the necessary identities.

`Case 2.1.` $s_1 > 1$, $s_2 > 1$.

$$\begin{aligned} \deg_{2X_2}(\Delta) &= 2s_1 + 2s_2 - 1\,, \\ \lambda(\Delta) &= \frac{D^{s_2}(-1)^{s_1}}{(2s_1-1)(2s_2-1)}\big(s_2 d_{s_2-1} f'_{s_1-1} - s_1 d'_{s_2-1} f_{s_1-1}\big)\,. \end{aligned}$$

Assuming that $\lambda(\Delta)$ vanishes identically, we find that

$$\frac{f^{s_2}_{s_1-1}(k)}{d^{s_1}_{s_2-1}(k)} = \mathrm{const}_k\,,$$

which is impossible since

$$\deg_k\big(f^{s_2}_{s_1-1}(k)\big) = 2s_1s_2 - 2s_2 \qquad \text{and} \qquad \deg_k\big(d^{s_1}_{s_2-1}(k)\big) = 2s_1s_2$$

by Lemma 8.

`Case 2.2.` $s_1 > 1$, $s_2 = 1$. We have

$$\begin{aligned} \zeta_Y(2) &= D(2x_2)x_3 + \frac{D}{2}(x_1^2-1)(2x_2)^2\,, \\ \deg_{2X_2}(\Delta) &= 1 + 2s_1\,, \\ \lambda(\Delta) &= \frac{D(-1)^{s_1-1}}{2s_1-1}\big((k^2-1)f'_{s_1-1} - 2s_1 k f_{s_1-1}\big)\,. \end{aligned}$$

Assuming that $\lambda(\Delta)$ vanishes identically, we find that

$$\frac{f_{s_1-1}(k)}{(1-k^2)^{s_1}} = \mathrm{const}_k\,,$$

which is impossible since

$$\deg_k\big(f_{s_1-1}(k)\big) = 2s_1 - 2 \qquad \text{and} \qquad \deg_k\big((1-k^2)^{s_1}\big) = 2s_1$$

by Lemma 8.

`Case 2.3.` $s_1 = 1$, $s_2 > 1$. We have $\zeta_X(2) = 2x_2^2 - 2x_2x_3$, hence

$$\Delta(x_1,x_2,x_3) = (2x_3 - 4x_2)\frac{\partial\zeta_Y(2s_2)}{\partial X_1}\,.$$

With $\deg_{2X_2}(\Delta) = 2s_2 + 1 \geq 5$ and

$$\lambda(\Delta) = \frac{D^{s_2}}{2s_2-1}d'_{s_2-1} \neq 0$$

by $\deg_k(d'_{s_2-1}(k)) = 2s_2 - 1 \geq 3$, it follows again that $\Delta(x_1,x_2,x_3) \neq 0$.

`Case 2.4.` $s_1 = 1$, $s_2 = 1$. Here,

$$\Delta(x_1,x_2,x_3) = 8Dx_1x_2^2(x_3 - 2x_2) \neq 0\,.$$

This completes the proof of Theorem 3. □

5. Auxiliary Results for Sets of Three Reciprocal Sums

In the following we need beside $A_{2j+1}(q)$, $B_{2j+1}(q)$, $C_{2j+1}(q)$, and $D_{2j+1}(q)$ three further q-series (cf. [3, Sec. 1]).

$$\begin{aligned} L_1(q) &= \sum_{n=1}^{\infty} \frac{nq^{2n}}{1+q^{2n}}, \\ M_1(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} nq^{2n}}{1+q^{2n}}, \\ J_0(q) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n-1/2}}{1-q^{2n-1}}. \end{aligned}$$

The numbers $\alpha = X_1 + Y_1\sqrt{D}$ and $\beta = X_1 - Y_1\sqrt{D}$ were already defined in Section 1.

5.1. Series Formed by Solutions of $X^2 - DY^2 = 1$

We have $\alpha\beta = 1$. For $s = 1$ we obtain by (21) and (22) the identities

$$\sum_{n=1}^{\infty} \frac{1}{X_n^2} = 4\sum_{n=1}^{\infty} \frac{1}{\left(\alpha^n + \beta^n\right)^2} = 4\sum_{n=1}^{\infty} \frac{1}{V_n^2} = 4B_1(\beta) \tag{38}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Y_n^2} = 4D\sum_{n=1}^{\infty} \frac{1}{\left(\alpha^n - \beta^n\right)^2} = \frac{4D}{\left(\alpha - \beta\right)^2}\sum_{n=1}^{\infty} \frac{1}{U_n^2} = 4DA_1(\beta). \tag{39}$$

It follows by (6) and (7) for $m = n$ that

$$X_nY_n = \frac{1}{4\sqrt{D}}\left(\alpha^{2n} - \beta^{2n}\right),$$

where $\alpha^2\beta^2 = \left(\alpha\beta\right)^2 = 1$. Similarly to the proof of (39) we obtain the formula

$$\sum_{n=1}^{\infty} \frac{1}{\left(X_nY_n\right)^2} = 16D\sum_{n=1}^{\infty} \frac{1}{\left(\alpha^{2n} - \beta^{2n}\right)^2} = 16DA_1(\beta^2). \tag{40}$$

The corresponding alternating series can be expressed in terms of q-series, too:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{X_n^2} = 4M_1(\beta), \tag{41}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{Y_n^2} = 4DL_1(\beta), \tag{42}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left(X_nY_n\right)^2} = 16DL_1(\beta^2). \tag{43}$$

5.2. Series Formed by Solutions of $X^2 - DY^2 = -1$

We have $\alpha\beta = -1$. For $s = 1$ we obtain by (23) and (24) the identities

$$\sum_{n=1}^{\infty} \frac{1}{X_n^2} = 4\sum_{n=1}^{\infty} \frac{1}{\left(\alpha^{2n-1} + \beta^{2n-1}\right)^2} = 4\sum_{n=1}^{\infty} \frac{1}{V_{2n-1}^2} = 4C_1(\beta^2) \tag{44}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Y_n^2} = 4D\sum_{n=1}^{\infty} \frac{1}{\left(\alpha^{2n-1} - \beta^{2n-1}\right)^2} = \frac{4D}{(\alpha-\beta)^2}\sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^2} = 4DD_1(\beta^2)\,. \tag{45}$$

It follows by (6) and (7) for $m = 2n-1$ that

$$X_nY_n = \frac{1}{4\sqrt{D}}\left(\alpha^{4n-2} - \beta^{4n-2}\right) = \frac{1}{4\sqrt{D}}\left(\left(\alpha^2\right)^{2n-1} - \left(\beta^2\right)^{2n-1}\right),$$

where $\alpha^2\beta^2 = \left(\alpha\beta\right)^2 = 1$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{\left(X_nY_n\right)^2} = 16D\sum_{n=1}^{\infty} \frac{1}{\left(\alpha^{4n-2} - \beta^{4n-2}\right)^2} = 16DC_1(\beta^4)\,. \tag{46}$$

Similarly we derive from [1] the identity

$$\sum_{n=1}^{\infty} \frac{1}{Y_n} = 2\sqrt{D}\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n-1} - \beta^{2n-1}} = \frac{2\sqrt{D}}{\alpha-\beta}\sum_{n=1}^{\infty} \frac{1}{U_{2n-1}} = -2\sqrt{D}J_0(\beta^2)\,. \tag{47}$$

6. An Algebraic Independence Result for Three Reciprocal Sums Formed by Solutions of Pell's Equation

All the statements in this section are going back to algebraic independence and dependence results on certain q-series proven in [3]. The three identities

$$A_1(q) - B_1(q) = 4A_1(q^2)\,, \tag{48}$$
$$C_1(q) - D_1(q) = 4C_1(q^2)\,, \tag{49}$$
$$L_1(q) - M_1(q) = 4M_1(q^2) \tag{50}$$

can easily be proven by using the underlying q-series. Now, let q be an algebraic number over $\mathbb{Q}$ with $0 < |q| < 1$. As particular cases of [3, Theorem 2] we know that in each set $\{A_1(q), B_1(q)\}$, $\{C_1(q), D_1(q)\}$, and $\{L_1(q), M_1(q)\}$, the two elements are algebraically independent numbers over $\mathbb{Q}$. Assume that $A_1(q)$ and $A_1(q^2)$ are algebraically dependent over $\mathbb{Q}$. Substituting the expression for $A_1(q^2)$ from (48) into the relation between $A_1(q)$ and $A_1(q^2)$, we obtain a nontrivial relation between $A_1(q)$ and $B_1(q)$. The contradiction shows that $A_1(q)$ and $A_1(q^2)$ are algebraically independent over $\mathbb{Q}$. It can be proven analogously that $B_1(q)$ and $A_1(q^2)$ are algebraically independent over $\mathbb{Q}$. In a similar vein (49) and (50) are applied to prove the following lemma.

Lemma 12. *Let q be an algebraic number over $\mathbb{Q}$ satisfying $0 < |q| < 1$. Then there are the following statements.*

1. *Any two numbers in the set*
$$\{A_1(q), B_1(q), A_1(q^2)\}$$
are algebraically independent over $\mathbb{Q}$.

2. *Any two numbers in the set*
$$\{C_1(q), D_1(q), C_1(q^2)\}$$
are algebraically independent over $\mathbb{Q}$.

3. *Any two numbers in the set*
$$\{L_1(q), M_1(q), L_1(q^2)\}$$
are algebraically independent over $\mathbb{Q}$.

The real number $\beta := X_1 - Y_1\sqrt{D}$ is a quadratic surd satisfying $0 < |\beta| < 1$. With (38), (39), (40), and (44), (45), (46), and (41), (42), (43), respectively, we deduce the following theorem applying Lemma 12.

Theorem 4. 1. *Let (X_n, Y_n) denote positive integer solutions either of Pell's equation $X^2 - DY^2 = 1$ or of Pell's equation $X^2 - DY^2 = -1$. Then any two numbers in the set*
$$\left\{ \sum_{n=1}^{\infty} \frac{1}{X_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{Y_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{\big(X_nY_n\big)^2} \right\}$$
are algebraically independent over $\mathbb{Q}$. The three numbers in this set are linearly dependent over $\mathbb{Q}$.

2. *Let (X_n, Y_n) denote positive integer solutions of Pell's equation $X^2-DY^2 = 1$. Then any two numbers in the set*
$$\left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{X_n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{Y_n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\big(X_nY_n\big)^2} \right\}$$
are algebraically independent over $\mathbb{Q}$. The three numbers in this set are linearly dependent over $\mathbb{Q}$.

The statements on the linear dependence of three numbers in each of the sets in Theorem 4 follow easily from

$$\frac{1}{Y_n^2} - D\frac{1}{X_n^2} = \frac{\pm 1}{X_n^2 Y_n^2} \qquad (n = 1, 2, \dots)\,.$$

But there are nontrivial identities for reciprocal sums formed by solutions of Pell's equation. Let q be any complex number satisfying $0 < |q| < 1$. We cite two identities from Theorem 3 in [3]:

$$A_1(q) + B_1(q) = 2L_1(q)\,, \tag{51}$$
$$C_1(q) + D_1(q) = 2J_0^2(q)\,. \tag{52}$$

Case 1: Let (X_n, Y_n) be positive integer solutions of $X^2 - DY^2 = 1$. Then we have from (51) (with $q = \beta$), (38), (39) and (42) the formula

$$\sum_{n=1}^{\infty} \frac{1}{Y_n^2} + D\sum_{n=1}^{\infty} \frac{1}{X_n^2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{Y_n^2}\,.$$

Case 2: Let (X_n, Y_n) be positive integer solutions of $X^2 - DY^2 = -1$. Here, (52) (with $q = \beta^2$), (44), (45) and (47) prove

$$\sum_{n=1}^{\infty} \frac{1}{Y_n^2} + D\sum_{n=1}^{\infty} \frac{1}{X_n^2} = 2\Big(\sum_{n=1}^{\infty} \frac{1}{Y_n}\Big)^2\,.$$

7. An Application to the Cattle Problem of Archimedes

Since many authors have treated the details of the famous cattle problem (cf. [12], [13]), we omit all introductory explanations and restrict to the diophantine equations and their solutions. Let

$w_{æ}$	:	number of white bulls	$w_{ß}$	:	number of white cows
$b_{æ}$	:	number of black bulls	$b_{ß}$	:	number of black cows
$d_{æ}$	:	number of dappled bulls	$d_{ß}$	:	number of dappled cows
$y_{æ}$	:	number of yellow bulls	$y_{ß}$	:	number of yellow cows

There are seven linear and two quadratic restrictions. The linear equations are given by

$$
\begin{aligned}
w_{\text{æ}} &= \left(\frac{1}{2}+\frac{1}{3}\right) b_{\text{æ}} + y_{\text{æ}}\,,\\
b_{\text{æ}} &= \left(\frac{1}{4}+\frac{1}{5}\right) d_{\text{æ}} + y_{\text{æ}}\,,\\
d_{\text{æ}} &= \left(\frac{1}{6}+\frac{1}{7}\right) w_{\text{æ}} + y_{\text{æ}}\,,\\
w_{\text{ß}} &= \left(\frac{1}{3}+\frac{1}{4}\right) (b_{\text{ß}} + b_{\text{æ}})\,,\\
b_{\text{ß}} &= \left(\frac{1}{4}+\frac{1}{5}\right) (d_{\text{ß}} + d_{\text{æ}})\,,\\
d_{\text{ß}} &= \left(\frac{1}{5}+\frac{1}{6}\right) (y_{\text{ß}} + y_{\text{æ}})\,,\\
y_{\text{ß}} &= \left(\frac{1}{6}+\frac{1}{7}\right) (w_{\text{ß}} + w_{\text{æ}})\,.
\end{aligned}
$$

The two quadratic equations are

$$w_{\text{æ}} + b_{\text{æ}} \in \text{˜}\,, \tag{53}$$

$$y_{\text{æ}} + d_{\text{æ}} \in \triangle\,, \tag{54}$$

where ˜ denotes the set of perfect squares, whereas $\triangle$ is the set of triangular numbers. There are positive integers X, Y solving Pell's equation

$$X^2 - 410\,286\,423\,278\,424\,Y^2 = 1\,. \tag{55}$$

Since X is an odd number, let m be defined by $X = 2m+1$. Moreover, set

$$k := 4\,456\,749\,Y^2\,.$$

Then, a solution of the cattle problem is given by

$$
\begin{aligned}
w_{\text{ß}} &= 7\,206\,360\,k\,,\\
b_{\text{ß}} &= 4\,893\,246\,k\,,\\
y_{\text{ß}} &= 5\,439\,213\,k\,,\\
d_{\text{ß}} &= 3\,515\,820\,k\,,\\
w_{\text{æ}} &= 10\,366\,482\,k\,,\\
b_{\text{æ}} &= 7\,460\,514\,k\,,\\
y_{\text{æ}} &= 4\,149\,387\,k\,,\\
d_{\text{æ}} &= 7\,358\,060\,k\,.
\end{aligned}
$$

The restrictions (53) and (54) are fulfilled by

$$\begin{aligned} w_{æ} + b_{æ} &= 17\,826\,996\,k = (8\,913\,498\,Y)^2 , \\ y_{æ} + d_{æ} &= 11\,507\,447\,k = \frac{m(m+1)}{2} . \end{aligned}$$

It is obvious that the cattle problem has infinitely many solutions, which can be computed effectively by solving Pell's equation (55). By

$$C^{[n]} := w_{æ}^{[n]} + b_{æ}^{[n]} + d_{æ}^{[n]} + y_{æ}^{[n]} + w_{ß}^{[n]} + b_{ß}^{[n]} + d_{ß}^{[n]} + y_{ß}^{[n]}$$

we denote the number of cattles of the nth herd satisfying the Archimedean restrictions, where $C^{[1]} < C^{[2]} < \dots$. One knows that the smallest herd consists of $C^{[1]} = 776 \dots 800$ cattles, where $C^{[1]}$ is a number formed by 206 545 digits.
From Theorem 4, (55), and from

$$C^{[n]} = 50\,389\,082\,k = 224\,571\,490\,814\,418\,Y^2 ,$$

we derive the following result.

Theorem 5. *The numbers*

$$\sum_{n=1}^{\infty} \frac{1}{C^{[n]}} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{C^{[n]}}$$

are transcendental.

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